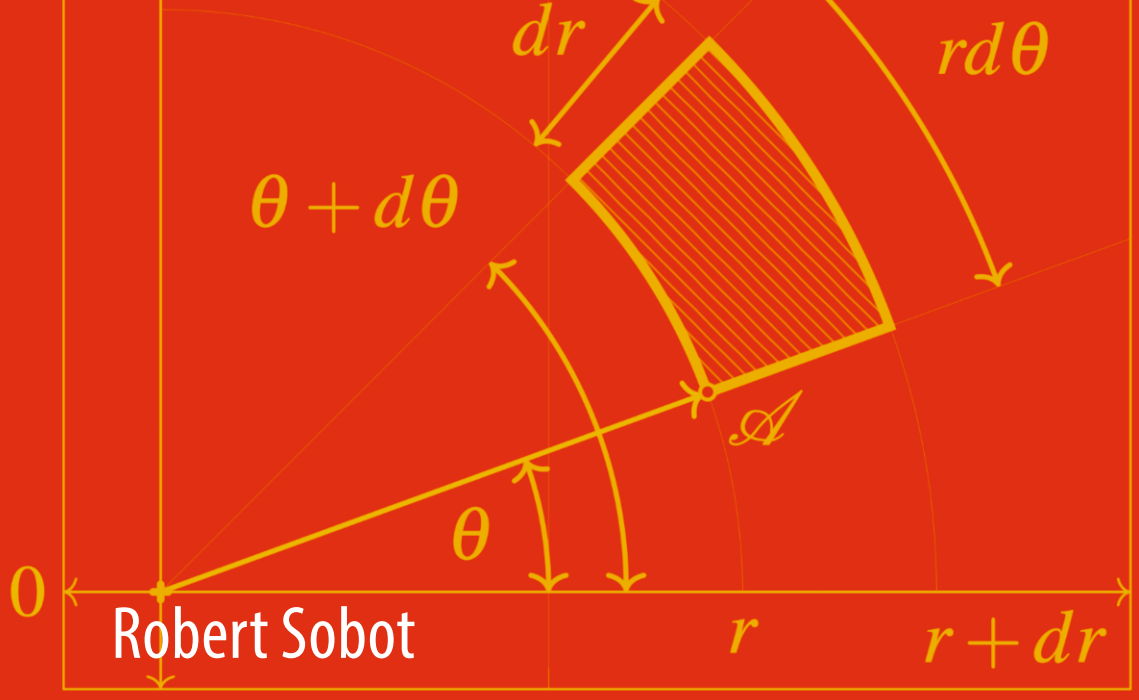

Robert Sobot

Engineering Mathematics by Example

Vol. II: Calculus

Second Edition

Springer

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Robert Sobot
ETIS-ENSEA
Cergy-Pontoise, France

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To my math teacher Mr. Miloš Belušević

Preface

Preface to the Second Edition

It is inevitable that first edition of any type of textbook, and especially textbooks for mathematics, includes quite a few errors that slipped by all preproduction reviews. In this second edition, errors are thoroughly reviewed and corrected, with hopes that not many new ones are created.

As the original volume doubled, this second edition is split into three separate books, Vol. I, II, III. In order to reinforce natural development in the study, best effort is put to logically organize the presented examples and techniques so that subsequent problems reference the once already solved.

Île-de-France, France — Robert Sobot
June 27, 2023

Preface to the First Edition

This tutorial book resulted from my lecture notes developed for undergraduate engineering courses in mathematics that I teach over the last several years at l'École Nationale Supérieure de l'Électronique et de ses Applications (ENSEA), Cergy in Val d'Oise department, France.

My main inspiration to write this tutorial type collection of solved problems came from my students who would often ask "How do I solve this? It is impossible to find the solution," while struggling to logically connect all the little steps and techniques that are required to combine together before reaching solution. In the traditional classical school systems mathematics used to be thought with the help of systematically organized volumes of problems that help us develop "the way of thinking." In other words, to learn how to apply the abstract mathematical concepts to everyday engineering problems. Same as for music, it is also true for mathematics that in order to reach high level of competence one must put daily effort into studying of typical forms over long period of time.

In this tutorial book I choose to give not only the complete solutions to the given problems, but also guided hints to techniques being used at the given moment. Therefore, problems presented in this book do not provide review of the rigorous mathematical theory, instead the theoretical background is assumed, while this set of classic problems provides a playground to play and to adopt some of the main problem-solving techniques.

The intended audience of this book are primarily undergraduate students in science and engineering. At the same time, my hope is that students of mathematics at any level will find this book to be useful source of practical problems to practice.

Îlele-de-France, France Robert Sobot
November 30, 2020

Acknowledgments

I would like to acknowledge all those classic wonderful collections of mathematical problems that I grew up with and used as the source of my knowledge, and I would like to say thank you to their authors for providing me with thousands of problems to work on. Specifically, I would like to acknowledge classic collections written in ex–Yugoslavian, Russian, English and French languages, some are listed in the bibliography. Hence, I do want to acknowledge their contributions that are clearly visible throughout this book, which are now being passed on to my readers.

I would like to thank all of my former and current students who I had opportunity to tutor in mathematics since my student years. Their relentless stream of questions posed with unconstrained curiosity forced me to open my mind, to broaden my horizons, and to improve my own understandings.

I want to acknowledge Mr. Allen Sobot for reviewing and verifying some of the problems, for very useful discussions and suggestions, as well as his work on technical redaction in first edition of this book.

Sincere gratitude goes to my publisher and editors for their support and making this book possible.

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Acronyms

$f(x)$, $g(x)$	Functions of x
$f'(x)$, $g'(x)$	First derivatives of functions of $f(x)$, $g(x)$
$f(g(x))$	Composite function
$\sin(x)$	Sine of x
$\cos(x)$	Cosine of x
$\tan(x)$	Tangent of x
$\arctan(x)$	Arctangent of x
$\text{sinc}\,(t)$	Sine cardinal of t
$\log(x)$	Logarithm of x, base 10
$\ln(x)$	Natural logarithm of x, base e
$\log_n(x)$	Logarithm of x, base n
$x * y$	Convolution product
i	Imaginary unit, $i^2 = -1$
j	Imaginary unit, $j^2 = -1$
$\Im(z)$	Imaginary part of a complex number z
$\Re(z)$	Real part of a complex number z
$\mathscr{F}(x(t))$	Fourier transform of $x(t)$
$X(\omega)$	Fourier transform of $x(t)$
$\mathscr{L}^{-1}\{F(s)\}$	Inverse Laplace transform of $F(s)$
$\mathscr{F}^{-1}(X(\omega))$	Inverse Fourier transform of $X(\omega)$
$\overset{F}{\rightarrow}$	Apply Fourier transform
$\overset{LT}{\longrightarrow}$	Apply Laplace transform
$\overset{\text{l'H}}{=}$	Apply l'Hôpital rule
DFT	Discrete Fourier transform
FFT	Fast Fourier transform
$\delta(x)$	Dirac distribution
$\Lambda(x)$	Triangular distribution
$\Pi(x)$	Square (rectangular) distribution
Ш_T	Dirac comb whose period is T
$\Gamma(t)$	Heaviside step function of t
$u(t)$	Heaviside step function of t
$r(t)$	Ramp function of t
$\text{sign}\,(t)$	Sign function of t
(a, b)	Pair of numbers
(a, b)	Open interval
$\equiv$	Equivalence

$\{a_1, a_2, a_3, \dots\}$	List of elements, vector, series
$\{a_n\}$	List of elements, vector, series
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational numbers
$a \in \mathbb{C}$	Number a is included in the set of complex numbers
$a \in \mathbb{R}$	Number a is included in the set of real numbers
$\forall x$	For all values of x
$\angle$	Angle, argument
$\Rightarrow$	It follows that
$\frac{dy}{dx}$	Derivative of y relative to x
u'_x	Partial derivative of $u(x, y, z, \dots)$ relative to x
$u''_{x,y}$	Second partial derivative of $u(x, y, z, \dots)$, relative to x then to y
z^*	Complex conjugate of number z
$\|x\|$	Absolute value of x
$A_{m,n}$	Matrix A whose size is $m \times n$
I_n	The identity square matrix whose size is $n \times n$
$\|A\|$	Determinant of matrix A
A^T	Transpose of matrix A
Δ	Difference between two variables
Δ	Main determinant of a matrix
Δ_x	Cramer's sub-determinant relative to variable x
$\vec{a}$	Vector a
$\sum_{i=0}^{\infty} a_i$	Sum of elements $\{a_0, a_1, \dots, a_\infty\}$
dB	Decibel
dBm	Decibel normalized to "10^{-3}", i.e., "milli"
$H(jx)$	Transfer function
$x \to \underset{\twoheadleftarrow}{a}$	Limiting to a from its right side (x-axis)
$x \to \underset{\twoheadrightarrow}{a}$	Limiting to a from its left side (x-axis)
$x \to a\downarrow$	Limiting to a from above (y-axis)
$x \to a\uparrow$	Limiting to a from below (y-axis)
$\{\cdot\}$	in–line hints

Limits

1

(a) *Limit properties and forms:*

Basic rules for calculating limits

$$\lim_{x\to x_0}\Big(C\,f(x)\Big) = C\lim_{x\to x_0} f(x)$$

$$\lim_{x\to x_0}\Big(f(x)\pm g(x)\Big) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$$

$$\lim_{x\to x_0}\Big(f(x)\,g(x)\Big) = \lim_{x\to x_0} f(x)\,\lim_{x\to x_0} g(x)$$

$$\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

$$\lim_{x\to x_0}\Big(f(x)\Big)^n = \left(\lim_{x\to x_0} f(x)\right)^n$$

$\lim\limits_{x\to \underset{\leftarrow}{a}} f(x)$ right-side limit, i.e., $a > x$

$\lim\limits_{x\to \underset{\rightarrow}{a}} f(x)$ left-side limit, i.e., $a < x$

Important limits to infinity

$$\lim_{x\to+\infty} x^n = +\infty$$

$$\lim_{x\to-\infty} x^n = -\infty \quad n \text{ is odd}$$

$$\lim_{x\to-\infty} x^n = +\infty \quad n \text{ is even}$$

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

$$\lim_{x\to 0} \ln x = -\infty$$

$$\lim_{x\to\infty} \ln x = +\infty$$

$$\lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to\infty} e^x = +\infty$$

(b) *Some of the "famous" limits:*

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \lim_{x\to 0}\frac{e^x-1}{x} = 1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

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Problems

1.1 Simple Limits

Calculate limits in P.1.1 to P.1.18.

1.1. $\lim_{x\to 2} 5$

1.2. $\lim_{x\to 2} x$

1.3. $\lim_{x\to -1} -\frac{x}{3}$

1.4. $\lim_{x\to -2} (2x+1)$

1.5. $\lim_{x\to 0} \left(x^2+2x+1\right)$

1.6. $\lim_{x\to 0} \left(-3x^4+2x^2-x\right)$

1.7. $\lim_{x\to 0} \frac{1}{x}$

1.8. $\lim_{x\to 1} \frac{x-1}{x}$

1.9. $\lim_{x\to 1} \frac{x-4}{x-2}$

1.10. $\lim_{x\to 0} \frac{x^2-2x}{x}$

1.11. $\lim_{x\to -5} \frac{2x-6}{x^2+x-2}$

1.12. $\lim_{x\to 1/2} \frac{\frac{2x-6}{x^2}+\frac{1}{2}}{x-2}$

1.13. $\lim_{x\to 1/2} \frac{x^3+2x-1}{x-2}$

1.14. $\lim_{x\to 1} \frac{x^2-1}{x-1}$

1.15. $\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$

1.16. $\lim_{x\to -5} \frac{x^2+2x-15}{x^2+8x+15}$

1.17. $\lim_{x\to -3} \frac{x-1}{x^2-5x+6}$

1.18. $\lim_{x\to -2} \frac{x^3}{x^2-4}$

1.2 Limits to Infinity

Calculate limits in P.1.19 to P.1.24.

1.19. $\lim_{x\to\infty} -\frac{x}{3}$

1.20. $\lim_{x\to\infty} \frac{1}{x}$

1.21. $\lim_{x\to\infty} \frac{x-1}{x}$

1.22. $\lim_{x\to\infty} \frac{x-2}{x^2}$

1.23. $\lim_{x\to\infty} \frac{2x-6}{x^2+x-2}$

1.24. $\lim_{x\to\infty} \frac{x^2+2x-15}{x^2+8x+15}$

1.3 Limits Involving Radicals

Calculate limits in P.1.25 to P.1.38.

1.25. $\lim_{x\to -2} \frac{2x}{\sqrt{x+2}}$

1.26. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

1.27. $\lim_{x\to 8} \frac{\sqrt{x+1}-3}{x-8}$

1.28. $\lim_{x\to 9} \frac{3-\sqrt{x}}{x-9}$

1.29. $\lim_{x\to -\infty} \frac{\sqrt{x^2-1}}{2x+5}$

1.30. $\lim_{(x,y)\to(3,3)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

1.31. $\lim\limits_{x\to 3} \dfrac{x-3}{\sqrt{x+1}-2}$

1.32. $\lim\limits_{x\to 5} \dfrac{x^2-25}{\sqrt{5x}-5}$

1.33. $\lim\limits_{x\to 1} \dfrac{\sqrt[4]{x}-\sqrt[6]{x}}{\sqrt[8]{x}-\sqrt[12]{x}}$

1.34. $\lim\limits_{x\to 6} \dfrac{5(2-\sqrt{x-2})}{x^2-36}$

1.35. $\lim\limits_{x\to 2} \dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$

1.36. $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x}}{\sqrt{3x+\sqrt{3x+\sqrt{3x}}}}$

1.37. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x-1}-\sqrt{x^2-x+1}\right)$

1.38. $\lim\limits_{x\to\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

1.4 The Squeeze Theorem

Reminder: Given three functions where $f(x)$, $g(x)$, and $h(x)$ so that

$$f(x) \le g(x) \le h(x)$$

then, if both $f(x)$ and $h(x)$ limit to the same value a when $x \to x_0$, as

$$\lim_{x\to x_0} f(x) = a = \lim_{x\to x_0} h(x)$$

then, because $g(x)$ is bound (i.e., "squeezed") in between $f(x)$ and $h(x)$, it must be true that

$$\lim_{x\to x_0} g(x) = a$$

In other words, by calculating $f(x)$ and $h(x)$ limits, then the limit of the function $g(x)$ that they bound must have the same limit.

Calculate limits in P.1.39 to P.1.46 using the squeeze theorem.

1.39. $\lim\limits_{x\to 2} f(x)$ given that $2x \le f(x) \le x^2$

1.40. $\lim\limits_{x\to 1} \dfrac{(x-1)^5}{x-1}$ given that $0 \le \dfrac{(x-1)^5}{x-1} \le -x(x-2)(x-1)^2$

1.41. $\lim\limits_{\alpha\to 0} \sin\alpha$

1.42. $\lim\limits_{x\to 0} x\cos x$

1.43. $\lim\limits_{x\to 0} \left(x\,\sin\dfrac{1}{x}\right)$

1.44. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

1.45. $\lim\limits_{x\to\infty} \dfrac{2x+\sin x}{x}$

1.46. $\lim\limits_{x\to 0} x\,e^{\sin(1/x)}$

1.5 Limits Based on sinc (x) Function

Calculate limits in P.1.47 to P.1.52.

1.47. $\lim\limits_{x\to 0} \dfrac{x+\sin(x)}{x}$

1.48. $\lim\limits_{x\to 0} \dfrac{2x+\sin x}{\sin x}$

1.49. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

1.50. $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{x}$

1.51. $\lim\limits_{x\to 0} \dfrac{4\sin(5x)}{\sin(4x)}$

1.52. $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)}$

1.6 Limits Based on the Constant e

Reminder: One of the most common derivations of the mathematical constant e is as the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e \quad \text{or, in general} \quad \lim_{f(x)\to\infty}\left(1+\frac{1}{f(x)}\right)^{f(x)}=e$$

or, equivalently after the change of variable $1/x = t$, as

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}}=e$$

Calculate limits in P.1.53 to P.1.60.

1.53. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{3x}$

1.54. $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{x}$

1.55. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{2x}\right)^{x}$

1.56. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x+5}$

1.57. $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^{x}$

1.58. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{\frac{1}{x}}$

1.59. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^2}\right)^{\frac{1}{x}}$

1.60. $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x-1}\right)^{x+1}$

1.61. $\lim\limits_{x\to 0}\dfrac{e^{-2x}-1}{x}$

1.7 Limits Involving Trigonometric Functions

Calculate limits in P.1.62 to P.1.70.

1.62. $\lim\limits_{x\to 0} x\ \sin x$

1.63. $\lim\limits_{x\to 0} \dfrac{\sin(\sin x)}{x}$

1.64. $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x^2}$

1.65. $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{x\sin x}$

1.66. $\lim\limits_{x\to 2} \sin\left(\dfrac{x^2-4}{x-2}\right)$

1.67. $\lim\limits_{x\to 0} (1+\sin x)^{1/x}$

1.68. $\lim\limits_{x\to 0} \dfrac{e^{x^2} - \cos x}{x^2}$

1.69. $\lim\limits_{x\to 0} \dfrac{1 - \cos(1 - \cos x)}{x^4}$

1.70. $\lim\limits_{x\to 0} \dfrac{1 - \sqrt{\cos 2x}\ \cos x}{x^2}$

1.8 Limits of Piecewise Functions

Calculate limits in P.1.71 to P.1.76.

1.71. $\lim\limits_{x\to -2} |x|$

1.72. $\lim\limits_{x\to -1} |x + 1|$

1.73. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

1.74. $\lim\limits_{x\to 2} \dfrac{|x - 2|}{x - 2}$

1.75. $\lim\limits_{x\to 0} \begin{cases} 1 & (x \geq 0) \\ 0 & (x < 0) \end{cases}$

1.76. $\lim\limits_{x\to 2} \begin{cases} 3 & (x > 2) \\ 1 & (x < 2) \end{cases}$

1.9 Asymptotes

Reminder: Function asymptotes are found by calculating limits of that given function $f(x)$. There are three types of asymptotes: vertical, horizontal, and oblique.

(a) *Vertical asymptotes*: if point of discontinuity is found at $x = x_0$, then the vertical line at $x = x_0$ is said to be vertical asymptote of $f(x)$. Formal syntax for calculating vertical asymptote at this point is

$$\lim_{x\to x_0^{\rightarrow}} f(x) = \pm\infty \quad \text{(from the left ("below") side of } x_0\text{)}$$

$$\lim_{x\to x_0^{\leftarrow}} f(x) = \pm\infty \quad \text{(from the right ("above") side of } x_0\text{)}$$

where "$x \to x_0^{\rightarrow}$" reads as "x approaches x_0 from the left side" and "$x \to x_0^{\leftarrow}$" reads as "x approaches x_0 from the right side". If the left-side limit is not equal to the right-side limit, it is said that the limit does not exist; however, the vertical asymptote does exist.

(b) *Horizontal asymptotes*: if there is a real number a_0 so that

$$\lim_{x\to\infty} f(x) = a_0$$

then it is said that the horizontal line at $y = a_0$ is the horizontal asymptote of $f(x)$.

(c) *Oblique asymptotes*: if the following two limits exist

$$\lim_{x\to\infty} \frac{f(x)}{x} = a \quad \text{and} \quad \lim_{x\to\infty} \big(f(x) - ax = b$$

where a, b are real numbers, then it is said that the linear function $y = ax + b$ is the oblique asymptote of $f(x)$.

Calculate asymptotes of functions in P.1.77 to P.1.85.

1.77. $\frac{1}{x}$

1.78. $\frac{x}{x^2+1}$

1.79. $\frac{x^3}{x^2-4}$

1.80. $\frac{x-1}{x^2+5x+6}$

1.81. $\frac{x^3}{2(x+1)^2}$

1.82. $x+1+\frac{1}{x}$

1.83. $\frac{x-1}{\sqrt{x^2+1}}$

1.84. $\frac{x+1}{\sqrt{x^2-4}}$

1.85. $xe^{\frac{1}{x}}$

Answers

1.1 Simple Limits

1.1. A constant is a constant; it does not change its value. It does not matter what is the value of variable x; a constant keeps its own value. That is to say, $\lim_{x\to 2} 5 = 5$.

1.2. Limit of a function $f(x)$ when x tend to certain point a is simply the value of that function at that point, i.e., $f(a)$, *if defined*. That is to say, calculate $f(a)$ by direct substitution, in this case

$$\lim_{x\to 2} x = \{\text{set } x = 2\} = 2$$

The resulting value ("2") is well defined; therefore, no other calculation is needed.

1.3. By direct substitution, set $x = -1$, as

$$\lim_{x\to -1} -\frac{x}{3} = -\frac{-1}{3} = \frac{1}{3}$$

The resulting value "1/3" is well defined; thus, no further calculation is needed.

1.4. By direct substitution, set $x = -2$, i.e.,

$$\lim_{x\to -2} (2x+1) = 2(-2)+1 = -3$$

and no other calculation is needed.

1.5. Limits of polynomials are straightforward, as

$$\lim_{x\to 0} \left(x^2 + 2x + 1\right) = \{\text{set } x = 0\} = (0)^2 + 2(0)x + 1 = 1$$

1.6. Once x "arrives" to $x = 0$ value, the function equals

$$\lim_{x\to 0} \left(-3x^4 + 2x^2 - x\right) = -3(0)^4 + 2(0)^2(0) = 0$$

and there is not much else left to do.

1.7. Intuitively, as the first attempt, by direct substitution $x = 0$, it follows that

$$\lim_{x\to 0} \frac{1}{x} = \frac{1}{0} = \infty$$

However, this result is not complete; "approaching" a certain number may be done either from its "upper" or from its "lower" side. The two outcomes may be vastly different. Let us consider the following values of x "approaching" $x = 0$, first from its "lower" and then from its "upper" side, and then calculate the corresponding values of $f(x) = 1/x$ as $x \to 0$ (which reads as "x tends to 0")

x	$\cdots$	-10	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	0
$f(x) = 1/x$	$\cdots$	-0.1	-1	-10	-100	$-1\,000$	$-10\,000$	$\cdots$	$-\infty$

or from the "upper" side of $x = 0$

x	$\cdots$	$+10$	$+1$	$+0.1$	$+0.01$	$+0.001$	$+0.0001$	$\cdots$	0
$f(x) = 1/x$	$\cdots$	$+0.1$	$+1$	$+10$	$+100$	$+1\,000$	$+10\,000$	$\cdots$	$+\infty$

The two final results for $x = 0$, "$-\infty$" and "$+\infty$", are vastly different. Thus, *two separate limits* are to be calculated, one by approaching from its "lower" and one by approaching from its "upper" side relative to the limiting point (here, $x = 0$), as indicated by underscore arrow symbols,

$$\lim_{x \to \underset{\rightarrow}{0}} \frac{1}{x} = \frac{1}{\underset{\rightarrow}{0}} = -\infty \quad \text{and} \quad \lim_{x \to \underset{\leftarrow}{0}} \frac{1}{x} = \frac{1}{\underset{\leftarrow}{0}} = +\infty$$

where, in this case, the only debating issue was the positive/negative sign of the infinity; see Fig. 1.1. It is evident that by approaching $x = 0$ from the left side, this function delivers smaller and smaller negative numbers leading into the negative infinity. Similarly, while approaching $x = 0$ from the right side, this function delivers larger and larger positive numbers leading into the positive infinity. When two limits are not equal, it is said that limit at $x = 0$ does not exist.

1.8. Given a rational function form, by direct substitution, it follows that

$$\lim_{x \to 1} \frac{x - 1}{x} = \frac{1 - 1}{1} = \frac{0}{1} = 0$$

This continuous function is well defined in all points, including for $x = 1$.

1.9. Given a rational function form, by direct substitution, it follows that

$$\lim_{x \to 1} \frac{x - 4}{x - 2} = \frac{1 - 4}{1 - 2} = \frac{-3}{-1} = 3$$

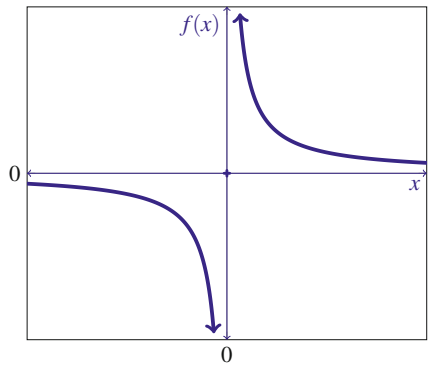


Fig. 1.1 Example P.1.7

This continuous function is well defined in all points, including for $x = 1$. The final result of this limit is therefore well defined, and there is nothing else to do.

1.10. Given this rational function $f(x)$, after factorization and by direct substitution, its limit is calculated as

$$\lim_{x\to 0} f(x) = \frac{x^2 - 2x}{x} = \lim_{x\to 0} \frac{\cancel{x}(x-2)}{\cancel{x}} \quad \{(x \neq 0)\} = \lim_{x\to 0}(x-2) = 0 - 2 = -2$$

where this rational function is *not* continuous at $x = 0$ point (due to the ambiguity of possible $0/0$ division). However, approaching this point of discontinuity from the left or the right side leads to the same value $f(0) = -2$, as

x	$\cdots$	-10	-1	-0.1	-0.01	-0.001	-0.0001	$\cdots$	0
$f(x)$	$\cdots$	-12	-3	-2.1	-2.01	-2.001	-2.0001	$\cdots$	-2

as well as

x	$\cdots$	$+10$	$+1$	$+0.1$	$+0.01$	$+0.001$	$+0.0001$	$\cdots$	0
$f(x)$	$\cdots$	8	-1	-1.9	-1.99	-1.999	-1.9999	$\cdots$	-2

Considering that both left-side and right-side limits are equal, it is said that the limit at $x = 0$ exists, despite the fact the $f(0)$ is not defined (i.e., $x \neq 0$), as

$$\lim_{x\to \underset{\rightarrow}{0}} \frac{x^2 - 2x}{x} = \lim_{x\to \underset{\rightarrow}{0}} (x-2) = \underset{\rightarrow}{0} - 2 = -2$$

and

$$\lim_{x\to \underset{\leftarrow}{0}} \frac{x^2 - 2x}{x} = \lim_{x\to \underset{\leftarrow}{0}} (x-2) = \underset{\leftarrow}{0} - 2 = -2$$

where the single discontinuity point $f(0)$ is marked by a crossed circle; see Fig. 1.2. It is evident that, while approaching $x = 0$ point, this function tends to $f(0) = -2$ from both the left and right sides.

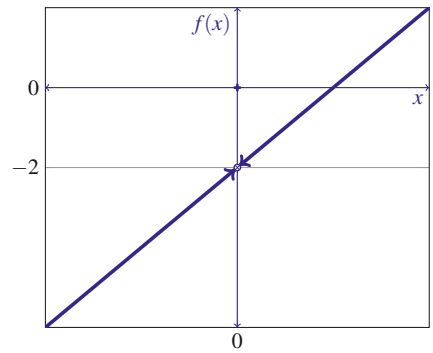


Fig. 1.2 Example P.1.10

1.11. Given rational function may be factorized, as

$$\lim_{x\to -5}\frac{2x-6}{x^2+x-2}=\lim_{x\to -5}\frac{2(x-3)}{(x+2)(x-1)}$$

so that its discontinuity points are evident, at $x=-2$ and $x=1$. Therefore, this rational function is continuous at $x=5$ where the limit is calculated by direct substitution

$$\lim_{x\to -5}\frac{2(x-3)}{(x+2)(x-1)}=\frac{2(-5-3)}{(-5+2)(-5-1)}=\frac{-16}{18}=-\frac{8}{9}$$

1.12. Given rational function may be factorized as

$$\lim_{x\to 1/2}\frac{\frac{2x-6}{x^2}+\frac{1}{2}}{x-2}=\lim_{x\to 1/2}\frac{\frac{2(2x-6)+x^2}{2x^2}}{x-2}=\lim_{x\to 1/2}\frac{x^2+4x-12}{2x^2(x-2)}=\lim_{x\to 1/2}\frac{(x-2)(x+6)}{2x^2(x-2)}=\lim_{x\to 1/2}\frac{x+6}{2x^2}$$

where $x=2$ is the discontinuity point, thus excluded. Obviously, this rational function is continuous at $x=1/2$ point (because its denominator equals to zero for $x=0$ not for $x=1/2$); therefore, the limit may be found by direct substitution as

$$\lim_{x\to 1/2}\frac{x+6}{2x^2}=\frac{1/2+6}{2(1/2)^2}=\frac{13}{4\,\frac{1}{4}}=13$$

1.13. Given rational function is discontinuous at $x=2$; therefore, its limit to $1/2$ is found by direct substitution as

$$\lim_{x\to 1/2}\frac{x^3+2x-1}{x-2}=\frac{(1/2)^3+2(1/2)-1}{(1/2)-2}=\frac{1/8+1-1}{-3/2}=-\frac{1}{12}$$

1.14. Evidently, given rational function has discontinuity at $x=1$, and in addition, it is not defined at that point because it takes the form of $0/0$ division as

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\frac{1^2-1}{1-1}=\binom{0}{0}\quad\{\text{(not defined)}\}$$

Therefore, its limiting value at $x=1$ must be calculated, for example, after algebraic simplification as

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)\quad\{(x\neq 1)\}=(1+1)=2$$

In conclusion, even though this function is not defined at $x=1$ point, its limiting value is "2".

1.15. Similar to problem A. 1.14, the limiting value of this rational function may be calculated as

$$\lim_{x\to 1}\frac{x^2+2x-3}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+3)}{x-1}=\lim_{x\to 1}(x+3)=1+3=4\quad\{(x\neq 1)\}$$

In summary, $f(1)$ is not defined; however, the limiting value at that point equals "4".

1.16. Algebraic method for calculating given limit, for example, is

$$\lim_{x\to -5}\frac{x^2+2x-15}{x^2+8x+15}=\lim_{x\to -5}\frac{x^2-3x+5x-15}{x^2+3x+5x+15}=\lim_{x\to -5}\frac{x(x-3)+5(x-3)}{x(x+3)+5(x+3)}$$
$$=\lim_{x\to -5}\frac{\cancel{(x+5)}(x-3)}{\cancel{(x+5)}(x+3)}$$
$$=\lim_{x\to -5}\frac{x-3}{x+3}\quad\{(x\neq -5)\}=\frac{-5-3}{-5+3}=4$$

In summary, $f(-5)$ is not defined; however, the limiting value at that point equals "4".

1.17. First, given rational function may be factorized as

$$f(x)=\frac{x-1}{x^2-5x+6}=\frac{x-1}{x^2+2x+3x+6}=\frac{x-1}{x(x+2)+3(x+2)}=\frac{x-1}{(x+2)(x+3)}=\frac{P(x)}{Q(x)}$$

Evidently, there are two values when denominator $Q(x)=0$, namely, $x=-2$ and $x=-3$. At each of these two points, this function tends to infinity; therefore, these points must be approached from both left ("lower") and right ("upper") sides. Algebraic method to calculate given limit may be as follows.

Case $(x\to -3)$: only binomial term $(x+3)$ becomes zero, and by consequence, $f(x)\to\infty$. At the same time, the other two binomial terms do not cause neither infinity nor ambiguous values, therefore may be calculated by direct substitution as

$$\lim_{x\to -3^{+}}\frac{x-1}{(x+2)(x+3)}=\lim_{x\to -3^{+}}\frac{-3-1}{(-3+2)(x+3)}=\lim_{x\to -3^{+}}\frac{-4}{-(x+3)}=\frac{-4}{-((\underset{\to}{-3})+3)}$$
$$=\{\text{if } x<-3\Rightarrow (x+3)<0\}=\frac{-4}{-(\underset{\to}{0})}=\frac{-4}{0_{+}}=-\infty$$

$$\lim_{x\to -3^{-}}\frac{x-1}{(x+2)(x+3)}=\lim_{x\to -3^{-}}\frac{-3-1}{(-3+2)(x+3)}=\lim_{x\to -3^{-}}\frac{-4}{-(x+3)}=\frac{-4}{-((\underset{\leftarrow}{-3})+3)}$$
$$=\{\text{if } x>-3\Rightarrow (x+3)>0\}=\frac{-4}{-(\underset{\leftarrow}{0})}=\frac{-4}{0_{-}}=+\infty$$

At the beginning, this technique may appear a bit intimidating. The reason is that it may not be so evident why some of the binomial terms are "allowed to arrive" exactly to the point of discontinuity, in this case $(x+2)$ and $(x-1)$, while at the same time some other terms are not, such as $(x+3)$. It is because, at the end, it is the *product and/or ratio* of polynomial terms that is the source of discontinuity. The infinity is certain; the only question is to find its sign. Thus, to determine the left-/right-side *signs* of each term is the goal of this analysis, followed by the overall sign.

Similarly, for the sake of the completeness,

Case $(x \to -2)$: only binomial term $(x+2)$ becomes zero, and by consequence, $f(x) \to \infty$. At the same time, the other two binomial terms do not cause neither infinity nor ambiguous values, therefore may be calculated by direct substitution as

$$\lim_{x \to \underset{\to}{-2}} \frac{x-1}{(x+2)(x+3)} = \lim_{x \to \underset{\to}{-2}} \frac{-2-1}{(x+2)(-2+3)} = \lim_{x \to \underset{\to}{-2}} \frac{-3}{(x+2)}$$

$$= \left\{ \text{if } x < -2 \Rightarrow (x+2) < 0 \right\} = \frac{-3}{0_-} = +\infty$$

$$\lim_{x \to \underset{\leftarrow}{-2}} \frac{x-1}{(x+2)(x+3)} = \lim_{x \to \underset{\leftarrow}{-2}} \frac{-2-1}{(x+2)(-2+3)} = \lim_{x \to \underset{\leftarrow}{-2}} \frac{-3}{(x+2)}$$

$$= \left\{ \text{if } x > -2 \Rightarrow (x+2) > 0 \right\} = \frac{-3}{0_+} = -\infty$$

Note that left- and right-side limits around each of the points $x = -2$ and $x = -3$ tend to the opposite sides of infinity.

1.18. First, given rational function may be factorized as

$$f(x) = \frac{x^3}{x^2-4} = \frac{x^3}{(x-2)(x+2)} = \frac{P(x)}{Q(x)}$$

Evidently, there are two values when denominator $Q(x) = 0$, namely, $x = +2$ and $x = -2$. At each of these two points, this function tends to infinity; therefore, these points must be approached from both left ("lower") and right ("upper") sides. An algebraic method to calculate this limit may be as follows.

Case $(x \to -2)$: only binomial term $(x+2)$ becomes zero, thus responsible for $f(x) \to \infty$. The other two binomial terms are well defined at $x - -2$; hence,

$$\lim_{x \to \underset{\to}{-2}} \frac{x^3}{x^2-4} = \lim_{x \to \underset{\to}{-2}} \frac{x^3}{(x-2)(x+2)} = \lim_{x \to \underset{\to}{-2}} \frac{(-2)^3}{(-2-2)(x+2)} = \lim_{x \to \underset{\to}{-2}} \frac{-8}{-4(x+2)}$$

$$= \left\{ \text{if } x < -2 \Rightarrow (x+2) < 0 \right\} = \frac{-8}{-4(0_-)} = \frac{-8}{(0_+)} = -\infty$$

$$\lim_{x \to \underset{\leftarrow}{-2}} \frac{x^3}{x^2-4} = \lim_{x \to \underset{\leftarrow}{-2}} \frac{x^3}{(x-2)(x+2)} = \lim_{x \to \underset{\leftarrow}{-2}} \frac{(-2)^3}{(-2-2)(x+2)} = \lim_{x \to \underset{\leftarrow}{-2}} \frac{-8}{-4(x+2)}$$

$$= \left\{ \text{if } x > -2 \Rightarrow (x+2) > 0 \right\} = \frac{-8}{-4(0_+)} = \frac{-8}{0_-} = +\infty$$

Case $(x \to +2)$: only binomial term $(x-2)$ becomes zero, thus responsible for $f(x) \to \infty$.

$$\lim_{x \to \underset{\to}{2}} \frac{x^3}{x^2-4} = \lim_{x \to \underset{\to}{2}} \frac{x^3}{(x-2)(x+2)} = \lim_{x \to \underset{\to}{2}} \frac{(2)^3}{(x-2)(2+2)} = \lim_{x \to \underset{\to}{2}} \frac{8}{4(x-2)}$$

$$= \left\{ \text{if } x < 2 \Rightarrow (x-2) < 0 \right\} = \frac{8}{4(0_-)} = \frac{8}{0_-} = -\infty$$

$$\lim_{x \to \underset{\leftarrow}{2}} \frac{x^3}{x^2-4} = \lim_{x \to \underset{\leftarrow}{2}} \frac{x^3}{(x-2)(x+2)} = \lim_{x \to \underset{\leftarrow}{2}} \frac{(2)^3}{(x+2)(2+2)} = \lim_{x \to \underset{\leftarrow}{2}} \frac{8}{4(x+2)}$$

$$= \left\{ \text{if } x > 2 \Rightarrow (x-2) > 0 \right\} = \frac{8}{4(0_+)} = \frac{8}{0_+} = +\infty$$

1.2 Limits to Infinity

1.19. By direct substitution, it follows that

$$\lim_{x \to \infty} -\frac{x}{3} = -\frac{\infty}{3} = -\infty$$

1.20. In the case of inverse functions (and often encountered in practice), note that

$$\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

1.21. Slightly more complicated form relative to A. 1.20 evidently has one discontinuity point at $x = 0$, and it may be resolved as

$$\lim_{x \to \infty} \frac{x-1}{x} = \lim_{x \to \infty} \left(\frac{\cancel{x}}{\cancel{x}} - \frac{1}{x} \right) = \lim_{x \to \infty} \left(1 - \frac{1}{x} \right) \quad \{(x \neq 0)\}$$

Note that this point of discontinuity is not problematic at all: positive infinity is very far away from $x = 0$; therefore, in that region, it is always valid that $x/x = 1$. That being said,

$$\lim_{x \to \infty} \left(1 - \frac{1}{x} \right) = \lim_{x \to \infty} (1) - \lim_{x \to \infty} \left(\frac{1}{x} \right) = 1 - 0 = 1$$

because the second term tends to zero; see A. 1.20. In the shorthand syntax, the last equation may be written as

$$\lim_{x \to \infty} \left(1 - \frac{1}{\overset{0}{\cancel{x}}} \right) = \lim_{x \to \infty} (1) = 1$$

1.22. This rational form has one discontinuity at $x = 0$; however, as $(x \to \infty)$, there is no problem in dividing $x/x = 1$. Similar to A. 1.20 and A. 1.21,

$$\lim_{x\to\infty} f(x) = \frac{x-2}{x^2} = \lim_{x\to\infty}\left(\frac{x}{x^2}\overset{1}{\cancel{\frac{}{}}} - \frac{2}{x^2}\overset{0}{\nearrow}\right) = \lim_{x\to\infty}\frac{1}{x} = \frac{1}{\infty} = 0$$

1.23. This rational form may be transformed by factorizing the highest power in both numerator and denominator polynomials, so that its limit is calculated as follows:

$$\lim_{x\to\infty}\frac{2x-6}{x^2+x-2} = \lim_{x\to\infty}\frac{\cancel{x^2}(2/x - 6/x^2)}{\cancel{x^2}(1+1/x-2/x^2)} = \lim_{x\to\infty}\frac{\overset{0}{\nearrow}\quad\overset{0}{\nearrow}}{1+\underset{0}{\nearrow}\quad\underset{0}{\nearrow}} = \lim_{x\to\infty}\frac{0}{1} = \lim_{x\to\infty}(0) = 0$$

It should be evident that, as $(x \to \infty)$, this rational form tends to zero because its denominator polynomial is of higher order than its numerator polynomial.

1.24. Similar to A. 1.23,

$$\lim_{x\to\infty}\frac{x^2+2x-15}{x^2+8x+15} = \lim_{x\to\infty}\frac{\cancel{x^2}(1+2/x-15/x^2)}{\cancel{x^2}(1+8/x+15/x^2)} = \lim_{x\to\infty}\frac{1+\overset{0}{\nearrow}\quad\overset{0}{\nearrow}}{1+\underset{0}{\nearrow}\quad\underset{0}{\nearrow}} = \lim_{x\to\infty}(1) = 1$$

1.3 Limits Involving Radicals

1.25. Given

$$\lim_{x\to -2}\frac{2x}{\sqrt{x+2}}$$

note that this rational function includes radical in its denominator. That being the case, the domain of radical function term as well as denominator not equal to zero constraints must be considered. Here,

$$\pm\sqrt{x+2} \Rightarrow \underline{x \geq -2}$$

which is to say that for $(x < -2)$, this radical is not defined, and by consequence, the left-side limit of $\sqrt{x+2}$ does not exist. By extension, the left-side limit of the given function also does not exist.

The right-side limit, however, may be calculated as follows:

$$\lim_{\substack{x\to -2\\ \leftarrow}}\frac{2x}{+\sqrt{x+2}} = \lim_{\substack{x\to -2\\ \leftarrow}}\frac{-4}{+\sqrt{0_+}} = -\infty$$

assuming positive value of the square root; see Fig. 1.3. For the negative value of the square root,

$$\lim_{\substack{x\to -2\\ \leftarrow}}\frac{2x}{-\sqrt{x+2}} = \lim_{\substack{x\to -2\\ \leftarrow}}\frac{-4}{-\sqrt{0_+}} = +\infty$$

In conclusion, due to the disparate left-side (does not exist) and right-side (∞) limits, it is said that the overall limit when $(x \to -2)$ does not exist.

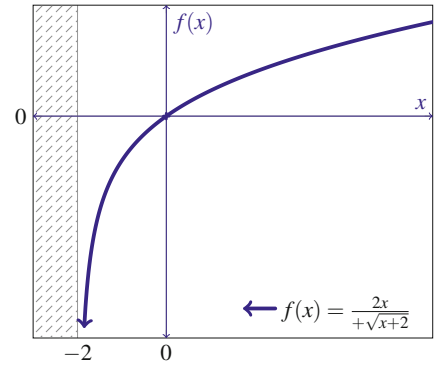


Fig. 1.3 Example P.1.25

1.26. Given

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$$

this rational form is limited to $x \geq 0$ (due to the radical) in addition to $x \neq 1$ that creates discontinuity. Furthermore, direct substitution illustrates that as $x \to 1$, this function is not determined as it takes the form of $0/0$ division

$$\lim_{x\to 1}\frac{\sqrt{1}-1}{1-1}=\frac{0}{0}$$

In general, rational functions that are not defined at certain point(s) and that include radicals should be converted into some other form by means of algebraic transformations. It is highly recommended to review techniques in Vol. I related to the factorization techniques, radical functions, and complex numbers. Sometimes, it is possible to "move" radical function from denominator to numerator by taking advantage of difference of squares identity $a^2-b^2=(a-b)(a+b)$, that is to say, to multiply both numerator and denominator by the appropriate binomial term.

In this example, "moving" radical term from numerator down to denominator results in the form that is defined, $(x \geq 0, x \neq 1)$, as

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$$

$$\left\{a^2-b^2=(a+b)(a-b)\right\}$$

$$\therefore$$

$$\left\{x-1=(\sqrt{x})^2-1^2=(\sqrt{x}+1)(\sqrt{x}-1)\right\}$$

$$=\lim_{x\to 1}\frac{\cancel{\sqrt{x}-1}}{\cancel{(\sqrt{1}-1)}(\sqrt{1}+1)}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}$$

$$=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

Fig. 1.4 Example P.1.26

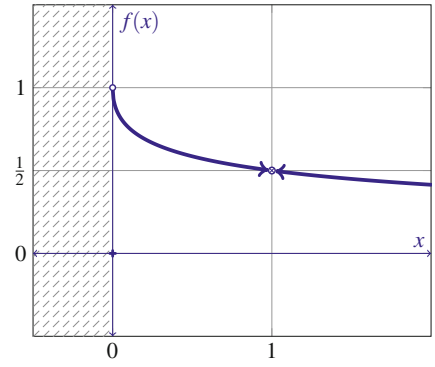
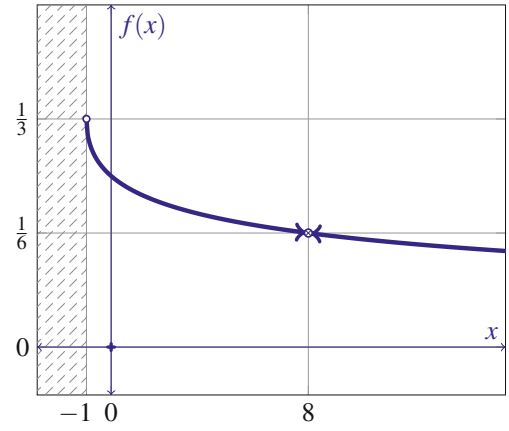


Fig. 1.5 Example P.1.27



In conclusion, even though function is not defined for $x = 1$ (see Fig. 1.4), this limit exists because both left- and right-side limits tend to $1/2$.

1.27. Given

$$\lim_{x\to 8} \frac{\sqrt{x+1}-3}{x-8} = \frac{\sqrt{8+1}-3}{8-8} = \frac{0}{0}$$

it is evident that $x \neq 8$ and $x \geq -1$. Algebraic transformations result in (Fig. 1.5)

$$\lim_{x\to 8} \frac{\sqrt{x+1}-3}{x-8} \frac{\sqrt{x+1}+3}{\sqrt{x+1}+3} = \lim_{x\to 8} \frac{(x+1)-9}{(x-8)\left(\sqrt{x+1}+3\right)} = \lim_{x\to 8} \frac{\cancel{x-8}}{\cancel{(x-8)}\left(\sqrt{x+1}+3\right)}$$

$$= \lim_{x\to 8} \frac{1}{\sqrt{x+1}+3} = \frac{1}{\sqrt{8+1}+3} = \frac{1}{6} \quad (x \neq 8)$$

1.28. Given

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}$$

where, evidently, in addition to $x \geq 0$, this function is discontinuous at $x = 9$. Similar to A. 1.26, algebraic transformations facilitate the limit calculation as

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9} = \left\{x-9=(\sqrt{x}-3)(\sqrt{x}+3)\right\} = \lim_{x\to 9}\frac{3-\sqrt{x}}{(\sqrt{x}-3)(\sqrt{x}+3)}$$
$$= \lim_{x\to 9}-\frac{\cancel{\sqrt{x}-3}\,1}{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)} = \lim_{x\to 9}-\frac{1}{\sqrt{x}+3} = -\frac{1}{\sqrt{9}+3} = -\frac{1}{6} \quad (x\neq 9)$$

1.29. Given

$$\lim_{x\to-\infty}\frac{\sqrt{x^2-1}}{2x+5} = \frac{\sqrt{(-\infty)^2-1}}{2(-\infty)+5} = \frac{\infty}{-\infty}$$

that is to say, this form of limit is not determined.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2-1}}{2x+5} = \lim_{x\to-\infty}\frac{\sqrt{x^2(1-1/x^2)}}{x(2+5/x)} = \left\{\sqrt{x^2}=|x|\right\} = \lim_{x\to-\infty}\frac{|x|\sqrt{(1-1/x^2)}}{x(2+5/x)}$$

As $(x\to-\infty)$, which means $x<0$, then $|x|=-x$, so that

$$\lim_{x\to-\infty}\frac{|x|\sqrt{(1-1/x^2)}}{x(2+5/x)} = \lim_{x\to-\infty}\frac{-\cancel{x}\sqrt{(1-\cancel{1/x^2}^{\,0})}}{\cancel{x}(2+\cancel{5/x}^{\,0})} = -\lim_{x\to-\infty}\frac{1}{2} = -\frac{1}{2}$$

1.30. Given two-variable function,

$$\lim_{(x,y)\to(3,3)}\frac{x-y}{\sqrt{x}-\sqrt{y}} = \frac{3-3}{\sqrt{3}-\sqrt{3}} = \frac{0}{0}$$

that is to say, this form of limit is not determined. Algebraic method to rearrange the radical terms results in

$$\lim_{(x,y)\to(3,3)}\frac{x-y}{\sqrt{x}-\sqrt{y}}\,\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(3,3)}\frac{\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} = \lim_{(x,y)\to(3,3)}(\sqrt{x}+\sqrt{y})$$
$$= (\sqrt{3}+\sqrt{3}) = 2\sqrt{3}; \quad (x,y)\neq(3,3)$$

1.31. Given

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2} = \frac{3-3}{\sqrt{3+1}-2} = \frac{0}{0}$$

it is evident that $x\neq 3$ and $x\geq -1$ (due to domain of the square root term). Algebraic transformations result in

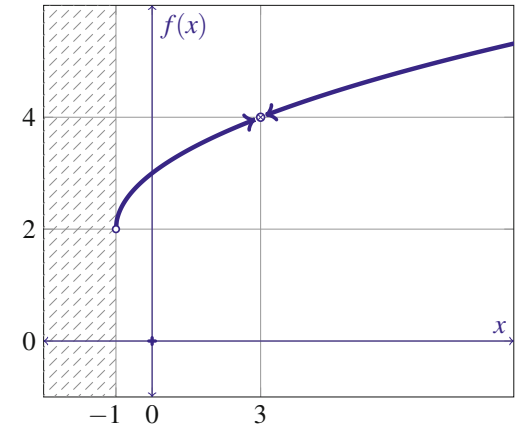


Fig. 1.6 Example P.1.31

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}\,\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}=\lim_{x\to 3}\frac{(x-3)(\sqrt{x+1}+2)}{x-3}$$
$$=\lim_{x\to 3}(\sqrt{x+1}+2)=(\sqrt{3+1}+2)=4$$

Even though given function is not continuous at $x=3$, this limit exists because both left- and right-side limits tend to the same point $(3,4)$; see Fig. 1.6.

1.32. Given

$$\lim_{x\to 5}\frac{x^2-25}{\sqrt{5x}-5}=\lim_{x\to 5}\frac{(5)^2-25}{\sqrt{5\times 5}-5}=\frac{0}{0}$$

that is to say, this form of limit is not determined.

$$\lim_{x\to 5}\frac{x^2-25}{\sqrt{5x}-5}=\lim_{x\to 5}\frac{(x-5)(x+5)}{\sqrt{5x}-5}\,\frac{\sqrt{5x}+5}{\sqrt{5x}+5}=\lim_{x\to 5}\frac{(x-5)(x+5)(\sqrt{5x}+5)}{5x-25}$$
$$=\lim_{x\to 5}\frac{(x-5)(x+5)(\sqrt{5x}+5)}{5\,(x-5)}=\frac{1}{5}\lim_{x\to 5}(x+5)(\sqrt{5x}+5)$$
$$=\frac{1}{5}\,(5+5)(\sqrt{5\times 5}+5)=20$$

1.33. Given

$$\lim_{x\to 1}\frac{\sqrt[4]{x}-\sqrt[6]{x}}{\sqrt[8]{x}-\sqrt[12]{x}}=\frac{\sqrt[4]{1}-\sqrt[6]{1}}{\sqrt[8]{1}-\sqrt[12]{1}}=\frac{0}{0}$$

that is to say, this form of limit is not determined. Radicals are simplified by algebraic transformations as

$$\lim_{x\to 1}\frac{\sqrt[4]{x}-\sqrt[6]{x}}{\sqrt[8]{x}-\sqrt[12]{x}}\ \frac{\sqrt[8]{x}+\sqrt[12]{x}}{\sqrt[8]{x}+\sqrt[12]{x}}=\lim_{x\to 1}\frac{(\sqrt[4]{x}-\sqrt[6]{x})(\sqrt[8]{x}+\sqrt[12]{x})}{\sqrt[4]{x}-\sqrt[6]{x}}=\lim_{x\to 1}(\sqrt[8]{x}+\sqrt[12]{x})=1+1=2$$

1.34. Given

$$\lim_{x\to 6}\frac{5(2-\sqrt{x-2})}{x^2-36}=\frac{5(2-\sqrt{6-2})}{6^2-36}=\frac{0}{0}$$

that is to say, this form of limit is not determined.

$$\begin{aligned}\lim_{x\to 6}\frac{5(2-\sqrt{x-2})}{x^2-36}&=\lim_{x\to 6}\frac{5(2-\sqrt{x-2})}{(x-6)(x+6)}\ \frac{2+\sqrt{x-2}}{2+\sqrt{x-2}}=\lim_{x\to 6}\frac{5\big(4-(x-2)\big)}{(x-6)(x+6)(2+\sqrt{x-2})}\\&=\lim_{x\to 6}\frac{-5\,(x-6)}{(x-6)(x+6)(2+\sqrt{x-2})}=\lim_{x\to 6}\frac{-5}{(x+6)(2+\sqrt{x-2})}\\&=\frac{-5}{(6+6)(2+\sqrt{6-2})}=-\frac{5}{48}\end{aligned}$$

1.35. Given

$$\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}=\frac{2^2+2^{3-2}-6}{\sqrt{2^{-2}}-2^{1-2}}=\frac{0}{0}$$

that is to say, this form of limit is not determined. Among other methods (see Sect. 2.6), this limit form may be resolved by the change of variable method as follows ($x\neq 2$).

$$\begin{aligned}\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}&=\lim_{x\to 2}\frac{2^x+2^3\,2^{-x}-6}{2^{-x/2}-2\cdot 2^{-x}}=\big\{2^x=t\ \ \therefore\ \text{ if } x\to 2\ \ \text{ then }\ \ t\to 4\big\}\\&=\lim_{t\to 4}\frac{t+8/t-6}{1/t^{1/2}-2/t}\ \frac{t}{t}=\lim_{t\to 4}\frac{t^2+8-6t}{t^{1/2}-2}=\lim_{t\to 4}\frac{t^2-6t+8}{\sqrt{t}-2}\ \frac{\sqrt{t}+2}{\sqrt{t}+2}\\&=\lim_{t\to 4}\frac{(t-2)(t-4)(\sqrt{t}+2)}{t-4}=\lim_{t\to 4}(t-2)(\sqrt{t}+2)\\&=\lim_{t\to 4}(4-2)(\sqrt{4}+2)=8\end{aligned}$$

1.36. Given that direct substitution method results in

$$\lim_{x\to\infty}\frac{\sqrt{2x}}{\sqrt{3x+\sqrt{3x+\sqrt{3x}}}}=\frac{\sqrt{2\,\infty}}{\sqrt{3\,\infty+\sqrt{3\,\infty+\sqrt{3\,\infty}}}}=\frac{\infty}{\infty}$$

that is to say, this form of limit is not determined. Among other methods, this limit form may be resolved by the change of variable method. That being said, note that as $t\to\infty$ (thus t is always positive), then it is true that $\sqrt{t^2}=|t|=t$. Furthermore, it is often possible for infinite limits to take advantage of the inverse function property, i.e., if $x\to\infty$, then $1/x\to 0$.

$$\lim_{x\to\infty}\frac{\sqrt{2x}}{\sqrt{3x+\sqrt{3x+\sqrt{3x}}}}=\left\{\sqrt{3x}=t\ \ \therefore\ \ x=\frac{t^2}{3},\quad \text{as well as, if}\quad x\to\infty\quad \text{, then}\quad t\to\infty\right\}$$

$$=\lim_{t\to\infty}\frac{\sqrt{2\,\frac{t^2}{3}}}{\sqrt{\cancel{3}\,\frac{t^2}{\cancel{3}}+\sqrt{\cancel{3}\,\frac{t^2}{\cancel{3}}+t}}}=\sqrt{\frac{2}{3}}\lim_{t\to\infty}\frac{|t|}{\sqrt{t^2+\sqrt{t^2+t}}}=\sqrt{\frac{2}{3}}\lim_{t\to\infty}\frac{t}{\sqrt{t^2+\sqrt{t^2\left(1+\frac{1}{t}\right)}}}$$

$$=\sqrt{\frac{2}{3}}\lim_{t\to\infty}\frac{t}{\sqrt{t^2+t\sqrt{1+\frac{1}{t}}}}=\sqrt{\frac{2}{3}}\lim_{t\to\infty}\frac{\cancel{t}}{\cancel{t}\sqrt{1+\overset{0}{\cancel{\frac{1}{t}}}}\ \sqrt{1+\overset{0}{\cancel{\frac{1}{t}}}}}=\sqrt{\frac{2}{3}}\lim_{t\to\infty}1=\sqrt{\frac{2}{3}}$$

1.37. Given that direct substitution method results in

$$\lim_{x\to\infty}\left(\sqrt{x^2+x-1}-\sqrt{x^2-x+1}\right)=\sqrt{\infty^2+\infty-1}-\sqrt{\infty^2-\infty+1}=\infty-\infty$$

that is to say, this form of limit is not determined. Algebraic transformation may be used as

$$\lim_{x\to\infty}\left(\sqrt{x^2+x-1}-\sqrt{x^2-x+1}\right)\frac{\sqrt{x^2+x-1}+\sqrt{x^2-x+1}}{\sqrt{x^2+x-1}+\sqrt{x^2-x+1}}$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}+x-1-\cancel{x^2}+x-1}{\sqrt{x^2+x-1}+\sqrt{x^2-x+1}}$$

$$=\lim_{x\to\infty}\frac{2x-2}{x\left(\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}\right)}=\lim_{x\to\infty}\frac{\cancel{x}\left(2-\frac{2}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}\right)}$$

{all inverse terms tend to zero}

$$=\frac{2}{\sqrt{1}+\sqrt{1}}=1$$

1.38. Given that direct substitution method results in

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)=\sqrt{\infty+\sqrt{\infty+\sqrt{\infty}}}-\sqrt{\infty}=\infty-\infty$$

that is to say, this form of limit is not determined. Algebraic transformation may be used as

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=\lim_{x\to\infty}\frac{\cancel{x}+\sqrt{x+\sqrt{x}}-\cancel{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}+\sqrt{x}}} = \left\{\text{expand by } \frac{\sqrt{x}}{\sqrt{x}}\right\} = \lim_{x\to\infty} \frac{\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x}}}{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}+\sqrt{x}}}{\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\sqrt{1/x}}}{\sqrt{1+\sqrt{(1/x)+\sqrt{1/x^2}}+1}} \quad \{1/x \to 0\} = \frac{\sqrt{1}}{\sqrt{1}+1} = \frac{1}{2}$$

1.4 The Squeeze Theorem

1.39. Given that $2x \le f(x) \le x^2$, it is said that $f(x)$ is bound by function on the left (here: $2x$) and by function on the right (here: x^2) side. In other words, $f(x)$ cannot take any value neither smaller than $2x$ nor greater than x^2. It is permanently "imprisoned" between these two walls. By consequence, if both left- and right-side "walls" move toward a certain point, $f(x)$ is squeezed between the two and must end up at the same spot. (There are a number of popular names for this theorem: two policemen and a thief theorem, the sandwich theorem, etc.)

Note that, as given, it is customary to write double inequalities in a single line, that is to say, if

$$2x \le f(x) \quad \text{and} \quad f(x) \le x^2$$

then these two inequalities may be written in a single line as

$$2x \le f(x) \le x^2$$

The squeeze theorem is applied by calculating the left- and right-side limits first and then by concluding that $f(x)$ must take the same value. Given double inequality

$$2x \le f(x) \le x^2$$

$$\therefore$$

$$\lim_{x\to 2} 2x \le \lim_{x\to 2} f(x) \le \lim_{x\to 2} x^2 \quad \{\text{left and right sides are moving toward } x = 2\}$$

$$\therefore$$

$$4 \le \lim_{x\to 2} f(x) \le 4 \quad \Rightarrow \lim_{x\to 2} f(x) = 4$$

as the only possible conclusion.

1.40. Direct substitution method results in a non-defined result as

$$\lim_{x\to 1} \frac{(x-1)^5}{x-1} = \frac{(1-1)^5}{1-1} = \frac{0}{0}$$

Thus, it is necessary to use some other methods to resolve this limit. Given that

$$0 \le \frac{(x-1)^5}{x-1} \le \left(-x(x-2)(x-1)^2\right)$$

the squeeze theorem is applied by calculating the left- and right-side limits first, as

$$\lim_{x\to 1} 0 \le \lim_{x\to 1} \frac{(x-1)^5}{x-1} \le \lim_{x\to 1}\left(-x(x-2)(x-1)^2\right)$$

$$10 \le \lim_{x\to 1} \frac{(x-1)^5}{x-1} \le \left(-1(1-2)(1-1)^2\right)$$

$$10 \le \lim_{x\to 1} \frac{(x-1)^5}{x-1} \le 0 \quad \Rightarrow \quad \lim_{x\to 1} \frac{(x-1)^5}{x-1} = 0$$

As $x \to 1$, the squeezing process is illustrated in Fig. 1.7 showing $\dfrac{(x-1)^5}{x-1}$ being squeezed between regions bounded by the other two functions and therefore "forced" to settle at 0 value.

1.41. Consider the unit circle in Fig. 1.8, where value of $\sin\alpha$ is evidently equal to the length of the vertical blue projection line. Its length varies between zero, when $\alpha = 0$, and one, when $\alpha = \pi/2$. At the same time, the arc length (in red) is equal to $r\alpha = \alpha$ (because the unit circle radius $r = 1$).

In Euclidean geometry, an arc connecting two points is always longer than a line (i.e., the shortest path) connecting the same two points. At any given point along the circle, the blue line length (i.e. $\sin\alpha$) is bound by zero length on one side and is always inferior to the arc length α on the other side. That relation may be formalized as

$$0 \le \sin\alpha < \alpha$$

The left-side function equals zero, and the right-side function equals α, while $\sin\alpha$ is squeezed between them. Therefore, left- and right-side limits are bounds for $\sin\alpha$ function, as

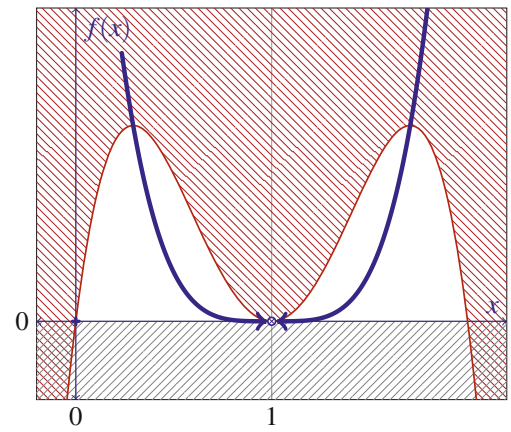
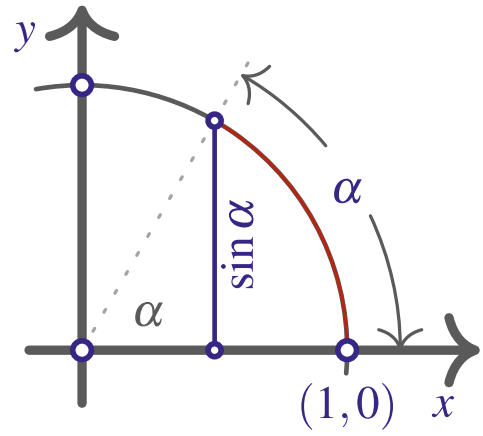


Fig. 1.7 Example P.1.40

Fig. 1.8 Example P.1.41



$$\lim_{\alpha\to 0} 0 \leq \lim_{\alpha\to 0} \sin\alpha < \lim_{\alpha\to 0} \alpha$$

Left- and right-side limits are solved first, as

$$0 \leq \lim_{\alpha\to 0} \sin\alpha < 0$$

Therefore, there is only one possible conclusion:

$$\lim_{\alpha\to 0} \sin\alpha = 0$$

This example is a classic illustration of the squeeze theorem and a proof of a very important trigonometric limit. As extension to this example, note that

$$\lim_{\alpha\to 0} \cos\alpha = \cos 0 = 1$$

is well defined at $\alpha = 0$.

1.42. In this case, direct substitution method does produce correct answer because the function is continuous at $x = 0$, as

$$\lim_{x\to 0} x\cos x = 0 \cdot 1 = 0$$

However, for the sake of exercise, the squeezing method may be applied as

$$-1 \leq \cos x \leq 1$$

$$-x \leq x\cos x \leq x$$

$$\therefore$$

$$\lim_{x\to 0}(-x) \leq \lim_{x\to 0}(x\cos x) \leq \lim_{x\to 0} x$$

$$0 \leq \lim_{x\to 0}(x\cos x) \leq 0 \quad \Rightarrow \quad \leq \lim_{x\to 0}(x\cos x) = 0$$

1.43. Direct substitution method does not produce correct limit, due to division by zero, as

$$\lim_{x\to 0}\left(x\ \sin\frac{1}{x}\right) = 0\ \sin\frac{1}{0} = 0\ \sin(\infty)$$

which is a non-determined form. By definition, sinusoidal functions are bound between ± 1 for any argument; thus, it is true to write

$$-1 \leq \sin\frac{1}{x} \leq 1$$

Then, multiplication of all three functions by x leads to

$$-x \leq x\sin\frac{1}{x} \leq x$$

and therefore to limits

$$\lim_{x\to 0}(-x) \leq \lim_{x\to 0}\left(x\sin\frac{1}{x}\right) \leq \lim_{x\to 0} x$$

$$\therefore$$

$$0 \leq \lim_{x\to 0}\left(x\sin\frac{1}{x}\right) \leq 0$$

$$\therefore$$

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0$$

As $x \to 0$, the squeezing process is illustrated in Fig. 1.9 showing $x\sin(1/x)$ being squeezed between "x" and "$-x$" functions and therefore "forced" to settle at $\sin 0 = 0$ value.

1.44. Given sinus cardinal function $\text{sinc}\,(x)$, direct substitution results in

$$\lim_{x\to 0}\text{sinc}\,(x) = \lim_{x\to 0}\frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0}$$

that is to say, not defined. Among other methods (see Sect. 2.6), this limit may be solved by the squeezing theorem after applying similar logic as in A. 1.41 where it was shown that $\sin\alpha < \alpha$. With the same argument, by creating Pythagorean triangle $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ in Fig. 1.10 (left), it is evident that the line whose length equals to $\tan\alpha$ is always greater than the arc whose length equals to α.

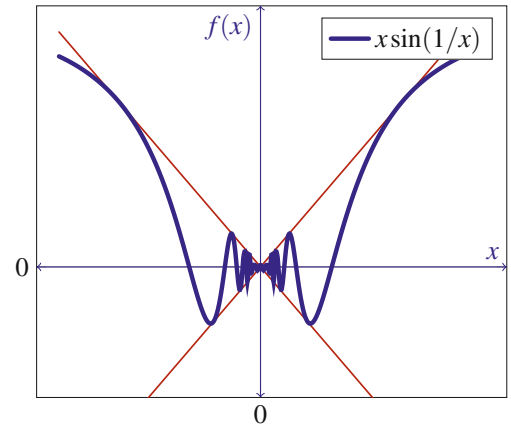


Fig. 1.9 Example P.1.43

Reminder: Note that Pythagorean triangle $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ has the length of one of the two catheti $(\mathscr{A}, \mathscr{B})$ equal to one. By definition, the tangent of angle α is the ratio of the opposite cathetus relative to the adjunct cathetus, i.e.,

$$\tan\alpha = \frac{(\mathscr{B},\mathscr{C})}{(\mathscr{A},\mathscr{B})} = \frac{(\mathscr{B},\mathscr{C})}{1} \quad \therefore \quad \tan\alpha = (\mathscr{B},\mathscr{C})$$

This relation among the intervals and arc lengths may be formalized with double inequality

$$\sin\alpha \leq \alpha \leq \tan\alpha$$

where it is said that $\sin\alpha$ is lower bound and $\tan\alpha$ is higher bound of α value. Thus,

$$\sin\alpha \leq \alpha \leq \frac{\sin\alpha}{\cos\alpha} \quad \{\text{divide all three terns by } \sin\alpha\}$$

$$\frac{\cancel{\sin\alpha}\ 1}{\cancel{\sin\alpha}} \leq \frac{\alpha}{\sin\alpha} \leq \frac{\cancel{\sin\alpha}}{\cos\alpha}\ \frac{1}{\cancel{\sin\alpha}}$$

$$\therefore$$

$$1 \leq \frac{\sin\alpha}{\alpha} \leq \cos\alpha$$

First, calculate left- and right-side limits

$$\lim_{x\to 0} 1 \leq \lim_{x\to 0} \frac{\sin\alpha}{\alpha} \leq \lim_{x\to 0} \cos\alpha$$

$$1 \leq \lim_{x\to 0} \frac{\sin\alpha}{\alpha} \leq 1$$

with the only conclusion that

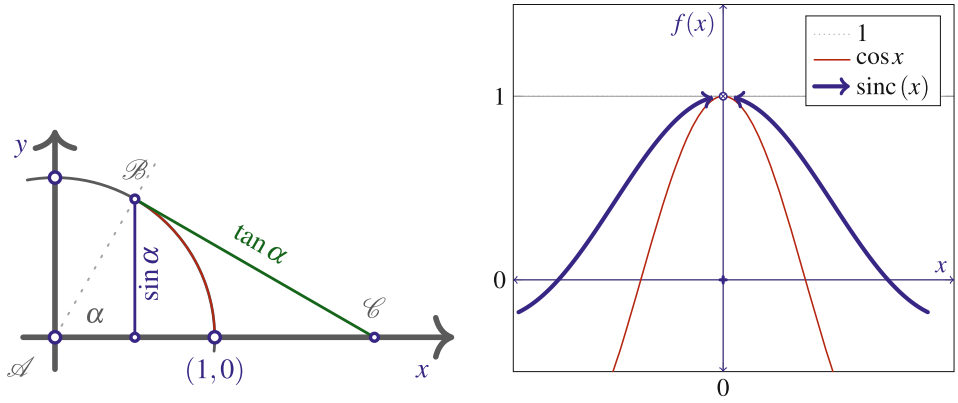


Fig. 1.10 Example P.1.44

$$\lim_{x\to 0}\frac{\sin\alpha}{\alpha}=1$$

As $x\to 0$, the squeezing process is illustrated in Fig. 1.10 (right) showing sinc (x) being squeezed between "1" and "cos x" functions and therefore "forced" to settle at sinc $(0)=1$ value.

1.45. Direct substitution method results in a non-defined form as

$$\lim_{x\to\infty}\frac{2x+\sin x}{x}=\frac{2\infty+\sin(\infty)}{\infty}$$

Therefore, this limit must be resolved by some other methods. By definition, the sinusoidal functions are bound between ± 1; thus, the required form is forced by algebraic transformations as

$$\begin{aligned}
-1\le\sin x&\le 1\\
2x-1\le 2x+\sin x&\le 2x+1\\
\frac{2x-1}{x}\le\frac{2x+\sin x}{x}&\le\frac{2x+1}{x}\\
2-\frac{1}{x}\le\frac{2x+\sin x}{x}&\le 2+\frac{1}{x}\\
\therefore&\\
\lim_{x\to\infty}\left(2-\overset{0}{\cancel{\frac{1}{x}}}\right)\le\lim_{x\to\infty}\left(\frac{2x+\sin x}{x}\right)&\le\lim_{x\to\infty}\left(2+\overset{0}{\cancel{\frac{1}{x}}}\right)\\
2\le\lim_{x\to\infty}\left(\frac{2x+\sin x}{x}\right)\le 2\ &\Rightarrow\ \lim_{x\to\infty}\left(\frac{2x+\sin x}{x}\right)=2
\end{aligned}$$

1.46. Direct substitution method results in a non-defined form as

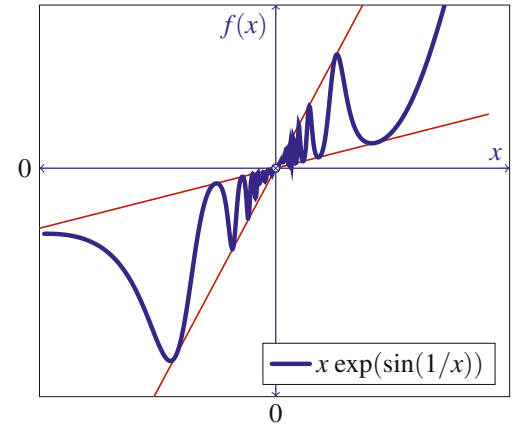


Fig. 1.11 Example P.1.46

$$\lim_{x\to 0} x\, e^{\sin(1/x)} = 0\, e^{\sin(1/0)} = 0\, e^{\sin(\infty)}$$

By definition, sinusoidal functions are bound by ± 1 for any argument. Thus, it follows that

$$\begin{aligned} -1 &\le \sin(1/x) \le 1 \\ e^{-1} &\le e^{\sin(1/x)} \le e^{1} \\ \frac{x}{e} &\le x\, e^{\sin(1/x)} \le e\, x \\ &\therefore \\ \lim_{x\to 0} \frac{x}{e} &\le \lim_{x\to 0} x\, e^{\sin(1/x)} \le \lim_{x\to 0} e\, x \\ 0 \le \lim_{x\to 0} x\, e^{\sin(1/x)} &\le 0 \quad \Rightarrow \quad \lim_{x\to 0} x\, e^{\sin(1/x)} = 0 \end{aligned}$$

As $x \to 0$ (see Fig. 1.11), this exponential trigonometric function is squeezed between "x/e" and "$e\,x$" linear functions and therefore "forced" to settle at 0 value, where it is not defined.

1.5 Limits Based on sinc (x) Function

1.47. Limit of sinc (x) function is resolved in A. 1.44 as well in A. 2.86; thus, it follows

$$\lim_{x\to 0} \frac{x + \sin(x)}{x} = \lim_{x\to 0} \left(1 + \frac{\sin(x)}{x} \right) = 2$$

1.48. Direct substitution method results in a non-defined form as

$$\lim_{x\to 0} \frac{2x + \sin x}{\sin x} = \frac{2\cdot 0 + \sin 0}{\sin 0} = \frac{0}{0}$$

Thus, given that this function is not defined for $x = 0$, at all the other points, it may be simplified as

$$\lim_{x\to 0}\frac{2x+\sin x}{\sin x} = \lim_{x\to 0}\left(\frac{2x}{\sin x}+1\right) = \{x\neq 0\} = \lim_{x\to 0}\left(\frac{2x}{\sin x}\right)+\lim_{x\to 0}1 = \lim_{x\to 0}\left(\frac{2x}{\sin x}\right)+1$$
$$= \lim_{x\to 0}\left(\frac{2}{(\sin x)/x}\right)+1 = \frac{2}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)}+1 = \left\{\lim_{x\to 0}\frac{\sin x}{x}=1;\quad \text{see A.1.44}\right\}$$
$$= 2+1 = 3$$

1.49. Direct substitution method results in a non-defined form as

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \frac{1-1}{0} = \frac{0}{0}$$

Classic algebraic resolution of this form is as follows:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x}\,\frac{1+\cos x}{1+\cos x} = \{(a-b)(a+b)=a^2-b^2\} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$
$$= \{1-\cos^2 x = \sin^2 x\} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0}\left(\frac{\sin x}{x}\;\frac{\sin x}{1+\cos x}\right)$$
$$= \lim_{x\to 0}\frac{\sin x}{x}\;\lim_{x\to 0}\frac{\sin x}{1+\cos x} = 1\times\frac{\sin 0}{1+\cos 0} = 0$$

while one other method is shown in Sect. 2.6.

1.50. Note that sinc (x) function is defined so that argument of sinus is equal to the denominator. If that is not the case, it is necessary to use algebraic transformations followed by the change of variables technique, as

$$\lim_{x\to 0}\frac{\sin(2x)}{x} = \lim_{x\to 0}\frac{2}{2}\frac{\sin(2x)}{x} = 2\lim_{x\to 0}\frac{\sin(2x)}{2x} = \{t=2x\} = 2\lim_{t\to 0}\frac{\sin t}{t}^{1} = 2$$

1.51. Both numerator and denominator may be forced into sinc (x) form; as $x\neq 0$, it follows that

$$\lim_{x\to 0}\frac{4\sin(5x)}{\sin(4x)} = \lim_{x\to 0}\frac{\sin(5x)\;\frac{5x}{5x}}{\frac{\sin(4x)}{4}\;\frac{x}{x}} = \lim_{x\to 0}\frac{5x}{x}\;\frac{\frac{\sin(5x)}{5x}^{1}}{\frac{\sin(4x)}{4x}_{1}} = 5$$

1.52. Both numerator and denominator may be forced into sinc (x) form; as $x\neq 0$, it follows that

$$\lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)} = \lim_{x\to 0}\frac{\frac{2x}{2x}\sin(2x)}{\frac{3x}{3x}\sin(3x)} = \lim_{x\to 0}\frac{2x}{3x}\;\frac{\frac{\sin(2x)}{2x}^{1}}{\frac{\sin(3x)}{3x}_{1}} = \frac{2}{3}$$

1.6 Limits Based on the Constant e

1.53. As the e limit requires x in the exponent, it follows that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{3}=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\to e\right]^{3}=e^{3}$$

1.54. By definition, the numerator of the inverse term in the sum must be equal to one; therefore, the change of variable is applied as

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}=\left\{\begin{array}{l}\frac{2}{x}=\frac{1}{t}\quad\therefore\quad x=2t,\\ x\to\infty\quad\therefore\quad t\to\infty\end{array}\right\}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{2t}=\left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\to e\right]^{2}=e^{2}$$

1.55. By definition, the denominator of the inverse term in the sum must equal to one; therefore, the change of variable method is applied as

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{x}=\left\{\begin{array}{l}\frac{1}{2x}=\frac{1}{t}\quad\therefore\quad x=\frac{t}{2},\\ x\to\infty\quad\therefore\quad t\to\infty\end{array}\right\}=\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{\frac{t}{2}}=\left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\to e\right]^{\frac{1}{2}}=\sqrt{e}$$

1.56. The exponential term may be resolved as

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+5}=\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{5}\right]=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5}$$

$$=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\to e\left(1+\lim_{x\to\infty}\frac{1}{x}\to 0\right)^{5}=e$$

1.57. After algebraic transformations, it follows that

$$\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}=\lim_{x\to\infty}\left(\frac{1}{\frac{1+x}{x}}\right)^{x}=\frac{\lim_{x\to\infty}1^{x}\to 1}{\lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{x}}=\frac{1}{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\to e}=\frac{1}{e}$$

1.58. Evidently, this form hints to some relationship with e; however, the exponential term is not right – instead of x, there is $1/x$ function. One possible way to transform this limit's form may be by exploiting properties of logarithmic functions as follows.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{1}{x}} = \left\{\begin{array}{ll}\text{if } x\geq 0 \quad \text{then:} & e^{\ln f(x)} = f(x)\\ & \ln a^b = b\ln a\\ & \lim \ln f(x) = \ln \lim f(x)\end{array}\right\}$$

$$= \lim_{x\to\infty}\exp\left(\ln\left(1+\frac{1}{x}\right)^{1/x}\right) = \lim_{x\to\infty}\exp\left(\frac{1}{x}\ln\left(1+\frac{1}{x}\right)\right)$$

$$= \exp\left(\lim_{x\to\infty}\underbrace{\frac{1}{x}}_{0}\lim_{x\to\infty}\ln\left(1+\underbrace{\frac{1}{x}}_{0}\right)\right)$$

$$= e^{0\cdot\ln 1} = e^0 = 1$$

1.59. With the same ideas as in A. 1.58, it follows that

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{\frac{1}{x}} = \lim_{x\to\infty}\exp\left(\ln\left(1+\frac{1}{x^2}\right)^{1/x}\right) = \lim_{x\to\infty}\exp\left(\frac{1}{x}\ln\left(1+\frac{1}{x^2}\right)\right)$$

$$= \exp\left(\lim_{x\to\infty}\underbrace{\frac{1}{x}}_{0}\lim_{x\to\infty}\ln\left(1+\underbrace{\frac{1}{x^2}}_{0}\right)\right)$$

$$= e^{0\cdot\ln 1} = e^0 = 1$$

1.60. The combination of algebraic transformations and the change of variables leads to

$$\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+1} = \left\{\begin{array}{l}x-1=t \quad \therefore \quad x=t+1\\ x\to\infty \quad \therefore \quad t\to\infty\end{array}\right\} = \lim_{t\to\infty}\left(\frac{t+4}{t}\right)^{t+2}$$

$$= \lim_{t\to\infty}\left(\frac{t+4}{t}\right)^{t}\lim_{t\to\infty}\left(\frac{t+4}{t}\right)^{2} = \lim_{t\to\infty}\left(1+\frac{4}{t}\right)^{t}\lim_{t\to\infty}\left(1+\underbrace{\frac{4}{t}}_{0}\right)^{2}$$

$$= \lim_{t\to\infty}\left(1+\frac{4}{t}\right)^{t}\underbrace{\lim_{t\to\infty}1^2}_{1} = \left\{\begin{array}{ll}\frac{4}{t}=\frac{1}{p} & \therefore \quad t=4p\\ t\to\infty & \therefore \quad p\to\infty\end{array}\right\} = \lim_{p\to\infty}\left(1+\frac{1}{p}\right)^{4p}$$

$$= \left[\underbrace{\lim_{p\to\infty}\left(1+\frac{1}{p}\right)^{p}}_{e}\right]^4 = e^4$$

1.61. Direct substitution method results in a non-defined form as

$$\lim_{x\to 0}\frac{e^{-2x}-1}{x} = \frac{e^{-2\cdot 0}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$$

Among other techniques, algebraic resolution of this limit by the change of variables method may be as follows.

$$e^{-2x} - 1 = t \quad \therefore \quad \text{if } x \to 0 \quad \text{, then} \quad t \to 0$$

$$e^{-2x} = t + 1 \Rightarrow x = -\frac{1}{2}\ln(1+t)$$

$$\therefore$$

$$\frac{e^{-2x}-1}{x} = \frac{t}{-\frac{1}{2}\ln(1+t)} = -2\,\frac{1}{\frac{1}{t}\ln(1+t)} = -2\,\frac{1}{\ln(1+t)^{1/x}}$$

so that the original problem is transformed into

$$\lim_{x\to 0}\frac{e^{-2x}-1}{x} = -2\lim_{t\to 0}\frac{1}{\ln(1+t)^{1/x}} = -2\,\frac{1}{\ln\left(\lim_{t\to 0}(1+t)^{1/x}\right)} = -2\,\frac{1}{\ln e} = -2\cdot 1 = -2$$

1.7 Limits Involving Trigonometric Functions

1.62. Direct substitution method results in

$$\lim_{x\to 0} x\,\sin x = 0\,\sin 0 = 0$$

1.63. Direct substitution method results in a non-defined value at $x = 0$, as

$$\lim_{x\to 0}\frac{\sin(\sin x)}{x} = \frac{\sin(\sin 0)}{0} = \frac{0}{0}$$

Algebraic transformation may be as follows.

$$\lim_{x\to 0}\frac{\sin(\sin x)}{x}\quad \frac{\sin x}{\sin x} = \lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}\lim_{x\to 0}\frac{\sin x}{x} = 1$$

1.64. Direct substitution method results in a non-defined value as

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \frac{1-\cos 2\cdot 0}{0^2} = \frac{0}{0}$$

which is to say that this function is not continuous at $x = 0$. Therefore, this limit must be resolved by other methods. The change of variables method combined with the basic trigonometric identity $\sin^2 x + \cos^2 x = 1$ as well as result in A. 1.44 may be used as follows.

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \left\{2x = t \quad \therefore \quad x = \frac{t}{2} \quad \text{as well as, if} \quad x \to 0 \Rightarrow t \to 0\right\}$$

$$= 4 \lim_{t\to 0} \frac{1-\cos t}{t^2} \quad \frac{1+\cos t}{1+\cos t} = 4 \lim_{t\to 0} \frac{1-\cos^2 t}{t^2\,(1+\cos t)} = 4 \lim_{t\to 0} \frac{\sin^2 t}{t^2} \frac{1}{(1+\cos t)}$$

$$= 4 \left(\lim_{t\to 0} \frac{\sin t}{t}\right)^2 \lim_{t\to 0} \frac{1}{1+\cos t} = 4 \cdot 1^2 \frac{1}{1+1} = 2$$

1.65. As an extension to A. 1.44 and A. 1.64, it is straightforward to write

$$\lim_{x\to 0} \frac{1-\cos 2x}{x \sin x} = \lim_{x\to 0} \frac{1-\cos 2x}{\frac{x}{x}\, x \sin x} = \lim_{x\to 0} \frac{1-\cos 2x}{x^2} \quad \frac{1}{\lim_{x\to 0} \frac{\sin x}{x}} = 2 \cdot 1 = 2$$

1.66. Direct substitution method results in a non-defined value at $x = 2$, as

$$\lim_{x\to 2} \sin\left(\frac{x^2-4}{x-2}\right) = \sin\left(\frac{2^2-4}{2-2}\right) = \sin\frac{0}{0}$$

Algebraic identities may be applied as follows:

$$\lim_{x\to 2} \sin\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2} \sin\left(\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}\right) = \sin 4$$

1.67. As given, $\lim_{x\to 0} (1+\sin x)^{1/x}$, this limit is not defined at $x = 0$ due to the inverse exponent term. There are multiple techniques that may be applied. For example,

Method 1: Resolution of this limit may be reached by applying a non-rigorous but correct argument as follows. For small argument, i.e., in the proximity of zero, there is a well-known "small angle approximation" (see Taylor expansion examples) stating that as $\to 0$, then $\sin x \to x$. That being said, it is correct to conclude that

$$\lim_{x\to 0} \left(1+\cancel{\sin x}^{\,x}\right)^{1/x} = \lim_{x\to 0} (1+x)^{1/x} = e$$

Method 2: Otherwise, algebraic transformations may be applied as follows.

$$\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}\,\frac{\sin x}{\sin x}} = \lim_{x\to 0} (1+\sin x)^{\frac{1}{\sin x}\,\frac{\sin x}{x}} = \left(\cancel{\lim_{x\to 0} (1+\sin x)^{\frac{1}{\sin x}}}^{\,e}\right)^{\cancel{\lim_{x\to 0}\frac{\sin x}{x}}^{\,1}} = e$$

1.68. Direct substitution method results in a non-defined value at $x = 0$, as

$$\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2} = \frac{e^{0^2}-\cos 0}{0^2} = \frac{0}{0}$$

Algebraic transformations may be as follows:

$$\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2} = \lim_{x\to 0} \frac{e^{x^2}-1+1-\cos x}{x^2} = \lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

where the first limit may be resolved as

$$\lim_{x\to 0}\frac{e^{x^2}-1}{x^2}=\left\{\begin{array}{ll} e^{x^2}-1=t \;\;\therefore & x^2=\ln(t+1)\\ & x\to 0\Rightarrow t\to 0\end{array}\right\}$$

$$=\lim_{t\to 0}\frac{t}{\ln(t+1)}=\lim_{t\to 0}\frac{1}{\frac{1}{t}\ln(t+1)}=\lim_{t\to 0}\frac{1}{\ln(t+1)^{1/t}}=\frac{1}{\ln\left(\lim_{t\to 0}(t+1)^{1/t}\right)^{e}}$$

$$=\frac{1}{\ln e}=1$$

and the second limit may be resolved as

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}\;\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2\,(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{x^2\,(1+\cos x)}$$

$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^{1\,2}\lim_{x\to 0}\frac{1}{(1+\cos x)}=\frac{1}{2}$$

Therefore, in total,

$$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=\lim_{x\to 0}\frac{e^{x^2}-1}{x^2}+\lim_{x\to 0}\frac{1-\cos x}{x^2}=1+\frac{1}{2}=\frac{3}{2}$$

1.69. Direct substitution method results in a non-defined value at $x=0$, as

$$\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}=\frac{1-\cos(1-\cos 0)}{0^4}=\frac{1-\cos(0)}{0}=\frac{0}{0}$$

Using half-angle trigonometric identity, algebraic transformation may be as follows.

$$\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}=\left\{1-\cos(2\alpha)=2\sin^2(\alpha)\right\}=\lim_{x\to 0}\frac{1-\cos\left(2\sin^2(x/2)\right)}{x^4}$$

$$=\left\{\sin^2(x/2)=\theta\right\}=\lim_{x\to 0}\frac{1-\cos(2\theta)}{x^4}=\lim_{x\to 0}\frac{2\sin^2\theta}{x^4}$$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\sin^2(x/2)\right)}{x^4}=\lim_{x\to 0}\left[\frac{\sin^2\left(\sin^2(\frac{x}{2})\right)}{\frac{x^4}{2}}\;\frac{\left(\sin^2(\frac{x}{2})\right)^2}{\left(\sin^2(\frac{x}{2})\right)^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{\sin\left(\sin^2(\frac{x}{2})\right)}{\sin^2(\frac{x}{2})}\;\frac{\sin\left(\sin^2(\frac{x}{2})\right)}{\sin^2(\frac{x}{2})}\;\frac{\sin^2(\frac{x}{2})}{\frac{2}{2}\,\frac{x^2}{2}}\;\frac{\sin^2(\frac{x}{2})}{\frac{4\,x^2}{4}}\right]$$

$$=\frac{1}{8}\left(\lim_{x\to 0}\frac{\sin\left(\sin^2(x/2)\right)}{\sin^2(x/2)}\right)^2\left(\lim_{x\to 0}\frac{\sin(x/2)}{x/2}\right)^4$$

$$= \frac{1}{8}\left(\lim_{x\to 0}\frac{\sin a}{a}\overset{1}{}\right)^2\left(\lim_{x\to 0}\frac{\sin b}{b}\overset{1}{}\right)^4 = \frac{1}{8}$$

1.70. Direct substitution method results in a non-defined value at $x = 0$, as

$$\lim_{x\to 0}\frac{1-\sqrt{\cos 2x}\ \cos x}{x^2} = \frac{1-\sqrt{\cos(2\cdot 0)}\ \cos(0)}{0^2} = \frac{1-1}{0} = \frac{0}{0}$$

Algebraic transformation may be as follows.

$$\lim_{x\to 0}\frac{1-\sqrt{\cos 2x}\ \cos x}{x^2}\ \frac{1+\sqrt{\cos 2x}\ \cos x}{1+\sqrt{\cos 2x}\ \cos x} = \lim_{x\to 0}\frac{1-\cos^2 x\ \cos 2x}{x^2\left(1+\sqrt{\cos 2x}\ \cos x\right)}$$

$$= \left\{1-\cos(2\alpha) = 2\sin^2(\alpha)\right\} = \lim_{x\to 0}\frac{1-\cos^2 x\left(1-2\sin^2 x\right)}{x^2\left(1+\sqrt{\cos 2x}\ \cos x\right)}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x\ +2\cos^2 x\ \sin^2 x}{x^2\left(1+\sqrt{\cos 2x}\ \cos x\right)} = \lim_{x\to 0}\frac{\sin^2 x\ +2\cos^2 x\ \sin^2 x}{x^2\left(1+\sqrt{\cos 2x}\ \cos x\right)}$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\overset{1}{}\right)^2\lim_{x\to 0}\frac{1}{\left(1+\sqrt{\cos 2x}\ \cos x\right)} + \left(\lim_{x\to 0}\frac{\sin x}{x}\overset{1}{}\right)^2\lim_{x\to 0}\frac{2\cos^2 x}{\left(1+\sqrt{\cos 2x}\ \cos x\right)}$$

$$= \frac{1}{\left(1+\sqrt{\cos 2\cdot 0}\ \cos 0\right)} + \frac{2\cos^2 0}{\left(1+\sqrt{\cos 2\cdot 0}\ \cos 0\right)} = \frac{1}{2}+\frac{2}{2} = \frac{3}{2}$$

1.8 Limits of Piecewise Functions

1.71. Give absolute function, by definition, if $x \le 0$, then $|x| = -x$; therefore, direct substitution method results in

$$\lim_{x\to -2}|x| = \lim_{x\to -2}(-x) = \big(-(-2)\big) = 2$$

1.72. Given absolute function, direct substitution method results in

$$\lim_{x\to -1}|x+1| = |-1+1| = 0$$

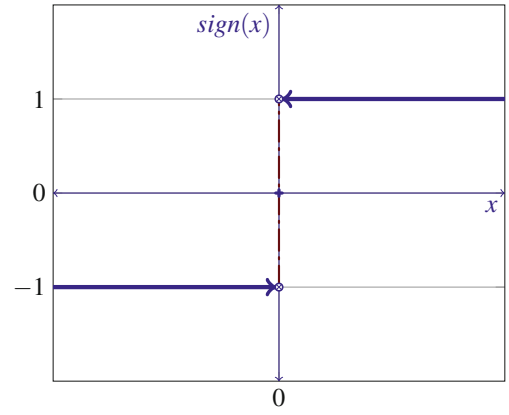
This function is continuous at $x = 0$.

1.73. Given absolute function, direct substitution method results in

$$\lim_{x\to 0}\frac{|x|}{x} = \frac{0}{0}$$

because $\text{sign}\,(x) = |x|/x = x/|x|$ is not continuous at $x = 0$. Therefore, limits must be resolved separately from the left and right sides of $x = 0$, as

Fig. 1.12 Example P.1.73



$$\lim_{x\to 0}\frac{|x|}{x}=\begin{cases}\underline{x<0:}\quad \therefore\ |x|=-x & \Rightarrow \lim_{x\to 0}\dfrac{-x}{x}=\lim\limits_{x\to 0}\ -1=-1\\[2ex] \underline{x>0:}\quad \therefore\ |x|=x & \Rightarrow \lim_{x\to 0}\dfrac{x}{x}=\lim\limits_{x\to 0}\ 1=1\end{cases}$$

which is to say that, because the left- and right-side limits are not equal (see Fig. 1.12), this limit does not exist. Function sign (x) belongs to the family of special functions that is very important in mathematics and engineering.

1.74. Given absolute function, direct substitution method results in

$$\lim_{x\to 2}\frac{|x-2|}{x-2}=\frac{|2-2|}{2-2}=\frac{0}{0}$$

Because this function is not continuous at $x=2$, limits on the left and right sides of $x=2$ are

$$\lim_{x\to 2}\frac{|x-2|}{x-2}$$

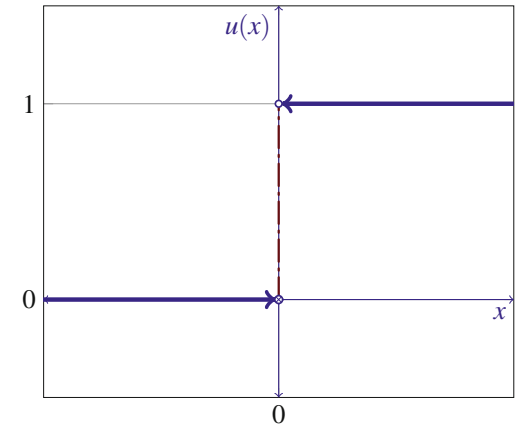
$$=\begin{cases}\underline{x-2<0:}\ \therefore x<2\quad \text{then}\quad |x-2|=-(x-2) & \Rightarrow \lim_{x\to 2}\dfrac{-(x-2)}{x-2}=\lim\limits_{x\to 2}\ -1=-1\\[2ex] \underline{x-2>0:}\ \therefore x>2\quad \text{then}\quad |x-2|=x-2 & \Rightarrow \lim_{x\to 2}\dfrac{x-2}{x-2}=\lim\limits_{x\to 2}\ 1=1\end{cases}$$

which is to say that, because the left- and right-side limits are not equal, this limit does not exist.

1.75. Given piecewise linear "unit step (Heaviside step)" function $u(x)$ as

$$u(x)=\lim_{x\to 0}\begin{cases}1 & (x\geq 0)\\ 0 & (x<0)\end{cases}$$

Fig. 1.13 Example P.1.75



which is to say, limits of the constants "1" and 0 are simply the constants themselves. It is evident that the left- and right-side limits are not equal (see Fig. 1.13); therefore, this limit does not exist. The point of discontinuity $x = 0$ is included in the equality part of $x \geq 0$ definition.

1.76. As defined, this function

$$\lim_{x\to 2} \begin{cases} 3 & (x > 2) \\ 1 & (x < 2) \end{cases}$$

is not continuous at $x = 2$. For the reason that the left-side and right-side limits are not equal, $\lim_{x\to 2}$ of this function does not exist.

1.9 Asymptotes

1.77. Given rational function may be factorized as

$$f(x) = \frac{1}{x} = \frac{P(x)}{Q(x)}$$

There is one point of discontinuity that is calculated as the root of denominator $Q(x) = 0$, i.e., $x = 0$, that separates total domain into two intervals: $(-\infty, 0)$ and $(0, +\infty)$ (see Fig. 1.14). By consequence, there is one vertical asymptote at $x = 0$. The study of a function with one discontinuity therefore requires calculation of four limits in total.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to -\infty} \frac{1}{x} = 0\uparrow$$

2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to\infty}\frac{1}{x}=0\rfloor$$

In conclusion: as these two left-/right-side infinite limits are equal, it is said that horizontal asymptote does exist at $y=0$.

Vertical asymptote(s):

3. Left-side limit around vertical asymptote $x=\underset{\rightarrow}{0}$:

$$\lim_{x\to\underset{\rightarrow}{0}}\frac{1}{x}=-\infty$$

4. Right-side limit around vertical asymptote $x=\underset{\leftarrow}{0}$:

$$\lim_{x\to\underset{\leftarrow}{0}}\frac{1}{x}=+\infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to 0}$ does not exist.

Oblique asymptote(s):
In order to determine the existence of oblique asymptotes $y_{\text{bb}}=ax+b$ to the given function, it is necessary to evaluate the following limit(s).

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{1}{x^2}=0$$

Given that $a=0$, it follows that $y_{\text{bb}}=b$, that is to say, there is no oblique asymptote.

The sketch of this function (see Fig. 1.14) illustrates tendencies of these four limits, as well as positions of vertical and horizontal asymptotes. Evidently, just by knowing function's limits, it is possible to sketch its rough graph.

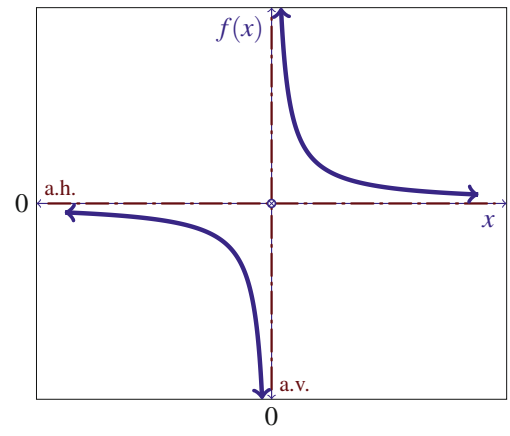


Fig. 1.14 Example P.1.77

1.78. Given rational function may be factorized as

$$f(x) = \frac{x}{x^2+1} = \frac{P(x)}{Q(x)}$$

Possible discontinuities would be at $Q(x) = 0$ points. In this case, $x^2+1 \neq 0$ in $\mathbb{R}$ domain. Therefore, there are no vertical asymptotes. The study of a function without discontinuities therefore requires calculation of two limits in total.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{\cancel{x}}{\cancel{x}\,(x+1/x)} = \frac{1}{-\infty} = 0\uparrow$$

   where " $0\uparrow$" reads as "approaching $y = 0$ from below (i.e., negative y-axis side)".
2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\,(x+1/x)} = \frac{1}{\infty} = 0\downarrow$$

   where " $0\downarrow$" reads as "approaching $y = 0$ from above (i.e., positive y-axis side)".

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\text{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x}{x\,(x^2+1)} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\,(x^2+1)} = \frac{1}{\infty} = 0$$

Given that $a = 0$, it follows that $y_{\text{bb}} = b$, that is to say, there is not oblique asymptote; see Fig. 1.15. However, note that function can cross its horizontal asymptote.

1.79. Given rational function may be factorized as

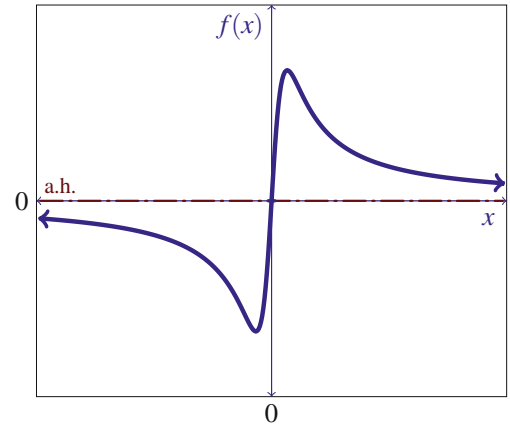
$$f(x) = \frac{x^3}{x^2-4} = \frac{x^3}{(x-2)(x+2)} = \frac{P(x)}{Q(x)}$$

There are two points of discontinuity that are calculated as roots of denominator $Q(x) = 0$:

$$(x+2)(x-2) = 0 \;\;\therefore\;\; x_1 = -2, \;\; x_2 = 2$$

that separate total domain into three intervals, $(-\infty, -2)$, $(-2, 2)$, and $(2, +\infty)$ (see Fig. 1.16), where within each interval, there are two limits to calculate (left and right sides). By consequence, there are two vertical asymptotes: at $x = -2$ and $x = 2$. The study of a function with two discontinuities therefore requires calculation of six limits in total.

Fig. 1.15 Example P.1.78



Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} \frac{x^3}{x^2-4} = \lim_{x\to-\infty} \frac{\cancel{x^2}\,x}{\cancel{x^2}\left(1-\cancel{4/x^2}^{\,0}\right)} = \lim_{x\to-\infty} x = -\infty$$

2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to\infty} \frac{x^3}{x^2-4} = \lim_{x\to\infty} \frac{\cancel{x^2}\,x}{\cancel{x^2}\left(1-\cancel{4/x^2}^{\,0}\right)} = \lim_{x\to\infty} x = \infty$$

In conclusion: as these two left-/right-side infinite limits are not equal, it is said that horizontal asymptote does not exist.

Vertical asymptote(s):

3. Left-side limit around vertical asymptote $x = -2$:

$$\lim_{x\to\underset{\to}{-2}} \frac{x^3}{x^2-4} = \lim_{x\to\underset{\to}{-2}} \frac{x^3}{(x-2)(x+2)} = \lim_{x\to\underset{\to}{-2}} \frac{(-2)^3}{(-2-2)(x+2)}$$

$$= \lim_{x\to\underset{\to}{-2}} \frac{-8}{-4(x+2)} = \left\{\text{if } x < -2 \Rightarrow (x+2) < 0\right\} = \frac{-8}{-4(\underset{\to}{0})} = -\infty$$

4. Right-side limit around vertical asymptote $x = -2$:

$$\lim_{x\to\underset{\leftarrow}{-2}} \frac{x^3}{x^2-4} = \lim_{x\to\underset{\leftarrow}{-2}} \frac{x^3}{(x-2)(x+2)} = \lim_{x\to\underset{\leftarrow}{-2}} \frac{(-2)^3}{(-2-2)(x+2)}$$

$$= \lim_{x \to -2 \atop \leftarrow} \frac{-8}{-4(x+2)} = \left\{ \text{if } x > -2 \ \ \therefore \ \ (x+2) > 0 \right\} = \frac{-8}{-4(\underset{\leftarrow}{0})} = +\infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x \to -2}$ does not exist.

5. Left-side limit around vertical asymptote $x = 2$:

$$\lim_{x \to 2 \atop \rightarrow} \frac{x^3}{x^2-4} = \lim_{x \to 2 \atop \rightarrow} \frac{x^3}{(x-2)(x+2)} = \lim_{x \to 2 \atop \rightarrow} \frac{(2)^3}{(x-2)(2+2)}$$

$$= \lim_{x \to 2 \atop \rightarrow} \frac{8}{4(x-2)} = \left\{ \text{if } x < 2 \ \ \therefore \ \ (x-2) < 0 \right\} = \frac{8}{4(\underset{\rightarrow}{0})} = -\infty$$

6. Right-side limit around vertical asymptote $x = 2$:

$$\lim_{x \to 2 \atop \leftarrow} \frac{x^3}{x^2-4} = \lim_{x \to 2 \atop \leftarrow} \frac{x^3}{(x-2)(x+2)} = \lim_{x \to 2 \atop \leftarrow} \frac{(2)^3}{(x-2)(2+2)}$$

$$= \lim_{x \to 2 \atop \leftarrow} \frac{8}{4(x-2)} = \left\{ \text{if } x > 2 \ \ \therefore \ \ (x-2) > 0 \right\} = \frac{8}{4(\underset{\leftarrow}{0})} = +\infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x \to 2}$ does not exist.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\texttt{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

$$a = \lim_{x \to \infty} \frac{\frac{x^3}{x^2-4}}{x} = \lim_{x \to \infty} \frac{x^3}{x(x^2-4)} = \lim_{x \to \infty} \frac{\cancel{x^3}}{\cancel{x^3}\left(1 - \cancel{4/x^2}^{\,0}\right)} = 1$$

and

$$b = \lim_{x \to \infty} \left(\frac{x^3}{x^2-4} - x \right) = \lim_{x \to \infty} \frac{x^3 - x(x^2-4)}{x^2-4} = \lim_{x \to \infty} \frac{\cancel{x^3} - \cancel{x^3} + 4x}{x^2-4}$$

$$= \lim_{x \to \infty} \frac{4\cancel{x}}{\cancel{x}\left(x - \cancel{4/x}^{\,0}\right)} = \lim_{x \to \infty} \frac{4}{x} = 0$$

Therefore, there is one oblique asymptote $y_{\texttt{aa}} = x$; see Fig. 1.16.

1.80. Given rational function may be factorized as

$$f(x) = \frac{x-1}{x^2+5x+6} = \frac{x-1}{x^2+2x+3x+6} = \frac{x-1}{x(x+2)+3(x+2)} = \frac{x-1}{(x+2)(x+3)} = \frac{P(x)}{Q(x)}$$

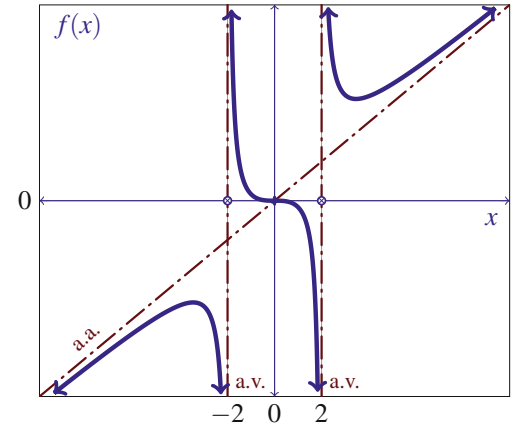


Fig. 1.16 Example P.1.79

There are two points of discontinuity that are calculated as roots of denominator $Q(x) = 0$:

$$(x+2)(x+3) = 0 \quad \therefore \quad x_1 = -2, \quad x_2 = -3$$

that separate total domain into three intervals, $(-\infty, -3)$, $(-3, -2)$, and $(-2, +\infty)$ (see Fig. 1.17 (left)), where within each interval, there are two limits to calculate (left and right sides). By consequence, there are two vertical asymptotes: at $x = -3$ and $x = -2$. The study of a function with two discontinuities therefore requires calculation of six limits in total.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} \frac{x-1}{x^2+5x+6} = \lim_{x\to-\infty} \frac{\cancel{x}\left(1-\frac{1}{x}\right)}{\cancel{x}\left(x+5+\frac{6}{x}\right)} = \lim_{x\to-\infty} \frac{1}{x+5} = \frac{1}{-\infty} = 0\uparrow$$

   where " $0\uparrow$" reads as "approaching $y = 0$ from below (i.e., negative y-axis side)".
2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to+\infty} \frac{x-1}{x^2+5x+6} = \lim_{x\to+\infty} \frac{\cancel{x}\left(1-\frac{1}{x}\right)}{\cancel{x}\left(x+5+\frac{6}{x}\right)} = \lim_{x\to+\infty} \frac{1}{x+5} == \frac{1}{+\infty}$$

$$= 0\downarrow$$

   where " $0\downarrow$" reads as "approaching $y = 0$ from above (i.e., positive y-axis side)".

   In conclusion: there is one horizontal asymptote at $y = 0$; see Fig. 1.17 (right). Note that graph does not show completely $x > 0$ side – after reaching its maximum, this function drops to zero as correctly calculated by $\lim_{x\to+\infty}$.

Vertical asymptote(s):

1. Left-side limit around vertical asymptote $x = -3$:

$$\lim_{\substack{x\to -3\\ \to}} \frac{x-1}{(x+2)(x+3)} = \lim_{\substack{x\to -3\\ \to}} \frac{-3-1}{(-3+2)(x+3)} = \lim_{\substack{x\to -3\\ \to}} \frac{-4}{-(x+3)}$$

$$= \left\{\text{if } x < -3 \Rightarrow (x+3) < 0 \Rightarrow -(x+3) > 0\right\} = \frac{-4}{\underset{\leftarrow}{0}} = -\infty$$

2. Right-side limit around vertical asymptote $x = -3$:

$$\lim_{\substack{x\to -3\\ \leftarrow}} \frac{x-1}{(x+2)(x+3)} = \lim_{\substack{x\to -3\\ \leftarrow}} \frac{-3-1}{(-3+2)(x+3)} = \lim_{\substack{x\to -3\\ \leftarrow}} \frac{-4}{-(x+3)}$$

$$= \left\{\text{if } x > -3 \Rightarrow (x+3) > 0 \Rightarrow -(x+3) < 0\right\} = \frac{-4}{\underset{\to}{0}} = +\infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to -3}$ does not exist.

3. Left-side limit around vertical asymptote $x = -2$:

$$\lim_{\substack{x\to -2\\ \to}} \frac{x-1}{(x+2)(x+3)} = \lim_{\substack{x\to -2\\ \to}} \frac{-2-1}{(x+2)(-2+3)} = \lim_{\substack{x\to -2\\ \to}} \frac{-3}{(x+2)}$$

$$= \left\{\text{if } x < -2 \Rightarrow (x+2) < 0\right\} = \frac{-3}{\underset{\to}{0}} = +\infty$$

4. Right-side limit around vertical asymptote $x = -2$:

$$\lim_{\substack{x\to -2\\ \leftarrow}} \frac{x-1}{(x+2)(x+3)} = \lim_{\substack{x\to -2\\ \leftarrow}} \frac{-2-1}{(x+2)(-2+3)} = \lim_{\substack{x\to -2\\ \leftarrow}} \frac{-3}{(x+2)}$$

$$= \left\{\text{if } x > -2 \Rightarrow (x+2) > 0\right\} = \frac{-3}{\underset{\leftarrow}{0}} = -\infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to -2}$ does not exist.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\text{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x-1}{x\,(x^2+5x+6)} = \lim_{x\to\infty} \frac{\cancel{x}\,(1-1/x)}{\cancel{x}\,(x^2+5x+6)} = \frac{1}{\infty} = 0$$

Given that $a = 0$, it follows that $y_{\text{bb}} = b$, that is to say, there is no oblique asymptote.

The sketch of this function (see Fig. 1.17 (right)) illustrates tendencies of the six limits, as well as positions of two vertical asymptotes. Evidently, just by knowing function's limits, it is possible to sketch its rough graph.

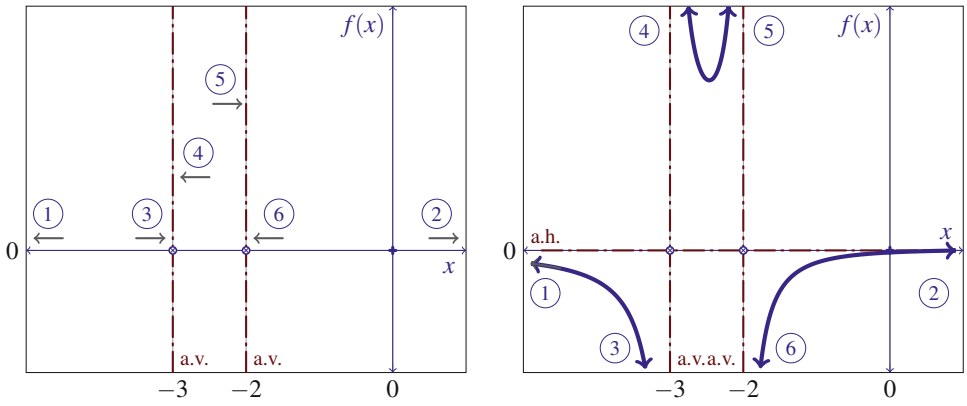


Fig. 1.17 Example P.1.80

1.81. Given rational function

$$f(x) = \frac{x^3}{2(x+1)^2} = \frac{P(x)}{Q(x)}$$

There is one point of discontinuity that is calculated as the root of denominator $Q(x) = 0$:

$$2(x+1)^2 = 0 \;\; \therefore \;\; x+1 = 0 \;\; \therefore \;\; x_1 = -1$$

that separates total domain into two intervals, $(-\infty, -1)$ and $(-1, +\infty)$ (see Fig. 1.18), where within each interval, there are two limits to calculate (left and right sides). By consequence, there is one vertical asymptote: at $x = -1$. The study of a function with one discontinuity therefore requires calculation of four limits in total.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} \frac{x^3}{2(x+1)^2} = \lim_{x\to-\infty} \frac{x\,\cancel{x^2}}{2\,\cancel{x^2}\left(1+\cancel{1/x}^{\,0}\right)^2} = \lim_{x\to-\infty} \frac{x}{2} = -\infty$$

2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to\infty} \frac{x^3}{2(x+1)^2} \lim_{x\to\infty} \frac{x^3}{2(x+1)^2} = \lim_{x\to\infty} \frac{x\,\cancel{x^2}}{2\,\cancel{x^2}\left(1+\cancel{1/x}^{\,0}\right)^2} = \lim_{x\to\infty} \frac{x}{2} = \infty$$

In conclusion: as both of these left-/right-side limits tend to infinity, it is said that horizontal asymptote does not exist.

Vertical asymptote(s):

3. Left-side limit around vertical asymptote $x = \underset{\twoheadrightarrow}{-1}$:

$$\lim_{x \to \underset{\twoheadrightarrow}{-1}} \frac{x^3}{2(x+1)^2} = \lim_{x \to \underset{\twoheadrightarrow}{-1}} \frac{(-1)^3}{2(x+1)^2} = \left\{(x+1)^2 > 0\right\} = \frac{-1}{2 \cdot 0} = -\infty$$

4. Right-side limit around vertical asymptote $x = \underset{\twoheadleftarrow}{-1}$:

$$\lim_{x \to \underset{\twoheadleftarrow}{-1}} \frac{x^3}{2(x+1)^2} = \lim_{x \to \underset{\twoheadleftarrow}{-1}} \frac{(-1)^3}{2(x+1)^2} = \left\{(x+1)^2 > 0\right\} = \frac{-1}{2 \cdot 0} = -\infty$$

As these two left-/right-side limits are equal, it is said that limit $\lim_{x \to -2}$ does exist and, in this case, tends to $-\infty$; see Fig. 1.18.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\texttt{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

$$a = \lim_{x \to \infty} \frac{x^3}{2x\,(x+1)^2} = \lim_{x \to \infty} \frac{\cancel{x^3}\,1}{2\,\cancel{x^3}\left(1 + \cancel{1/x}^{\,0}\right)^2} = \frac{1}{2}$$

and

$$b = \lim_{x \to \infty} \left(\frac{x^3}{2(x+1)^2} - \frac{x}{2} \right) = \lim_{x \to \infty} \frac{x^3 - x\,(x+1)^2}{2(x+1)^2} = \lim_{x \to \infty} \frac{\cancel{2x^2}\left(-1 - \cancel{1/2x}^{\,0}\right)}{\cancel{2x^2}\left(1 + \cancel{1/x}^{\,0}\right)^2} = -1$$

Therefore, there is one oblique asymptote $y_{\texttt{aa}} = x/2 - 1$; see Fig. 1.18.

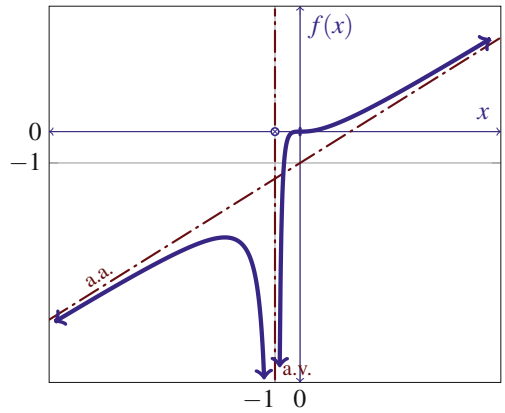


Fig. 1.18 Example P.1.81

1.82. Given rational function may be factorized as

$$f(x) = x + 1 + \frac{1}{x} = \frac{x^2 + x + 1}{x} = \frac{P(x)}{Q(x)}$$

There is one point of discontinuity that is calculated as the root of denominator $Q(x) = 0$, i.e., $x = 0$, that separates total domain into two intervals: $(-\infty, 0)$, and $(0, +\infty)$; see Fig. 1.19. By consequence, there is one vertical asymptote at $x = 0$. The study of a function with one discontinuity therefore requires calculation of four limits in total.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} \frac{x^2 + x + 1}{x} = \lim_{x\to-\infty} \frac{\cancel{x}\left(x + 1 + \cancel{1/x}^{\,0}\right)}{\cancel{x}} = \lim_{x\to-\infty} (x + 1) = -\infty$$

2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to\infty} \frac{x^2 + x + 1}{x} = \lim_{x\to\infty} \frac{\cancel{x}\,(x + 1 + \cancel{1/x}^{\,0})}{\cancel{x}} = \lim_{x\to\infty} (x + 1) = \infty$$

In conclusion: as these two left-/right-side infinite limits are not equal, it is said that horizontal asymptote does not exist.

Vertical asymptote(s):

3. Left-side limit around vertical asymptote $x = \underset{\to}{0}$:

$$\lim_{x\to\underset{\to}{0}} \left(x + 1 + \frac{1}{x}\right) = \lim_{x\to\underset{\to}{0}} \left(1 + \frac{1}{x}\right) = 1 + \lim_{x\to\underset{\to}{0}} \frac{1}{\underset{\to}{0}} = -\infty$$

4. Right-side limit around vertical asymptote $x = \underset{\leftarrow}{0}$:

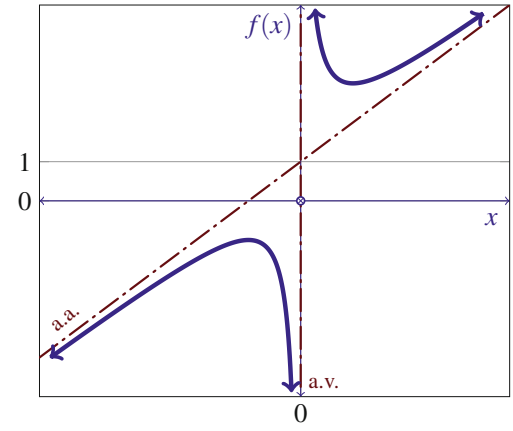
$$\lim_{x\to\underset{\leftarrow}{0}} \left(x + 1 + \frac{1}{x}\right) = \lim_{x\to\underset{\leftarrow}{0}} \left(1 + \frac{1}{x}\right) = 1 + \lim_{x\to\underset{\leftarrow}{0}} \frac{1}{\underset{\leftarrow}{0}} = +\infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to 0}$ does not exist.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\text{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

Fig. 1.19 Example P.1.82



$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2+x+1}{x^2} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1+\cancel{1/x}^{\to 0}+\cancel{1/x^2}^{\to 0}\right)}{\cancel{x^2}} = 1$$

and

$$b = \lim_{x\to\infty} \left(\cancel{x}+1+\frac{1}{x}-\cancel{x}\right) = 1+\lim_{x\to\infty}\frac{1}{x} = 1+\cancel{\frac{1}{\infty}}^{\to 0} = 1$$

Therefore, there is one oblique asymptote $y_{\mathtt{aa}} = x+1$; see Fig. 1.19.

1.83. Given rational function

$$f(x) = \frac{x-1}{\sqrt{x^2+1}} = \frac{P(x)}{Q(x)}$$

as $Q(x) \neq 0$ in $\mathbb{R}$ domain, this function is continuous. By consequence, there are no vertical asymptotes and therefore only two limits are calculated.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$: direct substitution method results in

$$\lim_{x\to-\infty} \frac{x-1}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x\left(1-\cancel{1/x}^{\to 0}\right)}{|x|\sqrt{1+\cancel{1/x^2}^{\to 0}}} = \lim_{x\to-\infty}\frac{x}{|x|} = -1$$

2. Right-side limit as $(x \to +\infty)$: similarly,

$$\lim_{x\to\infty} \frac{x-1}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x}{|x|} = 1$$

In conclusion: on the left side for $x < 0$, there is horizontal asymptote at $y = -1$, and on the right side for $x \geq 0$, there is horizontal asymptote at $y = 1$.

Vertical asymptote(s): as this function is continuous, there are no vertical asymptotes.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\text{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

$$a = \lim_{x\to\infty} \frac{x-1}{x\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\cancel{x}\left(1-\cancel{1/x}^{\,0}\right)}{\cancel{x}\,|x|\sqrt{1+\cancel{1/x^2}^{\,0}}} = \lim_{x\to\infty} \frac{1}{|x|} = 0$$

Therefore, there are no oblique asymptotes; see Fig. 1.20.

1.84. Given rational function may be factorized as

$$f(x) = \frac{x+1}{\sqrt{x^2-4}} = \frac{x+1}{\sqrt{(x-2)(x+2)}} = \frac{P(x)}{Q(x)}$$

There are two points of discontinuity that are calculated as roots of denominator $Q(x) = 0$:

$$(x+2)(x-2) = 0 \quad \therefore \quad x_1 = -2, \quad x_2 = 2$$

that separate total domain into three intervals: $(-\infty, -2)$, $(-2, 2)$, and $(2, +\infty)$; see Fig. 1.21. Note that this function is not defined within $(-2, 2)$ because the square root argument must be positive in $\mathbb{R}$ domain. Consequently, neither the right-side limit at $x = -2$ nor the left-side limit at $x = 2$ is defined. In this case, calculation of four limits in total is required.

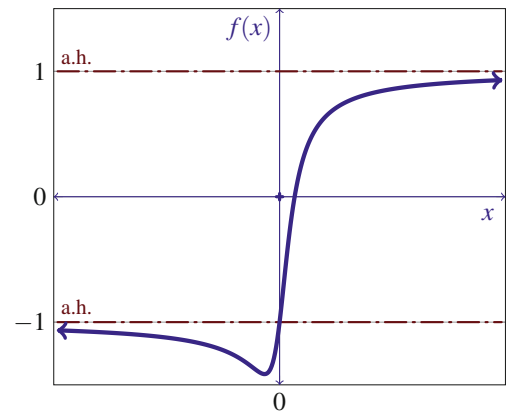


Fig. 1.20 Example P.1.83

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{(x-2)(x+2)}} = \lim_{x\to-\infty} \frac{x\left(1+1/x^{\to 0}\right)}{|x|\sqrt{\left(1-2/x^{\to 0}\right)\left(1+2/x^{\to 0}\right)}} = \lim_{x\to-\infty} \frac{x}{|x|} = -1$$

2. Right-side limit as $(x \to +\infty)$: similarly,

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{(x-2)(x+2)}} = \lim_{x\to\infty} \frac{x}{|x|} = 1$$

In conclusion: on the left side for $x < 0$, there is horizontal asymptote at $y = -1$, and on the right side for $x \geq 0$, there is horizontal asymptote at $y = 1$.

Vertical asymptote(s):

3. Left-side limit around vertical asymptote $x = -2$:

$$\lim_{x\to\overset{\to}{-2}} \frac{x+1}{\sqrt{(x-2)(x+2)}} = \lim_{x\to\overset{\to}{-2}} \frac{(-2)+1}{\sqrt{(-2-2)(x+2)}} = \lim_{x\to\overset{\to}{-2}} \frac{-1}{\sqrt{-4(x+2)}}$$

$$= \left\{\text{if } x < -2 \Rightarrow (x+2) < 0\right\} = \frac{-1}{\sqrt{-4(\overset{\to}{0})}} = -\infty$$

4. Right-side limit around vertical asymptote $x = -2$: not defined.

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to-2}$ does not exist.

5. Left-side limit around vertical asymptote $x = 2$: not defined.

6. Right-side limit around vertical asymptote $x = 2$:

$$\lim_{x\to\overset{\leftarrow}{2}} \frac{x+1}{\sqrt{(x-2)(x+2)}} = \lim_{x\to\overset{\leftarrow}{2}} \frac{2+1}{\sqrt{(x-2)(2+2)}} = \lim_{x\to\overset{\leftarrow}{2}} \frac{3}{\sqrt{4(x-2)}}$$

$$= \left\{\text{if } x > 2 \Rightarrow (x-2) > 0\right\} = \frac{3}{\sqrt{4(\overset{\leftarrow}{0})}} = \infty$$

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to2}$ does not exist.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\text{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

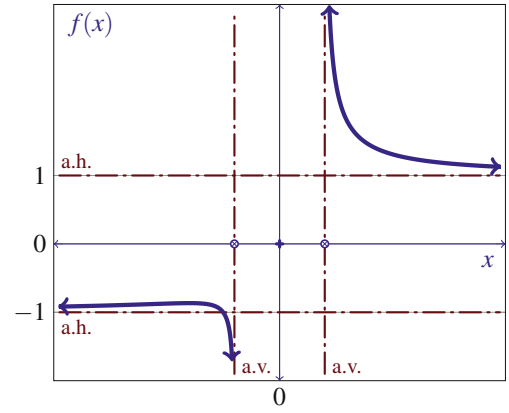


Fig. 1.21 Example P.1.84

$$a = \lim_{x\to\infty} \frac{x+1}{x\sqrt{(x-2)(x+2)}} = \lim_{x\to\infty} \frac{x\left(1+1/x^{\to 0}\right)}{x\,|x|\sqrt{\left(1-2/x^{\to 0}\right)\left(1+2/x^{\to 0}\right)}} = \lim_{x\to\infty} \frac{1}{|x|} = 0$$

Therefore, there are no oblique asymptotes; see Fig. 1.21.

1.85. Given function

$$f(x) = xe^{\frac{1}{x}}$$

There is one point of discontinuity at $x = 0$, due to $1/x$ power term, that separates total domain into two intervals, $(-\infty, 0)$ and $(0, +\infty)$ (see Fig. 1.22), where within each interval, there are two limits to calculate (left and right sides). By consequence, there is one vertical asymptote: at $x = 0$. The study of a function with one discontinuity therefore requires calculation of four limits in total.

Horizontal asymptote(s):

1. Left-side limit as $(x \to -\infty)$:

$$\lim_{x\to-\infty} xe^{\frac{1}{x}} = \lim_{x\to-\infty} x \lim_{x\to-\infty} e^{\frac{1}{x}} = -\infty \cdot \exp\left(\lim_{x\to-\infty} \frac{1}{x}\right) = -\infty \cdot e^{0^{\to 1}} = -\infty$$

2. Right-side limit as $(x \to +\infty)$:

$$\lim_{x\to\infty} xe^{\frac{1}{x}} = \lim_{x\to\infty} x \lim_{x\to\infty} e^{\frac{1}{x}} = \infty \cdot \exp\left(\lim_{x\to\infty} \frac{1}{x}\right) = \infty \cdot e^{0^{\to 1}} = \infty$$

In conclusion: as both of these two left-/right-side limits tend to infinity, it is said that horizontal asymptote does not exist.

Vertical asymptote(s):

3. Left-side limit around vertical asymptote $x = \underset{\rightarrow}{0}$: direct substitution method results in

$$\lim_{x\to \underset{\rightarrow}{0}} xe^{1/x} = \left(\lim_{x\to \underset{\rightarrow}{0}} x\right)\left(e^{\lim_{x\to \underset{\rightarrow}{0}} 1/x}\right) = (0)\ (e^{-\infty}) = 0\cdot 0 = 0$$

4. Right-side limit around vertical asymptote $x = \underset{\leftarrow}{0}$: direct substitution method results in

$$\lim_{x\to \underset{\leftarrow}{0}} xe^{1/x} = \left(\lim_{x\to \underset{\leftarrow}{0}} x\right)\left(e^{\lim_{x\to \underset{\leftarrow}{0}} 1/x}\right) = (0)\ (e^{+\infty}) = 0\cdot\infty$$

which is a non-determined form, thus must be resolved by some other methods. This classic form of limit is resolved most elegantly with L'Hôpital's rule (the seventeenth-century French mathematician Guillaume de l'Hôpital) after some algebraic transformations. For example,

$$\lim_{x\to \underset{\leftarrow}{0}} xe^{1/x} = \lim_{x\to \underset{\leftarrow}{0}} \frac{e^{1/x}}{1/x} = \frac{\infty}{\infty}$$

which is the form required by L'Hôpital's rule. Or, similarly,

$$\lim_{x\to \underset{\leftarrow}{0}} xe^{1/x} = \left\{\frac{1}{x} = t \Rightarrow x = \frac{1}{t} \quad \therefore\ x\to \underset{\leftarrow}{0} \Rightarrow t\to\infty\right\} = \lim_{x\to\infty} \frac{e^t}{t} \overset{\text{l'H}}{=} \lim_{x\to\infty} e^t = \infty$$

where the last limit to infinity is a classic form easily resolved by L'Hôpital's rule (see Sect. 2.6).

As these two left-/right-side limits are not equal, it is said that limit $\lim_{x\to 0}$ does not exist.

Oblique asymptote(s):

In order to determine the existence of oblique asymptotes $y_{\texttt{bb}} = ax + b$ to the given function, it is necessary to evaluate the following limit(s).

$$a = \lim_{x\to\infty} \frac{\cancel{x}\, e^{1/x}}{\cancel{x}} = \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

and

$$b = \lim_{x\to\infty} \left(xe^{1/x} - x\right) = \lim_{x\to\infty} \left(x\left(e^{1/x} - 1\right)\right) = \left\{\begin{array}{ll} e^{1/x} - 1 = t & \Rightarrow x = \dfrac{1}{\ln(t+1)} \\ & x\to\infty \Rightarrow t\to 0 \end{array}\right\}$$

$$= \lim_{x\to 0} \frac{t}{\ln(t+1)} = \lim_{x\to 0} \frac{1}{(1/t)\ln(t+1)} = \lim_{x\to 0} \frac{1}{\ln(t+1)^{1/t}} = \frac{1}{\ln\left(\lim_{x\to 0}(1+t)^{1/t}\right)} = \frac{1}{\cancel{\ln e}^{\,1}}$$

$$= 1$$

Therefore, there is one oblique asymptote $y_{\texttt{aa}} = x + 1$; see Fig. 1.22.

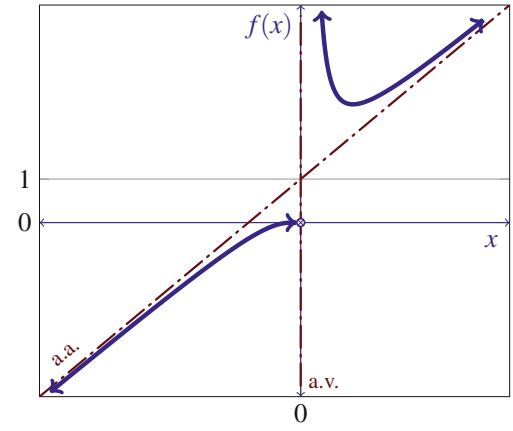


Fig. 1.22 Example P.1.85

Derivatives

2

Notation: there are two standard notations for the derivative operation:

1. In the case of single-variable functions, y' is a derivative of y assuming that the relevant variable is a priori known, such as x or t.
2. In the case of multiple-variable functions, for example, dy/dx explicitly specifies x as the relevant variable, where x may be one among possible variables.

Basic function derivatives are obtained from the fundamental theorem of calculus and are traditionally given in the form of "tabular derivatives", which are then simply used without additional proofs. Arguably, a minimal list of tabular derivatives and basic properties of derivation may be as follows (assuming that constant $a > 0$):

Tabular derivatives:

$$
\begin{array}{rl}
f(x) & f'(x) \\
a = \text{const.} & 0 \\
x^a & a\,x^{a-1} \\
\log_a x, \quad (x > 0, a > 0) & \dfrac{1}{x\,\ln a} \\
a^x, \quad (a > 0) & a^x\,\ln a \\
|x| & \text{sign}\,(x) \quad (x \neq 0) \\
\sin x & \cos x \\
\cos x & -\sin x \\
\arcsin x, \quad (|x| < 1) & \dfrac{1}{\sqrt{1-x^2}} \\
\arccos x, \quad (|x| < 1) & -\dfrac{1}{\sqrt{1-x^2}}
\end{array}
$$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41196-0_2

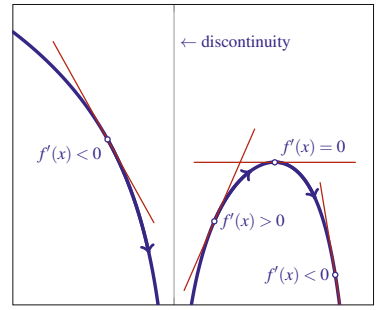


Fig. 2.1 Geometrical interpretation of derivative as a tangent at a given point

$\arctan x$	$\dfrac{1}{1+x^2}$

Basic properties and rules of derivation:

$$\Big[a\ f(x)\Big]' = a\ f'(x) \quad \text{(constant } a \text{ is factorized)}$$

$$\Big[f(x) \pm g(x)\Big]' = f'(x) \pm g'(x) \quad \text{(derivative of a sum is sum of derivatives)}$$

$$\Big[f(x)\ g(x)\Big]' = f'(x)\, g(x) + f(x)\, g'(x) \quad \text{(derivative of a product)}$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g^2(x)} \quad \text{(derivative of a ratio)}$$

$$\Big[f(g(x))\Big]' = f'(g(x))\, g'(x) \quad \text{(derivative of a composite function)}$$

Geometrical interpretation: Geometrical interpretation of a derivative at a given point (see Fig. 2.1) is as a tangent whose slope corresponds to:

- Function increase, i.e., $f'(x) > 0$
- Decrease, i.e., $f'(x) < 0$
- Constant, i.e., $f'(x) = 0$ (i.e., neither increase nor decrease)

Problems

2.1 Basic Derivatives

Calculate derivatives in P.2.1 to P.2.24 by using the list of basic derivatives.

2.1. x^5

2.2. $x^3 + 2x + 1$

2.3. $-\frac{1}{27}x^3 + \frac{1}{18}x^3$

2.4. $\frac{4}{x^3} - x^{-3}$

2.5. $\frac{1}{2}x^{\frac{1}{2}}$

2.6. $\frac{4}{3}x^{-\frac{3}{4}}$

2.7. $\sqrt{x}$

2.8. $\sqrt[3]{x^2}$

2.9. $5\sqrt[4]{x} - 2\sqrt[3]{x^3}$

2.10. $\frac{5}{\sqrt[4]{x}}$

2.11. $\frac{1}{\sqrt[3]{x^4}}$

2.12. $\frac{1}{3}x^3 - \frac{\sqrt{x}}{3} + 2$

2.13. 1^x

2.14. 2^x

2.15. e^x

2.16. 10^x

2.17. $a^x + x^a$

2.18. $e^x + x^3$

2.19. $\ln x$

2.20. $\log x$

2.21. $\log_5 x$

2.22. $\ln x + e^x$

2.23. $\ln x + 2\arctan x$

2.24. $\arcsin x - \arccos x$

2.2 Product Rule

Reminder: The product rule of derivation is

$$\big(f(x)\,g(x)\big)' = f'(x)\,g(x) + f(x)\,g'(x)$$

Calculate derivatives in P.2.25 to P.2.33.

2.25. $x \sin x$

2.26. $e^x \cos x$

2.27. $2x\sqrt{x}$

2.28. $\sqrt{x}\ln x$

2.29. $x^2\,3^x$

2.30. $(2x-1)(1-x)$

2.31. $x - \sin x \cos x$

2.32. $(x^3-1)(2x^2-3x+2)$

2.33. $(x^2+a^2)(x^2-a^2)$

2.3 Quotient Rule

Reminder: Derivative of a two-function ratio is done by the quotient rule,

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$

Calculate derivatives in P.2.34 to P.2.42.

2.34. $\dfrac{2x-1}{1-x}$ **2.35.** $\dfrac{x}{\sin x}$ **2.36.** $\dfrac{\sin x}{e^x}$

2.37. $\dfrac{\sqrt{x}}{\ln x}$ **2.38.** $\dfrac{\ln x - 2}{\ln x}$ **2.39.** $\dfrac{5-e^x}{e^x+2}$

2.40. $\tan x$ **2.41.** $\dfrac{\cos x}{1+2\sin x}$ **2.42.** $\dfrac{\tan x}{1-\tan x}$

2.4 Chain Rule

Reminder: Derivative of two or more composite functions is done by the chain rule,

$$\big(f\big(g(x)\big)\big)' = f'\big(g(x)\big)\,g'(x)$$

That is to say, in the first step, derivative of most external function form, in this case f(argument), is resolved while keeping (argument $= g(x)$) form as is. In the next step, the argument derivative, here $(g(x))$, itself is resolved. Finally, the two results are multiplied. In the case of multiple chained functions, for example, $f(f(h(x)))$, the same process is resolved until most inner function derivative is resolved and results are multiplied.

Calculate derivatives in P.2.43 to P.2.75.

2.43. e^{5x} **2.44.** $\sin\big(x^2\big)$ **2.45.** $\sin(abc\,x)$

2.46. $\cos(2x)$ **2.47.** $(1-2x)^5$ **2.48.** $\sqrt{x^2+2}$

2.49. $\sqrt[3]{\sqrt{2x-5}}$ **2.50.** $2\sin\big(\ln\big(3x^2\big)\big)$ **2.51.** e^{1-x}

2.52. 2^{x^2+2x-1} **2.53.** e^{x^2} **2.54.** $\sqrt[3]{e^{6x-3}}$

2.55. $\dfrac{1}{e^{-x^2/2}}$ **2.56.** $\dfrac{1}{e^{\sqrt{2-3x}}}$ **2.57.** $\ln(1-x)$

2.58. $\log_2\big(x^2+2x+1\big)$ **2.59.** $\ln\sqrt{2-x}$ **2.60.** $\sin(2x-1)$

2.61. $\sin\sqrt{x}$ **2.62.** $\cos(\sin(1-x))$ **2.63.** $\dfrac{1}{\ln(\sin x)}$

2.64. $\sin\left(\frac{1}{x}\right)$

2.65. $\ln\left(\sin x + \sqrt{1+\sin^2 x}\right)$

2.66. $\sin^2\left(\frac{2\pi}{3}+x\right)$

2.67. $\frac{1}{6}\ln\frac{2x-1}{x^2+x+1}$

2.68. $\sin(x-1)\cos(1-x)$

2.69. $\log(x-1)\cos(1-x)$

2.70. $\frac{\sin(x-1)}{\cos(1-x)}$

2.71. $\frac{\log(x-1)}{\cos(1-x)}$

2.72. $\frac{1}{\log(2x-3)}$

2.73. $\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$

2.74. $\ln(\ln(\ln x))$

2.75. $\ln\sqrt{\frac{\sin 2x}{1-\sin 2x}}$

2.5 Taylor Polynomial

Reminder: The Taylor polynomial of a function $f(x)$ that is differentiable at a point x_0 is in the form

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$= \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

where $n!$ denotes the factorial of n (by definition $0! = 1$) and $f^{(n)}(x_0)$ denotes the n-th derivative at the point x_0. By definition, zero derivative $f^{(0)}(x) = f(x)$ and $(x-x_0)^0 = 1$.

Develop the following functions in P.2.76 to P.2.84 into its equivalent Taylor polynomials, and calculate only the first four terms of the polynomials.

2.76. $e^x,\ x_0 = 0$

2.77. $\cos x,\ x_0 = 0$

2.78. $\sin x,\ x_0 = 0$

2.79. $\sqrt{x},\ x_0 = 1$

2.80. $\ln(1+x),\ x_0 = 0$

2.81. $\frac{x}{e^{x-1}},\ x_0 = 1$

2.82. $e^{2x-x^2},\ x_0 = 0$

2.83. $\frac{1}{x^2+1}, x_0 = 0$

2.84. $\frac{1}{x}, x_0 = 1$

2.6 L'Hôpital's Rule

Reminder: L'Hôpital's rule is a very powerful technique to calculate limits that take the following non-determined indeterminate forms:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad 0^0,\quad \infty^0,\quad \infty-\infty$$

However, note that L'Hôpital's rule is applicable *only* in the case of rational functions that take the form of

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{f(x_0)}{g(x_0)}=\left(\frac{0}{0}\quad\text{or,}\quad\frac{\infty}{\infty}\right)$$

Only in these two cases, the rational function's limit may be calculated after calculating one or more derivatives of its numerator and denominator separately, i.e.,

$$\text{if}\quad\lim_{x\to x_0}\frac{f(x)}{g(x)}=\left(\frac{0}{0}\quad\text{or}\quad\frac{\infty}{\infty}\right)\Rightarrow\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Orthogonality between two functions, for example, at a given point $(x,\,y)$, relation $f(x)\perp g(x)$ is satisfied for

$$F(x,\,y,\,y'_f)\quad\text{and}\quad G\left(x,\,y,\,-\frac{1}{y'_f}\right)$$

where y'_f is a derivative of $f(x)$ and $(-1/y'_f)$ is a derivative of $g(x)$, that is to say tangent, at a given point $(x,\,y)$.

Resolve limits in P.2.85 to P.2.93 by applying L'Hôpital's rule.

2.85. $\displaystyle\lim_{x\to1}\frac{x^2+2x-3}{x-1}$

2.86. $\displaystyle\lim_{x\to0}\frac{\sin(x)}{x}$

2.87. $\displaystyle\lim_{x\to0}\frac{x+\sin(x)}{x}$

2.88. $\displaystyle\lim_{x\to0}\frac{\sin(2x)}{x}$

2.89. $\displaystyle\lim_{x\to0}\frac{4\sin(5x)}{\sin(4x)}$

2.90. $\displaystyle\lim_{x\to0}\frac{\sin(2x)}{\sin(3x)}$

2.91. $\displaystyle\lim_{x\to\pi/2}\frac{\ln(\sin x)}{\cos x}$

2.92. $\displaystyle\lim_{x\to0}\frac{\ln(1+x)}{x}$

2.93. $\displaystyle\lim_{x\to0}\frac{e^{x^2}-\cos x}{x^2}$

2.94. $\displaystyle\lim_{x\to0}\frac{e^x-1}{x}$

2.95. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$

2.96. $\displaystyle\lim_{x\to-\infty}x\,e^x$

2.97. $\lim_{x\to 0} x \ln x$

2.98. $\lim_{x\to 0} x^x$

2.99. $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

2.100. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

2.101. $\lim_{x\to \infty} x^{1/x}$

2.102. $\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^{x+5}$

Answers

2.1 Basic Derivatives

2.1. $\left(x^5\right)' = 5x^{5-1} = 5x^4$

2.2. $\left(x^3 + 2x + 1\right)' = (x^3)' + (2x)' + (1)' = 3x^2 + 2 \ +0$

2.3. $\left(-\frac{1}{27}x^3 + \frac{1}{18}x^3\right)' = \left(-\frac{1}{27}x^3\right)' + \left(\frac{1}{18}x^3\right) = -\frac{3\,1}{27\,9}x^2 + \frac{3}{18}x^2 = -\frac{2}{18}x^2 + \frac{3}{18}x^2 = \frac{x^2}{18}$

2.4. $\left(\frac{4}{x^3} - x^{-3}\right)' = \left(4x^{-3} - x^{-3}\right)' = 4(-3)\,x^{-3-1} - (-3)\,x^{-3-1} = -12\,x^{-4} + 3\,x^{-4} = -\frac{9}{x^4}$

2.5. $\left(\frac{1}{2}x^{1/2}\right)' = \frac{1}{2}\frac{1}{2}x^{(1/2-1)} = \frac{1}{4}x^{(-1/2)} = \frac{1}{4x^{1/2}} = \frac{1}{4\sqrt{x}}$

2.6. $\left(\frac{4}{3}x^{-3/4}\right)' = \frac{4}{3}\left(-\frac{3}{4}\right)x^{-3/4-1} = -x^{-7/4} = -\frac{1}{x^{7/4}} = -\frac{1}{\sqrt[4]{x^7}}$

2.7. $\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

2.8. $\left(\sqrt[3]{x^2}\right)' = \left(x^{2/3}\right)' = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} = \frac{2}{3}\,\frac{1}{\sqrt[3]{x}}$

2.9. $\left(5\sqrt[4]{x} - 2\sqrt[3]{x^3}\right)' = \left(5x^{1/4} - 2x^{3/3}\right)' = \left(5x^{1/4} - 2x^{1}\right)' = \frac{5}{4}x^{-3/4} - 2\,x^0 = \frac{5}{4}\,\frac{1}{\sqrt[4]{x^3}} - 2$

2.10. $\left(\frac{5}{\sqrt[4]{x}}\right)' = \left(\frac{5}{x^{1/4}}\right)' = \left(5x^{-1/4}\right)' = 5\left(-\frac{1}{4}\right)x^{-5/4} = -\frac{5}{4}\,\frac{1}{\sqrt[4]{x^5}}$

2.11. $\left(\frac{1}{\sqrt[3]{x^4}}\right)' = \left(x^{-4/3}\right)' = -\frac{4}{3}x^{-7/3} = -\frac{4}{3}\,\frac{1}{\sqrt[3]{x^7}} = -\frac{4}{3}\,\frac{1}{\sqrt[3]{x^6\,x}} = -\frac{4}{3}\,\frac{1}{x^2\,\sqrt[3]{x}}$

2.12. $\left(\frac{1}{3}\,x^3 - \frac{\sqrt{x}}{3} + 2\right)' = \left(\frac{1}{3}\,x^3\right)' - \left(\frac{1}{3}\,\sqrt{x}\right)' + (2)' = x^2 - \frac{1}{6\sqrt{x}} \ +0$

2.13. $\left(1^x\right)' = 1^{x\,\to 1}\,\ln(1)^{\to 0} = 0$

2.14. $\left(2^x\right)' = 2^x\ \ln 2$

2.15. $\left(e^x\right)' = e^x\ \ln e^{\to 1} = e^x$

2.16. $\left(10^x\right)' = 10^x\ \ln 10$

2.17. $\left(a^x + x^a\right)' = a^x \ln a + a x^{a-1}$

2.18. $(e^x + x^3)' = (e^x)' + (x^3)' = e^x + 3x^2$

2.19. $(\ln x)' = \dfrac{1}{x \ln e} = \dfrac{1}{x}$

2.20. $(\log x)' = \dfrac{1}{x \ln 10}$

2.21. $\left(\log_5 x\right)' = \dfrac{1}{x \ln 5}$

2.22. $\left(\ln x + e^x\right)' = \dfrac{1}{x \ln e} + e^x \ln e = \dfrac{1}{x} + e^x = \dfrac{x e^x + 1}{x}$

2.23. $(\ln x + 2 \arctan x)' = \dfrac{1}{x \ln e} + \dfrac{2}{1 + x^2} = \dfrac{x^2 + 2x + 1}{x(x^2 + 1)}$

2.24. $(\arcsin x - \arccos x)' = \dfrac{1}{\sqrt{1 - x^2}} + \dfrac{1}{\sqrt{1 - x^2}} = \dfrac{2}{\sqrt{1 - x^2}}$

2.2 Product Rule

2.25. $(x \sin x)' = \left\{\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)\right\} = x' \sin x + x(\sin x)'$
$= \sin x + x \cos x$

2.26. $\left(e^x \cos x\right)' = \left(e^x\right)' \cos x + e^x (\cos x)' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$

2.27. $\left(2x\sqrt{x}\right) = \{x \geq 0\} = (2x)'\sqrt{x} + 2x\left(\sqrt{x}\right)' = 2\sqrt{x} + 2x \dfrac{1}{2\sqrt{x}} = 2\sqrt{x} + \dfrac{\sqrt{x^2}}{\sqrt{x}}$
$= 2\sqrt{x} + \sqrt{\dfrac{x^2}{x}} = 2\sqrt{x} + \sqrt{x} = 3\sqrt{x}$

2.28. $\left(\sqrt{x} \ln x\right)' = \left(\sqrt{x}\right)' \ln x + \sqrt{x}\ (\ln x)' = \dfrac{1}{2\sqrt{x}} \ln x + \sqrt{x} \dfrac{1}{x \ln e}$
$= \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x} = \dfrac{x \ln x + 2x}{2x\sqrt{x}}$

2.29. $\left(x^2\, 3^x\right)' = \left(x^2\right)' 3^x + x^2 \left(3^x\right)' = 2x\ 3^x + x^2\ 3^x \ln 3 = x\, 3^x (2 + x \ln 3)$

2.30. $\big((2x-1)(1-x)\big)' = (2x-1)'(1-x) + (2x-1)(1-x)' = 2(1-x) - (2x-1)$
$= 2 - 2x - 2x + 1 = -4x + 3$

Or, after polynomial development, the total derivative is calculated as

$$\big((2x-1)(1-x)\big)' = \big(-2x^2 + 3x - 1\big)' = -4x + 3$$

2.31. $(x - \sin x \cos x)' = x' - \big((\sin x)' \cos x + \sin x (\cos x)'\big) = 1 - \big(\cos x \cos x - \sin x \sin x\big)$
$= 1 - \big(\cos^2 x - \sin^2 x\big) = 1 - \cos^2 x + \sin^2 x = \big\{\cos^2 x + \sin^2 x = 1\big\}$
$= 2 \sin^2 x$

2.32. $\big((x^3-1)(2x^2-3x+2)\big)' = (x^3-1)'(2x^2-3x+2) + (x^3-1)(2x^2-3x+2)'$
$= 3x^2(2x^2-3x+2) + (x^3-1)(4x-3)$
$= 10x^4 - 12x^3 + 6x^2 - 4x + 3$

2.33. $\big((x^2+a^2)(x^2-a^2)\big)' = (x^2+a^2)'(x^2-a^2) + (x^2+a^2)(x^2-a^2)'$
$= 2x\,(x^2-a^2) + (x^2+a^2)\,2x = 2x\,(x^2 - \cancel{a^2} + x^2 + \cancel{a^2}) = 4x^3$

2.3 Quotient Rule

2.34. $\left(\dfrac{2x-1}{1-x}\right)' = \dfrac{(2x-1)'(1-x) - (2x-1)(1-x)'}{(1-x)^2} = \dfrac{2(1-x) + (2x-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$

2.35. $\left(\dfrac{x}{\sin x}\right)' = \dfrac{x' \sin x - x(\sin x)'}{\sin^2 x} = \dfrac{\sin x - x \cos x}{\sin^2 x}$

2.36. $\left(\dfrac{\sin x}{e^x}\right)' = \dfrac{(\sin x)'\, e^x - \sin x\, (e^x)'}{(e^x)^2} = \dfrac{\cos x\, e^x - \sin x\, e^x}{e^{2x}} = \cancel{e^x}\dfrac{\cos x - \sin x}{e^{\cancel{2}x}}$
$= \dfrac{\cos x - \sin x}{e^x}$

2.37. $\left(\dfrac{\sqrt{x}}{\ln x}\right)' = \dfrac{\big(\sqrt{x}\big)' \ln x - \sqrt{x}\,(\ln x)'}{\ln^2 x} = \dfrac{1/2\sqrt{x}\, \ln x - \sqrt{x}\, 1/x}{\ln^2 x} = \dfrac{\ln x/2\sqrt{x} - 1/\sqrt{x}}{\ln^2 x} = \dfrac{\ln x - 2}{2\sqrt{x} \ln^2 x}$

2.38. $\left(\dfrac{\ln x - 2}{\ln x}\right)' = \dfrac{(\ln x - 2)'\, \ln x - (\ln x - 2)(\ln x)'}{\ln^2 x} = \dfrac{1/x\, \ln x - (\ln x - 2)\, 1/x}{\ln^2 x}$
$= \dfrac{\ln x - (\ln x - 2)}{x\, \ln^2 x} = \dfrac{2}{x\, \ln^2 x}$

2.39. $\left(\dfrac{5 - e^x}{e^x + 2}\right)' = \dfrac{(5-e^x)'(e^x+2) - (5-e^x)(e^x+2)'}{(e^x+2)^2} = \dfrac{-e^x\,(e^x+2) - (5-e^x)\,e^x}{(e^x+2)^2}$
$= \dfrac{-e^x\,\big(\cancel{e^x} + 2 + 5 - \cancel{e^x}\big)}{(e^x+2)^2} = -\dfrac{7e^x}{(e^x+2)^2}$

2.40. $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \dfrac{\cos x\cos x + \sin x\sin x}{\cos^2 x}$

$$= \frac{\cancel{\cos^2 x + \sin^2 x}^{\,1}}{\cos^2 x} = \frac{1}{\cos^2 x}$$

or,

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x}}\,1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

2.41. $\left(\dfrac{\cos x}{1+2\sin x}\right)' = \dfrac{(\cos x)'(1+2\sin x) - (\cos x)(1+2\sin x)'}{(1+2\sin x)^2}$

$$= \frac{(-\sin x)(1+2\sin x) - (\cos x)(2\cos x)}{(1+2\sin x)^2}$$

$$= \frac{-\sin x - 2\sin^2 x - 2\cos^2 x}{(1+2\sin x)^2} = \frac{-\sin x - 2(\sin^2 x + \cos^2 x)}{(1+2\sin x)^2}$$

$$= -\frac{\sin x + 2}{(1+2\sin x)^2}$$

2.42. $\left(\dfrac{\tan x}{1-\tan x}\right)' = \dfrac{(\tan x)'(1-\tan x) - (\tan x)(1-\tan x)'}{(1-\tan x)^2} = \{\text{see A.2.40}\}$

$$= \frac{\frac{1-\tan x}{\cos^2 x} + \frac{\tan x}{\cos^2 x}}{(1-\tan x)^2} = \frac{1 - \cancel{\tan x} + \cancel{\tan x}}{\cos^2 x\,(1-\tan x)^2} = \frac{1}{\cos^2 x\,(1-\tan x)^2}$$

or,

$$= \frac{1+\tan^2 x}{(1-\tan x)^2}$$

2.4 Chain Rule

2.43. $\left(e^{5x}\right)' = \left\{\left(e^{f(x)}\right)' = e^{f(x)}\,f'(x)\right\} = e^{5x}\,(5x)' = 5\,e^{5x}$

2.44. $\left(\sin\left(x^2\right)\right)' = \left\{\left(\sin f(x)\right)' = \cos f(x)\,f'(x)\right\} = \cos\left(x^2\right)\left(x^2\right)' = 2x\,\cos\left(x^2\right)$

2.45. $\left(\sin(abc\,x)\right)' = \left\{\left(\sin f(x)\right)' = \cos f(x)\,f'(x)\right\} = abc\,\cos(abc\,x)$

2.46. $\left(\cos(2x)\right)' = -\sin(2x)\,(2x)' = -2\sin(2x)$

2.47. $\left((1-2x)^5\right)' = \left\{\left(\left(f(x)\right)^n\right)' = n\left(f(x)\right)^{n-1}\,f'(x)\right\} = 5\,(1-2x)^{5-1}\,(1-2x)'$

$$= 5\,(1-2x)^4\,(-2) = -10\,(1-2x)^4$$

2.48. $\left(\sqrt{x^2+2}\right)' = \left(\left(x^2+2\right)^{1/2}\right)' = \frac{1}{2}\left(x^2+2\right)^{-1/2}\left(x^2+2\right)' = \frac{1}{2}\frac{1}{\sqrt{x^2+2}}2x = \frac{x}{\sqrt{x^2+2}}$

2.49. $\left(\sqrt[3]{\sqrt{2x-5}}\right)' = \left\{x^{a^b} = x^{ab}\right\} = \left(\left((2x-5)^{1/2}\right)^{1/3}\right)' = \left((2x-5)^{\frac{1}{6}}\right)'$
$= \frac{1}{6}(2x-5)^{-5/6}(2x-5)' = \frac{1}{6}(2x-5)^{-5/6}(2) = \frac{1}{3}\frac{1}{\sqrt[6]{(2x-5)^5}}$

2.50. $\left(2\sin\left(\ln\left(3x^2\right)\right)\right)' = 2\cos\left(\ln\left(3x^2\right)\right)\left(\ln\left(3x^2\right)\right)' = 2\cos\left(\ln\left(3x^2\right)\right)\frac{1}{3x^2\ \ln e}\left(3x^2\right)'$
$= 2\cos\left(\ln\left(3x^2\right)\right)\frac{1}{3x^2}6x(x)' = \frac{4\cos\left(\ln\left(3x^2\right)\right)}{x}$

2.51. $\left(e^{1-x}\right)' = e^{1-x}\ \ln e^{1}(1-x)' = -e^{1-x}$

2.52. $\left(2^{x^2+2x-1}\right)' = 2^{x^2+2x-1}\ln(2)\left(x^2+2x-1\right)' = \ln(2)(2x+2)\,2^{x^2+2x-1}$
$= 2\ln(2)(x+1)\,2^{x^2+2x}2^{-1} = \ln(2)(x+1)\,2^{x(x+2)}$

2.53. $\left(e^{x^2}\right)' = e^{x^2}\ln e^{1}\left(x^2\right)' = 2x\,e^{x^2}$

2.54. $\left(\sqrt[3]{e^{6x-3}}\right)' = \left(\left(e^{6x-3}\right)^{1/3}\right)' = \left(e^{(6x-3)/3}\right)' = \left(e^{2x-1}\right)' = 2\,e^{2x-1}\ \ln e^{1}$

2.55. $\left(\frac{1}{e^{-x^2/2}}\right)' = \left(e^{x^2/2}\right)' = e^{x^2/2}\left(\frac{x^2}{2}\right)' = x\,e^{x^2/2}$

2.56. $\left(\frac{1}{e^{\sqrt{2-3x}}}\right)' = \left(e^{(2-3x)^{-1/2}}\right)' = \frac{1}{e^{\sqrt{2-3x}}}\ln e^{1}\left((2-3x)^{-1/2}\right)'$
$= \frac{1}{e^{\sqrt{2-3x}}}\left(-\frac{1}{2}\right)(2-3x)^{-3/2}(2-3x)' = \frac{3}{2}\frac{1}{e^{\sqrt{2-3x}}\sqrt{(2-3x)}}$

2.57. $\left(\ln(1-x)\right)' = \frac{1}{(1-x)\ln e^{1}}(1-x)' = -\frac{1}{1-x} = \frac{1}{x-1}$

2.58. $\left(\log_2\left(x^2+2x+1\right)\right)' = \frac{1}{\left(x^2+2x+1\right)\ln(2)}\left(x^2+2x+1\right)' = \frac{2x+2}{\ln(2)\left(x^2+2x+1\right)}$
$= \frac{2(x+1)}{\ln(2)(x+1)^2} = \frac{2}{\ln(2)(x+1)}$

2.59. $\left(\ln\sqrt{2-x}\right)' = \frac{1}{\sqrt{2-x}\ \ln e^{1}}\left(\sqrt{2-x}\right)' = \frac{1}{\sqrt{2-x}}\left((2-x)^{1/2}\right)'$
$= \frac{1}{2}\frac{1}{\sqrt{2-x}}(2-x)^{-1/2}(2-x)' = \frac{1}{2}\frac{1}{\sqrt{2-x}\sqrt{2-x}}(-1) = \frac{1}{2(x-2)}$

2.60. $(\sin(2x-1))' = \cos(2x-1)\,(2x-1)' = 2\,\cos(2x-1)$

2.61. $(\sin\sqrt{x})' = \cos(\sqrt{x})\,(\sqrt{x})' = \cos(\sqrt{x})\,(x^{1/2})' = \cos(\sqrt{x})\,\dfrac{1}{2}\,x^{-1/2} = \dfrac{\cos(\sqrt{x})}{2\sqrt{x}}$

2.62.
$$\begin{aligned}\big(\cos(\sin(1-x))\big)' &= -\sin\big(\sin(1-x)\big)\,\big(\sin(1-x)\big)'\\ &= -\sin\big(\sin(1-x)\big)\,\cos(1-x)\,(1-x)'\\ &= \sin\big(\sin(1-x)\big)\,\cos(1-x)\end{aligned}$$

2.63.
$$\begin{aligned}\left(\frac{1}{\ln(\sin x)}\right)' &= \Big(\big(\ln(\sin x)\big)^{-1}\Big)' = -\big(\ln(\sin x)\big)^{-2}\,\big(\ln(\sin x)\big)'\\ &= -\frac{1}{\ln^2(\sin x)}\,\frac{1}{\sin x\ \ln e^{1}}\,(\sin x)' = -\frac{1}{\ln^2(\sin x)}\,\frac{\cos x}{\sin x} = -\frac{\cot x}{\ln^2(\sin x)}\end{aligned}$$

2.64. $\left(\sin\left(\dfrac{1}{x}\right)\right)' = \cos\left(\dfrac{1}{x}\right)\left(\dfrac{1}{x}\right)' = \cos\left(\dfrac{1}{x}\right)\left(x^{-1}\right)' = \cos\left(\dfrac{1}{x}\right)(-1)\,x^{-2} = -\dfrac{\cos(1/x)}{x^2}$

2.65.
$$\begin{aligned}\left(\ln\left(\sin x+\sqrt{1+\sin^2 x}\right)\right)' &= \frac{1}{\sin x+\sqrt{1+\sin^2 x}}\left(\sin x+\sqrt{1+\sin^2 x}\right)'\\ &= \frac{1}{\sin x+\sqrt{1+\sin^2 x}}\left(\cos x+\frac{2\sin x\,\cos x}{2\sqrt{1+\sin^2 x}}\right)\\ &= \frac{\cos x}{\sin x+\sqrt{1+\sin^2 x}}\,\frac{\sqrt{1+\sin^2 x}+\sin x}{\sqrt{1+\sin^2 x}} = \frac{\cos x}{\sqrt{1+\sin^2 x}}\end{aligned}$$

2.66.
$$\begin{aligned}\left[\sin^2\left(\frac{2\pi}{3}+x\right)\right]' &= 2\sin\left(\frac{2\pi}{3}+x\right)\left(\sin\left(\frac{2\pi}{3}+x\right)\right)'\\ &= 2\sin\left(\frac{2\pi}{3}+x\right)\cos\left(\frac{2\pi}{3}+x\right)\left(\frac{2\pi}{3}+x\right)'\\ &= 2\sin\left(\frac{2\pi}{3}+x\right)\cos\left(\frac{2\pi}{3}+x\right) = \{2\sin x\cos x=\sin(2x)\}\\ &= \sin\left(\frac{4\pi}{3}+2x\right)\end{aligned}$$

2.67.
$$\begin{aligned}\left(\frac{1}{6}\ln\frac{2x-1}{x^2+x+1}\right)' &= \frac{1}{6}\,\frac{x^2+x+1}{(2x-1)\ln e}\left(\frac{2x-1}{x^2+x+1}\right)'\\ &= \frac{1}{6}\,\frac{x^2+x+1}{2x-1}\,\frac{1}{}\,\frac{(2x-1)'(x^2+x+1)-(2x-1)(x^2+x+1)'}{(x^2+x+1)^2}\\ &= \frac{1}{6}\,\frac{1}{2x-1}\,\frac{2(x^2+x+1)-(2x-1)(2x+1)}{x^2+x+1}\\ &= \frac{-2x^2+2x+3}{6(2x-1)(x^2+x+1)}\end{aligned}$$

2.68. $\big(\sin(x-1)\cos(1-x)\big)' = \big(\sin(x-1)\big)'\cos(1-x) + \sin(x-1)\big(\cos(1-x)\big)'$
$= \cos(x-1)(x-1)'\cos(1-x) - \sin(x-1)\sin(1-x)(1-x)'$
$= \cos(x-1)\cos(1-x) + \sin(x-1)\sin(1-x)$
$= \{\cos(x-y) = \cos x\cos y + \sin x \sin y\}$
$= \cos(2x-2) + \sin 0 \to 0$

2.69. $\big(\log(x-1)\cos(1-x)\big)' = \big(\log(x-1)\big)'\cos(1-x) + \log(x-1)\big(\cos(1-x)\big)'$
$= \dfrac{1}{(x-1)\ln 10}(x-1)'\cos(1-x)$
$- \log(x-1)\sin(1-x)(1-x)'$
$= \dfrac{1}{(x-1)\ln 10}\cos(1-x) + \log(x-1)\sin(1-x)$

2.70. $\left(\dfrac{\sin(x-1)}{\cos(1-x)}\right)' = \dfrac{\cos(x-1)\cos(1-x) + \sin(x-1)\sin(1-x)(-1)}{\cos^2(1-x)}$
$= \{\cos(-x) = \cos x \quad \text{and} \quad \sin(-x) = -\sin(x)\}$
$= \dfrac{\cos(x-1)\cos(x-1) + \sin(x-1)\sin(x-1)}{\cos^2(1-x)}$
$= \dfrac{\cos^2(x-1) + \sin^2(x-1)}{\cos^2(1-x)} = \dfrac{1}{\cos^2(1-x)}$

2.71. $\left(\dfrac{\log(x-1)}{\cos(1-x)}\right)' = \dfrac{\frac{\cos(1-x)}{(x-1)\ln 10} + \log(x-1)\sin(1-x)(-1)}{\cos^2(1-x)}$
$= \dfrac{\cos(1-x) - \ln 10\,(x-1)\,\log(x-1)\sin(1-x)}{\ln 10\,(x-1)\cos^2(1-x)} = \left\{\log a = \dfrac{\ln a}{\ln 10}\right\}$
$= \dfrac{\cos(1-x) - (x-1)\,\ln(x-1)\sin(1-x)}{\ln 10\,(x-1)\cos^2(1-x)}$

2.72. $\left(\dfrac{1}{\log(2x-3)}\right)' = \Big(\big(\log(2x-3)\big)^{-1}\Big)' = -\big(\log(2x-3)\big)^{-2}\big(\log(2x-3)\big)'$
$= -\dfrac{1}{\log^2(2x-3)}\dfrac{1}{(2x-3)\ln 10}(2x-3)'$
$= -\dfrac{2}{(2x-3)\ln 10\log^2(2x-3)}$

2.73. $\left(\dfrac{1}{\sqrt{3}}\arctan\dfrac{2x-1}{\sqrt{3}}\right)' = \dfrac{1}{\sqrt{3}}\dfrac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\left(\dfrac{2x-1}{\sqrt{3}}\right)' = \dfrac{1}{\sqrt{3}}\dfrac{(\sqrt{3})^2}{3+(2x-1)^2}\dfrac{2}{\sqrt{3}}$
$= \dfrac{1}{2(x^2-x+1)}$

2.74. $\left(\ln(\ln(\ln x))\right)'=\dfrac{1}{\ln(\ln x)\,\ln e}\left(\ln(\ln x)\right)'=\dfrac{1}{\ln(\ln x)}\,\dfrac{1}{\ln x\,\ln e}\left(\ln x\right)'=\dfrac{1}{\ln(\ln x)}\,\dfrac{1}{\ln x}\,\dfrac{1}{x}\,\dfrac{1}{\ln e}\,(x)'$

$$= \frac{1}{\ln(\ln x)}\,\frac{1}{\ln x}\,\frac{1}{x}$$

2.75. $$\left(\ln\sqrt{\frac{\sin 2x}{1-\sin 2x}}\right)' = \sqrt{\frac{1-\sin 2x}{\sin 2x}}\,\frac{1}{\ln e}\,\left(\sqrt{\frac{\sin 2x}{1-\sin 2x}}\right)'$$

$$= \sqrt{\frac{1-\sin 2x}{\sin 2x}}\,\frac{1}{2}\,\sqrt{\frac{1-\sin 2x}{\sin 2x}}\,\left(\frac{\sin 2x}{1-\sin 2x}\right)'$$

$$= \frac{1}{2}\,\frac{1-\sin 2x}{\sin 2x}\,\frac{(\sin 2x)'(1-\sin 2x)-(\sin 2x)(1-\sin 2x)'}{(1-\sin 2x)^2}$$

$$= \frac{1}{2}\,\frac{1}{\sin 2x}\,\frac{2\cos 2x\,(1-\sin 2x)-(\sin 2x)(-2\cos 2x)}{1-\sin 2x}$$

$$= \frac{1}{2}\,\frac{1}{\sin 2x}\,\frac{2\cos 2x-2\cos 2x\sin 2x+2\sin 2x\cos 2x}{1-\sin 2x}$$

$$= \frac{1}{2}\,\frac{1}{\sin 2x}\,\frac{2\cos 2x}{1-\sin 2x} = \frac{\cot 2x}{1-\sin 2x}$$

2.5 Taylor Polynomial

2.76. Given $x_0 = 0$ point and exponential function, the first four terms of Taylor polynomial are

$$f(x) = e^x \quad \therefore \quad f(0) = 1$$
$$f'(x) = e^x \quad \therefore \quad f'(0) = 1$$
$$f''(x) = e^x \quad \therefore \quad f''(0) = 1$$
$$f'''(x) = e^x \quad \therefore \quad f'''(0) = 1$$
$$\therefore$$
$$T(x) = \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

That is to say, as more and more higher-order monomial terms are added, Taylor polynomial becomes a better and better approximation of $f(x)$ (see Fig. 2.2 (left)), as

(1) one-term approximation: $T_1(x) = 1$ is correct at only one point $x = 0$;
(2) two-term approximation: $T_2(x) = 1 + x$ is a linear approximation around $x = 0$;
(3) three-term approximation: $T_3(x) = 1 + x + x^2/2$ is a quadratic approximation around $x = 0$;
(4) four-term approximation: $T_4(x) = 1 + x + x^2/2 + x^3/6$ is a cubic approximation around $x = 0$;

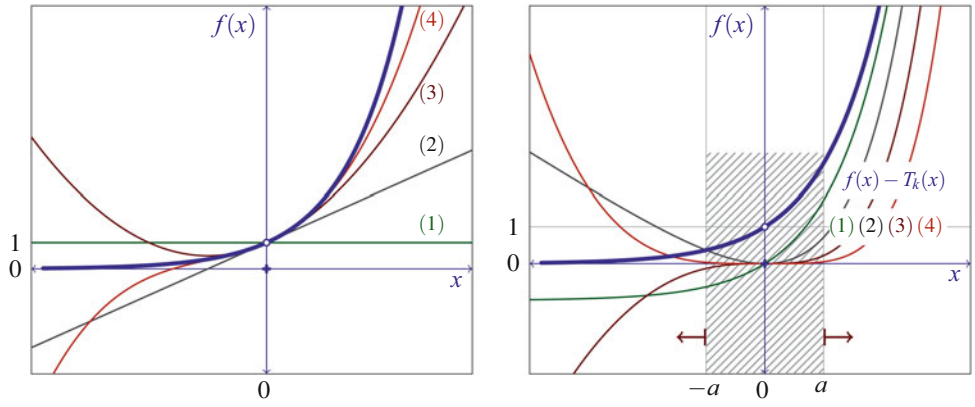


Fig. 2.2 Example P.2.76

and at the same time, as the polynomial order increases, the difference $f(x) - T(x)$ is reduced within the given interval $x \in (-a, a)$ (see Fig. 2.2 (right)). Obviously, if the interval is increased, the approximation's maximal error becomes larger.

Estimate of difference between the original functions and Taylor polynomial may be estimated by calculating what is referred to as the remainder,

$$R_k(x) = \frac{f^{(k+1)}(x - x_0)}{(k+1)!}|x - x_0|^{k+1}$$

where k is the highest derivative of $T_k(x)$ polynomial. For example, the upper bound of the four-term polynomial approximation within $x \in (-1, 1)$ interval may be estimated by calculating $(k+1)$-st derivative and applying the remainder formula as

$$f^{\text{IV}}(x) = e^x \quad \therefore \quad f^{\text{IV}}(0) = 1$$

$$\therefore$$

$$\frac{\left(f^{\text{IV}}(0)\right)^4}{4!}|(1-0)^4| = \frac{(1)^4}{24}|(1)^4| = \frac{1}{24} \approx 0.0417$$

2.77. Given $x_0 = 0$ point and a sinusoidal periodic function, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= \cos x, \quad \therefore \quad f(0) = \cos 0 = 1 \\ f'(x) &= -\sin x \quad \therefore \quad f'(0) = -\sin 0 = 0 \\ f''(x) &= -\cos x \quad \therefore \quad f''(0) = -\cos 0 = -1 \\ f'''(x) &= \sin x \quad \therefore \quad f'''(0) = \sin 0 = 0 \end{aligned}$$

that is to say,

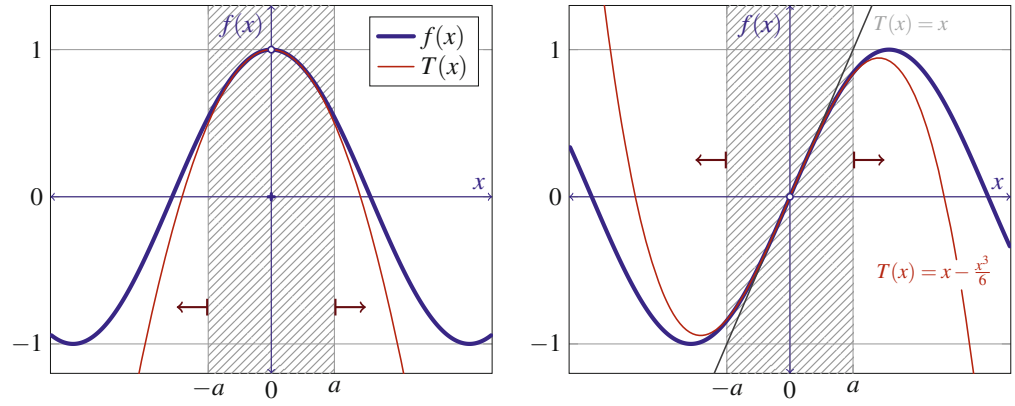


Fig. 2.3 Examples P.2.77 and P.2.78

$$T(x) = \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= \frac{1}{0!}x^0 + \frac{0}{1!}x^1 - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 = 1 - \frac{x^2}{2}$$

It should be noted that all odd-order polynomial terms are multiplied by zero. Therefore, quadratic polynomial reasonably well approximates $\cos x$ within around $x = 0$ point; see Fig. 2.3 (left).

2.78. Given $x_0 = 0$ point and a sinusoidal periodic function, then

$$\begin{aligned} f(x) &= \sin x \;\;\therefore\;\; f(0) = \sin 0 = 0 \\ f'(x) &= \cos x \;\;\therefore\;\; f'(0) = \cos 0 = 1 \\ f''(x) &= -\sin x \;\;\therefore\;\; f''(0) = -\sin 0 = 0, \\ f'''(x) &= -\cos x \;\;\therefore\;\; f'''(0) = -\cos 0 = -1 \\ &\therefore \end{aligned}$$

$$T(x) = \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$
$$= \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 = x - \frac{x^3}{6}$$

It should be noted that all even-order polynomial terms are multiplied by zero. Therefore, cubic polynomial reasonably well approximates $\sin x$ within around $x = 0$ point within $x \in (-a, a)$ interval; see Fig. 2.3 (right). It is evident that, for $x \approx 0$, function $\sin x$ is very well approximated with only the linear term, this linear approximation $\sin x \approx x$ is very often used.

2.79. Given $x_0 = 1$ point and a square root function, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= \sqrt{x} \quad \therefore \quad f(1) = 1 \\ f'(x) &= \frac{1}{2\sqrt{x}} \quad \therefore \quad f'(1) = \frac{1}{2} \\ f''(x) &= -\frac{1}{4\sqrt{x^3}} \quad \therefore \quad f''(1) = -\frac{1}{4} \\ f'''(x) &= \frac{3}{8\sqrt{x^5}} \quad \therefore \quad f'''(1) = \frac{3}{8} \end{aligned}$$

Therefore,

$$\begin{aligned} T(x) &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{4 \times 2!}(x-1)^2 + \frac{3}{8 \times 3!}(x-1)^3 \\ &= 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16} \end{aligned}$$

Comparison between $f(x)$ and $T(x)$ within a given interval $x \in (a, b)$ is shown in Fig. 2.4 (left).

2.80. Given $x_0 = 0$ point and a logarithmic function $f(x) = \ln(1+x)$, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= \ln(1+x) \quad \therefore \quad f(0) = 0 \\ f'(x) &= \frac{1}{1+x} \quad \therefore \quad f'(0) = 1 \\ f''(x) &= -\frac{1}{(1+x)^2} \quad \therefore \quad f''(0) = -1 \end{aligned}$$

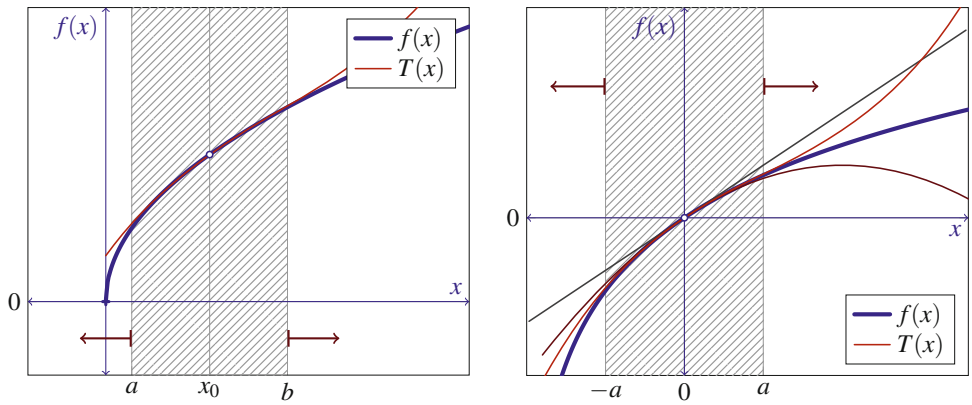


Fig. 2.4 Examples P.2.79 and P.2.80

$$f'''(x) = \frac{2}{(1+x)^3} \quad \therefore \quad f'''(0) = 2$$

Therefore,

$$\begin{aligned} T(x) &= \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \end{aligned}$$

Comparison between $f(x)$ and $T(x)$ within a given interval $x \in (-a, a)$ is shown in Fig. 2.4 (right).

2.81. Given $x_0 = 1$ point and an exponential rational function, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= \frac{x}{e^{x-1}} \quad \therefore \quad f(1) = 1 \\ f'(x) &= \frac{1-x}{e^{x-1}} \quad \therefore \quad f'(1) = 0 \\ f''(x) &= \frac{x-2}{e^{x-1}} \quad \therefore \quad f''(1) = -1 \\ f'''(x) &= \frac{3-x}{e^{x-1}} \quad \therefore \quad f'''(1) = 2 \end{aligned}$$

Therefore,

$$\begin{aligned} T(x) &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

Comparison between $f(x)$ and $T(x)$ within a given interval $x \in (a, b)$ is shown in Fig. 2.5 (left).

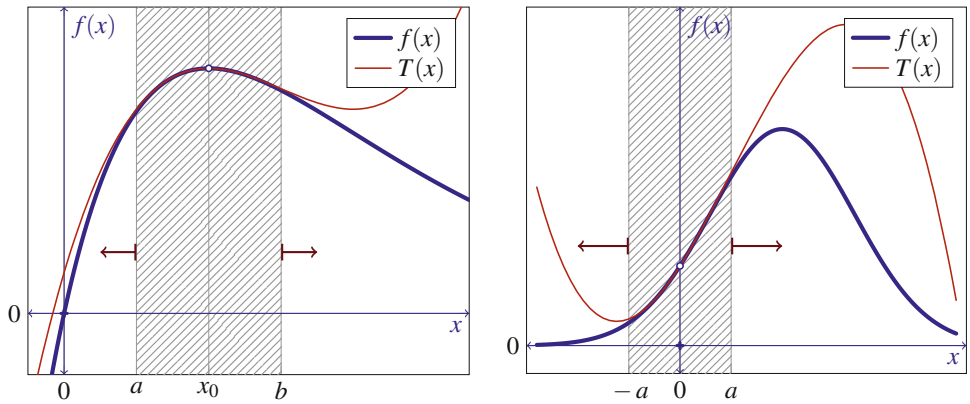


Fig. 2.5 Examples P.2.81 and P.2.82

2.82. Given $x_0 = 0$ point and an exponential function, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= e^{2x-x^2} \quad \therefore \quad f(0) = 1 \\ f'(x) &= e^{2x-x^2}(2-2x) \quad \therefore \quad f'(0) = 2 \\ f''(x) &= e^{2x-x^2}(4x^2-8x+2) \quad \therefore \quad f''(0) = 2 \\ f'''(x) &= e^{2x-x^2}(-8x^3+24x^2-12x-4) \quad \therefore \quad f'''(0) = -4 \end{aligned}$$

Therefore,

$$\begin{aligned} T(x) &= \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ &= 1 + 2x + x^2 - \frac{2}{3}x^3 \end{aligned}$$

Comparison between $f(x)$ and $T(x)$ within a given interval $x \in (-a, a)$ is shown in Fig. 2.5 (right).

2.83. Given $x_0 = a$ point and a rational function, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= \frac{1}{x^2+1} \quad \therefore \quad f(1) = \frac{1}{2} \\ f'(x) &= -\frac{2x}{(x^2+1)^2} \quad \therefore \quad f'(1) = -\frac{1}{2} \\ f''(x) &= \frac{2(3x^2-1)}{(x^2+1)^3} \quad \therefore \quad f''(1) = \frac{1}{2} \\ f'''(x) &= -\frac{24x(x-1)(x+1)}{(x^2+1)^4} \quad \therefore \quad f'''(1) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} T(x) &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{2}\frac{(x-1)^2}{2} + 0\,\frac{(x-1)^3}{6} = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 \end{aligned}$$

Comparison between $f(x)$ and $T(x)$ within a given interval $x \in (a, b)$ is shown in Fig. 2.6.

2.84. Given $x_1 = a$ point and the inverse function, the first four terms of Taylor polynomial are

$$\begin{aligned} f(x) &= \frac{1}{x} \quad \therefore \quad f(1) = 1 \\ f'(x) &= -\frac{1}{x^2} \quad \therefore \quad f'(1) = -1 \end{aligned}$$

Fig. 2.6 Example P.2.83

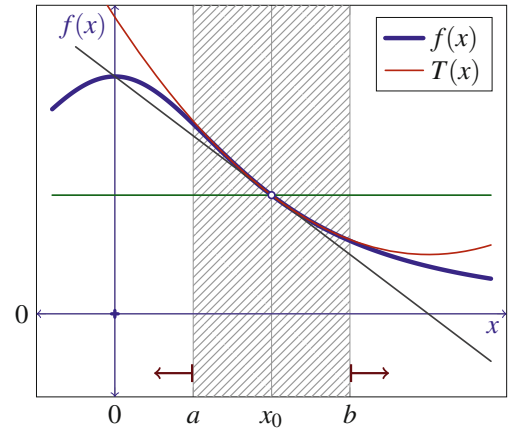
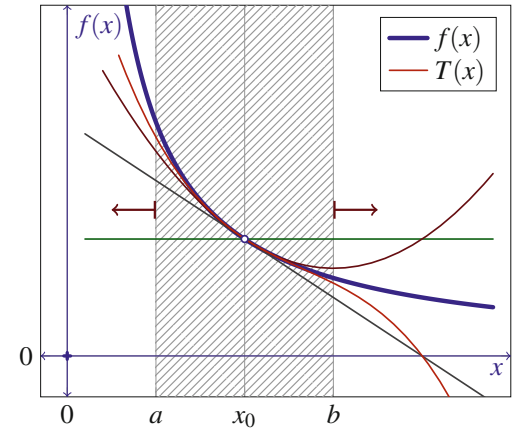


Fig. 2.7 Example P.2.84



$$f''(x) = \frac{2}{x^3} \quad \therefore \quad f''(1) = 2$$

$$f'''(x) = -\frac{6}{x^4} \quad \therefore \quad f'''(1) = -6$$

Therefore,

$$\begin{aligned} T(x) &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 - (x-1) + \cancel{2}\frac{(x-1)^2}{\cancel{2}} - \cancel{6}\frac{(x-1)^3}{\cancel{6}} = 1 - (x-1) + (x-1)^2 - (x-1)^3 \\ &= -x^3 + 4x^2 - 6x + 4 \end{aligned}$$

Comparison between $f(x)$ and $T(x)$ within a given interval $x \in (a, b)$ is shown in Fig. 2.7.

2.6 L'Hôpital's Rule

2.85. Direct substitution method shows that for $x = 1$, this limit tends to the one of the non-determined forms, as

$$\lim_{x\to 1}\frac{x^2+2x-3}{x-1}\underset{x=1}{=}\frac{1^2+2(1)-3}{1-1}=\frac{0}{0}$$

Therefore, L'Hôpital's rule may be used as

$$\lim_{x\to 1}\frac{x^2+2x-3}{x-1}\underset{x=1}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 1}\frac{(x^2+2x-3)'}{(x-1)'}=\lim_{x\to 1}\frac{2x+2}{1}=4$$

2.86. $$\lim_{x\to 0}\frac{\sin(x)}{x}\underset{x=0}{=}\frac{\sin(0)}{0}=\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(\sin(x))'}{(x)'}=\lim_{x\to 0}\frac{\cos(x)}{1}=\frac{\cos(0)}{1}=1$$

2.87. $$\lim_{x\to 0}\frac{x+\sin(x)}{x}\underset{x=0}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(x+\sin(x))'}{(x)'}=\lim_{x\to 0}\frac{1+\cos(x)}{1}=2$$

2.88. $$\lim_{x\to 0}\frac{\sin(2x)}{x}\underset{x=0}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(\sin(2x))'}{(x)'}=\lim_{x\to 0}\frac{2\cos(2x)}{1}=2$$

2.89. $$\lim_{x\to 0}\frac{4\sin(5x)}{\sin(4x)}\underset{x=0}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(4\sin(5x))'}{(\sin(4x))'}=\lim_{x\to 0}\frac{\cancel{4}\times 5\cos(5x)}{\cancel{4}\cos(4x)}=5$$

2.90. $$\lim_{x\to 0}\frac{\sin(2x)}{\sin(3x)}\underset{x=0}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{(\sin(2x))'}{(\sin(3x))'}=\lim_{x\to 0}\frac{2\cos(2x)}{3\cos(3x)}=\frac{2}{3}$$

2.91. $$\lim_{x\to \pi/2}\frac{\ln(\sin x)}{\cos x}\underset{x=\pi/2}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to \pi/2}\frac{(\ln(\sin x))'}{(\cos x)'}=\left\{[f(g(x))]'=f'(g(x))\,g'(x)\right\}$$
$$=\lim_{x\to \pi/2}\frac{(\cos x/\sin x)}{(-\sin x)}=\lim_{x\to \pi/2}\frac{\cos x}{-\sin^2 x}=\frac{0}{-1^2}=0$$

2.92. $$\lim_{x\to 0}\frac{\ln(1+x)}{x}\underset{x=0}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{\big(\ln(1+x)\big)'}{(x)'}=\left\{\big(f(g(x)\big)'=f'(g(x))\,g'(x)\right\}$$
$$=\lim_{x\to 0}\frac{1}{(1+x)\,(1)}=1$$

2.93. $$\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}\underset{x=0}{=}\left(\frac{0}{0}\right)\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{\left(e^{x^2}-\cos x\right)'}{\left(x^2\right)'}\lim_{x\to 0}\frac{2x\,e^{x^2}+\sin x}{2x}\underset{x=0}{=}\left(\frac{0}{0}\right)$$
$$\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{\left(2x\,e^{x^2}+\sin x\right)'}{(2x)'}$$

$$\stackrel{\text{l'H}}{=} \lim_{x\to 0} \frac{\left(2x\, e^{x^2} + \sin x\right)'}{(2x)'} = \lim_{x\to 0} \frac{\left(2e^{x^2} + 4x^2\, e^{x^2} + \cos x\right)'}{2}$$

$$= \frac{\left(2e^{0^2} + 4(0)^2\, e^{0^2} + \cos 0\right)'}{2} = \frac{3}{2}$$

2.94. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x} \underset{x=0}{=} \left(\frac{0}{0}\right) \stackrel{\text{l'H}}{=} \lim_{x\to 0} \frac{(e^x - 1)'}{(x)'} = \lim_{x\to 0} \frac{e^x}{1} = \frac{e^0}{1} = 1$

2.95. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2} \underset{x=\infty}{=} \left(\frac{\infty}{\infty}\right) \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to\infty} \frac{e^x}{2x} \underset{x=\infty}{=} \left(\frac{\infty}{\infty}\right) \stackrel{\text{l'H}}{=} \lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

2.96. $\displaystyle\lim_{x\to-\infty} x\, e^x \underset{x=-\infty}{=} (\infty\cdot 0) = \lim_{x\to-\infty} \frac{x}{e^{-x}} \underset{x=-\infty}{=} \left(\frac{\infty}{\infty}\right) \stackrel{\text{l'H}}{=} \lim_{x\to-\infty} \frac{(x)'}{(e^{-x})'} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$

2.97. $\displaystyle\lim_{x\to 0} x \ln x \underset{x=0}{=} (0\cdot\infty) = \lim_{x\to 0} \frac{\ln x}{1/x} \underset{x=0}{=} \left(\frac{\infty}{\infty}\right) \stackrel{\text{l'H}}{=} \lim_{x\to 0} \frac{(\ln x)'}{(1/x)'} = \lim_{x\to 0} \frac{1/\cancel{x}}{-1/\cancel{x}^2} = \lim_{x\to 0}(-x) = 0$

2.98. $\displaystyle\lim_{x\to 0_+} x^x \underset{x=0}{=} (0^0) = \left\{e^{\ln x} = x\right\} = \lim_{x\to 0_+} e^{\ln x^x} = \left\{\ln a^b = b\ln a\right\} = \lim_{x\to 0_+} e^{x\ln x}$

$$= \left\{\lim e^{f(x)} = e^{\lim f(x)}\right\} = e^{\lim_{x\to 0_+} x\ln x} = \{\text{see A.2.97}\} = e^0 = 1$$

2.99. $\displaystyle\lim_{x\to\pi/2} (\sin x)^{\tan x} \underset{x=\pi/2}{=} (1^\infty) = \left\{e^{\ln a} = a,\quad \ln a^b = b\ln a\right\}$

$$= \lim_{x\to\pi/2} \exp\left(\ln(\sin x)^{\tan x}\right) = \lim_{x\to\pi/2} \exp\left(\tan x\, \ln(\sin x)\right)$$

$$= \exp\left(\lim_{x\to\pi/2} \frac{\overset{\to 1}{\cancel{\sin x}}}{\cos x} \ln(\sin x)\right) = \exp\left(\lim_{x\to\pi/2} \frac{\ln(\sin x)}{\cos x}\right)$$

$$= \{\text{see A.2.91}\} = e^0 = 1$$

2.100. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) \underset{x=0}{=} (\infty - \infty) = \lim_{x\to 0}\left(\frac{\sin x - x}{x\, \sin x}\right) \underset{x=0}{=} \left(\frac{0}{0}\right)$

$$\stackrel{\text{l'H}}{=} \lim_{x\to 0}\left(\frac{(\sin x - x)'}{(x\, \sin x)'}\right) = \left\{[f(x)\, g(x)]' = f'(x)g(x) + f(x)g'(x)\right\}$$

$$= \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\, \cos x}\right) \underset{x=0}{=} \left(\frac{0}{0}\right) \stackrel{\text{l'H}}{=} \lim_{x\to 0}\left(\frac{(\cos x - 1)'}{(\sin x + x\, \cos x)'}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\sin x}{\cos x + \cos x - x\sin x}\right) = \frac{0}{2} = 0$$

2.101. $\displaystyle\lim_{x\to\infty} x^{1/x} \underset{x=\infty}{=} (\infty^0) = \left\{e^{\ln a} = a\right\} = \lim_{x\to\infty} \exp\left(\ln\, x^{1/x}\right) = \left\{\ln a^b = b\ln a\right\}$

$$= \lim_{x\to\infty} \exp\left(\frac{\ln x}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{\ln x}{x}\right) \underset{x=\infty}{=} \exp\left(\frac{\infty}{\infty}\right)$$

$$\stackrel{\text{l'H}}{=} \exp\left(\lim_{x\to\infty} \frac{(\ln x)'}{x'}\right) = \exp\left(\lim_{x\to\infty} \frac{1/x}{1}\right) = \exp\left(\frac{0}{1}\right) = e^0 = 1$$

2.102. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+5}=\lim\limits_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{5}\right]=\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{5}$

$=\limits_{x=\infty}\lim\limits_{x\to\infty}\left(1+\frac{1}{\infty}\right)^{\infty}\lim\limits_{x\to\infty}\left(1+\frac{1}{\infty}\right)^{5}=\limits_{x=\infty}\left(1^{\infty}\right)\ (1)=\left\{e^{\ln x}=x\right\}$

$=\lim\limits_{x\to\infty}\exp\left[\ln\left(1+\frac{1}{x}\right)^{x}\right]=\left\{\ln a^{b}=b\ln a\right\}$

$=\lim\limits_{x\to\infty}\exp\left[x\ \ln\left(1+\frac{1}{x}\right)\right]$

$=\limits_{x=\infty}(\exp(\infty\cdot 0))=\exp\left[\lim\limits_{x\to\infty}\frac{\ln(1+1/x)}{1/x}\right]=\limits_{x=\infty}\left(\exp\left(\frac{0}{0}\right)\right)$

$\overset{\text{l'H}}{=}\exp\left[\lim\limits_{x\to\infty}\frac{\left(\ln(1+1/x)\right)'}{(1/x)'}\right]=\exp\left(\lim\limits_{x\to\infty}\frac{x^{2}}{x^{2}+x}\right)=\limits_{x=\infty}\left(\exp\left(\frac{\infty}{\infty}\right)\right)$

$\overset{\text{l'H}}{=}\exp\left[\lim\limits_{x\to\infty}\frac{2x}{2x+1}\right]=\limits_{x=\infty}\left(\exp\left(\frac{\infty}{\infty}\right)\right)\overset{\text{l'H}}{=}\exp\left[\lim\limits_{x\to\infty}\frac{\cancel{2}}{\cancel{2}}\right]=e^{1}=e$

Function Analysis

3

Problems

The following tutorials are progressively more "sophisticated" and grouped by the similarity of functional forms. Basic polynomial forms are simplest, and they include linear, quadratic, or any higher-order polynomials. Their natural extension is rational polynomial forms, followed by radicals, exponential, logarithmic, and trigonometric forms. Important (and in reality, dominant) form of functions are known as "transcendental," and they are analyzed with the help of numerical techniques. Lastly, in the most general sense, one function may be argument of another, which by itself may be an argument of yet another function, etc., where this chain of input-output functions is referred to as a composite function.

3.1 Polynomial Functions

Reminder: this basic form of functions is created by adding monomials, i.e., ax^n terms as

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0 \\ &= a_0(x - x_1)^{\mu_1}(x - x_2)^{\mu_2} \cdots (x - x_n)^{\mu_r} \end{aligned}$$

where $x_1, x_2, \ldots, x_n$ are roots, i.e., solutions, of $f(x) = 0$ equation, and $\mu_1, \mu_2, \ldots, \mu_r$ are possible root multiplicities (i.e., repetitions), $\mu_i = 1, 2, \ldots$.

Theorem 3.1. *If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiplicities and complex roots $x_1, x_2, \ldots, x_n$.*

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https://doi.org/10.1007/978-3-031-41196-0_3

Reminder: Any polynomial of degree $n \geq 1$ can be therefore factorized into its linear factors as

$$\begin{aligned} P_n(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= (x - x_1)(x - x_2) \cdots (x - x_n) \end{aligned}$$

Even/odd orders (multiplicities) of polynomial roots control the function's shape and sign very differently. More precisely, odd roots force function to cross the horizontal axis (thus function changes its $\pm$ sign), while even roots force function to just touch the horizontal axis in one single point either from above or below the axis (thus, function does not change its $\pm$ sign). In addition, two different functions that are symmetrical (mirrored) relative to the horizontal axis do have identical set of zeros.

Reminder: Factoring techniques include:

1. Common factor: either a constant or variable can be factored from all polynomial terms, for example,

$$6x^3 + 12x^2y - 3xz = 3x(2x^2 + 4xy - z)$$

2. Grouping similar terms: either a constant or variable can be factored from a few polynomial terms at the time, for example,

$$10mx + 14nx - 15my - 21ny = 2x(5m + 7n) - 3y(5m + 7n) = (5m + 7n)(2x - 3y)$$

3. Binomial power forms: (recall Pascal's triangle):

$$x^2 + 2xy + y^2 = (x + y)^2 = (x + y)(x + y)$$
$$x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3 = (x + y)(x + y)(x + y)$$
$$\text{etc.}$$

4. Expanding with the "ghost" terms: sometimes it is useful to artificially expand a polynomial form to reach some of the known identities or factorizations, for example:

$$3x^2\underline{+10x} - 8 = 3x^2\underline{+12x - 2x} - 8 = 3x(x + 4) - 2(x + 4) = (x + 4)(3x - 2)$$

5. Algebraic identities: difference of two squares, completing the square, difference of two cubes, trinomials and cubic formulas, etc.. For example, completing the square:

$$x^2 + 6x + 7 = 0$$

(continued)

$$\underbrace{x^2 + 2 \cdot 3x}_{} \underbrace{+9}_{} - 9 + 7 = 0$$

$$(x+3)^2 - 2 = 0$$

$$x + 3 = \pm\sqrt{2} \quad \therefore \quad x_{1,2} = -3 \pm \sqrt{2}$$

Analyze polynomial functions in P.3.1 to P.3.19, and sketch their respective plots.

3.1. $f(x) = x$

3.2. $f(x) = -x$

3.3. $f(x) = 2x + 1$

3.4. $f(x) = x^2$

3.5. $f(x) = -x^2$

3.6. $f(x) = x^3$

3.7. $f(x) = -x^3$

3.8. $f(x) = x^4$

3.9. $f(x) = x^5$

3.10. $f(x) = x^2 - x - 2$

3.11. $f(x) = -x^2 + x + 2$

3.12. $f(x) = x^2 - 2x + 1$

3.13. $f(x) = -x^2 + 2x - 1$

3.14. $f(x) = x^2 - 2x + 2$

3.15. $f(x) = -x^2 + 2x - 2$

3.16. $f(x) = x^3 + 3x^2 + 2x$

3.17. $f(x) = x^3 + 2x^2 + x$

3.18. $f(x) = x^3 + x^2$

3.19. $f(x) = x^3 + 2x^2 - x - 2$

3.2 Rational Functions

Reminder: General form of a rational function $R(z)$ in its factorised form is:

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 z^0}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0 z^0} = \frac{a_n}{b_n} \frac{(z - z_{01})^{\mu_1} (z - z_{02})^{\mu_2} \cdots (z - z_{0r})^{\mu_r}}{(z - z_{p1})^{\xi_1} (z - z_{p2})^{\xi_2} \cdots (z - z_{pv})^{\xi_v}}$$

where μ_i are multiplicities of zeros and ξ_i are multiplicities of poles.

Zeros are roots of $R(z) = 0$ equation; therefore, the numerator $P(z) = 0$ equation. Main idea is to recall that if $A \cdot B \cdot D \cdots = 0$, then any or all of $A, B, C, \ldots$ must equal zero. Similarly, if $A/B = 0$, then it must be that $A = 0$. Also, note the ambiguous use of the word "zero": just because $R(z_0) = 0$, i.e., z_0 is a "zero" of $R(z)$, does not automatically mean that $z_0 = 0$ (it may as well be the case, but not necessarily).

(continued)

Poles are roots of $Q(z) = 0$ equation. Consequently, if $Q(z_p) = 0$, then $R(z_p) = \infty$; therefore, it is said that at z_p is a "pole" of $R(z)$ where $R(z)$ has discontinuity, i.e., a vertical asymptote.

Pole-zero cancelation: in the special case when one of the given zeros equals to one of the poles, i.e., $z_0 = z_p$, their rational function $R(z)$ is simplified (those two binomials may are removed; thus, numerator and denominator orders are reduced by one), and it is said that a *pole-zero cancelation* occurred. In that case, although the vertical asymptote is removed, the point $z = z_p$ is still excluded.

In summary, function's form is controlled by number and order of its poles and zeros, as well as the existence of pole-zero cancelations.

Analyze rational functions in P.3.20 to P.3.47, and sketch their respective plots. The problems progressively illustrate relations between multiple zeros and poles within a rational function.

3.20. $f(x) = \dfrac{1}{x}$

3.21. $f(x) = \dfrac{1}{x-1}$

3.22. $f(x) = \dfrac{x}{x-1}$

3.23. $f(x) = \dfrac{1}{x^2}$

3.24. $f(x) = \dfrac{1}{(x-1)^2}$

3.25. $f(x) = \dfrac{x}{(x-1)^2}$

3.26. $f(x) = \dfrac{1}{x^3}$

3.27. $f(x) = \dfrac{1}{(x+1)^3}$

3.28. $f(x) = \dfrac{x}{(x+1)^3}$

3.29. $f(x) = \dfrac{x+1}{x-1}$

3.30. $f(x) = \dfrac{x+1}{(x-1)^2}$

3.31. $f(x) = \dfrac{27x+27}{2(x-1)^3}$

3.32. $f(x) = \left(\dfrac{x+1}{x-1}\right)^2$

3.33. $f(x) = \dfrac{(x+1)^2}{(x-1)^3}$

3.34. $f(x) = \dfrac{(x+1)^2}{(x-1)^4}$

3.35. $f(x) = \dfrac{x+1}{x+1}$

3.36. $f(x) = \dfrac{x+1}{(x+1)^2}$

3.37. $f(x) = \dfrac{x+1}{(x+1)^3}$

3.38. $f(x) = \dfrac{(x+1)^3}{x+1}$

3.39. $f(x) = \dfrac{(x+1)^3}{(x+1)^2}$

3.40. $f(x) = \dfrac{(x+1)^3}{(x+1)^3}$

3.41. $f(x) = \dfrac{x}{x(x+1)^3}$

3.42. $f(x) = \dfrac{x(x+1)}{x(x+1)^3}$

3.43. $f(x) = \dfrac{x(x+1)^2}{x(x+1)^3}$

3.44. $f(x) = \dfrac{x^3}{(x-1)^2}$ **3.45.** $f(x) = 2 - \dfrac{2(1-x)}{x^2+1}$ **3.46.** $f(x) = \dfrac{x}{(x+1)(x-4)}$

3.47. $f(x) = \dfrac{4x}{4-x^2}$

3.3 Radical Functions

Reminder: for all *even* order ($2k, k = 1, 2, 3, \ldots$) radicals, it is true that their argument is limited to zero or positive values, as

$$\sqrt{f(x)}, \sqrt[4]{f(x)}, \sqrt[6]{f(x)}, \ldots \quad \therefore \quad \sqrt[2k]{f(x)} \Rightarrow f(x) \geq 0$$

On the other hand, for all *odd* order ($2k+1, k = 1, 2, 3, \ldots$) radicals, it is true that negative arguments produce negative results and positive arguments produce positive results, as

$$\sqrt[3]{f(x)}, \sqrt[5]{f(x)}, \sqrt[7]{f(x)}, \ldots \quad \begin{cases} \text{if} \quad f(x) < 0 \Rightarrow \sqrt[2k+1]{f(x)} < 0 \\ \text{if} \quad f(x) > 0 \Rightarrow \sqrt[2k+1]{f(x)} > 0 \end{cases}$$

(Try to calculate, e.g., $\sqrt[3]{-8}$ and $\sqrt[3]{8}$.)

Analyze radical functions in P.3.48 to P.3.67, and sketch their respective plots.

3.48. $f(x) = \sqrt{x}$ **3.49.** $f(x) = \sqrt{x+1}$ **3.50.** $f(x) = \sqrt{x^2}$

3.51. $f(x) = \sqrt{-x}$ **3.52.** $f(x) = \sqrt{-x-1}$ **3.53.** $f(x) = 1 - \sqrt{-x}$

3.54. $f(x) = \sqrt{x^2 - x^3}$ **3.55.** $f(x) = \sqrt[3]{x^2 - x^3}$ **3.56.** $f(x) = \sqrt[4]{1 + |x|}$

3.57. $f(x) = \sqrt{x + |x|}$ **3.58.** $f(x) = \sqrt{x - \sqrt{x}}$ **3.59.** $f(x) = \sqrt[3]{x^3 - 1}$

3.60. $f(x) = \sqrt{x + \sqrt{\dfrac{1}{x}}}$ **3.61.** $f(x) = \sqrt{x - \dfrac{1}{x}}$ **3.62.** $f(x) = \sqrt{x^2 + 2x - 1}$

3.63. $f(x) = \sqrt{x^2 + 2x + 1}$ **3.64.** $f(x) = \sqrt{x^3 + 2x^2 + 1}$ **3.65.** $f(x) = \sqrt{x^3 + 2x^2 - 1}$

3.66. $f(x) = \sqrt[3]{x^3 + 3x^2 + 3x + 1}$ **3.67.** $f(x) = \sqrt[3]{x^3 + 3x^2 - 3x - 1}$

3.4 Exponential and Log Functions

Reminder: by definition, logarithmic functions are defined only for strictly positive arguments, i.e., $\log f(x) \Rightarrow f(x) > 0$. Exponential functions are inverse to logarithms; thus $a^x > 0$, regardless of the argument's sign.

Analyze exp/log functions in P.3.68 to P.3.89, and sketch their respective plots.

3.68. $f(x) = 2^x$

3.69. $f(x) = e^x$

3.70. $f(x) = 10^x$

3.71. $f(x) = 2^{-x}$

3.72. $f(x) = e^{-x}$

3.73. $f(x) = 10^{-x}$

3.74. $f(x) = 2^x - 2$

3.75. $f(x) = 3^x - 3^{-x}$

3.76. $f(x) = \left|5^{|x|} - 2\right|$

3.77. $f(x) = \ln x$

3.78. $f(x) = \log_2 x$

3.79. $f(x) = \left|\log_2 x\right|$

3.80. $f(x) = e^{\ln x}$

3.81. $f(x) = \ln(e^x)$

3.82. $f(x) = \ln(x^2 - x)$

3.83. $f(x) = e^{2x} - 2e^x + 1$

3.84. $f(x) = 1 + e^{-x^2+x}$

3.85. $f(x) = e^x \ln(2x + 1)$

3.86. $f(x) = x \ln^2(x^2 + x)$

3.87. $f(x) = \log_{(x/2)}(x)$

3.88. $f(x) = x \ln^2 x$

3.89. $f(x) = \dfrac{1}{x} + \ln(x^2 + 2x + 1)$

3.5 Trigonometric Functions

Reminder: Trigonometric functions (aka "circular functions") are *transcendental*, because for a given argument x, the resulting value $y = f(x)$ cannot be calculated using algebraic methods, that is to say, based on addition, subtraction, multiplication, division, raising to a power, and root extraction operations.

Reminder: By definition, function $f(x)$ is *periodic* if it repeats its values at regular intervals. That is to say if $f(x + nT) = f(x)$ for all values of x in the domain, where $n = 1, 2, 3, \ldots$. The non-zero value T is then referred to as a *period* of this function. It is important to note that if T is period, so is any other integer multiple, i.e., $2T, 3T, 4T \ldots$.

Reminder: Periodic functions may be studied over one basic period only (i.e., $n = 1$). Periodicity of trigonometric functions is easily visualized by the unit circle: the one whose radius $r = 1$ thus its circumference equals $C = 2\pi r = 2\pi$. Consequently, a moving point that follows circular path is found at the same location every 2π distance, or in other words, $\sin(x + 2\pi) = \sin(x)$.

Reminder: Among many trigonometric identities,[a] some are used more often than the others,

$$a^2 + b = c^2 \quad \text{(Pythagoras' theorem)}$$

$$\sin^2\alpha + \cos^2\beta = 1 \quad \text{(Pythagoras' theorem in disguise, for } c = 1)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

Sum to product identities,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\{\text{if, } \alpha = \beta, \text{ note double angle identities}\}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^\alpha$$

Product to sum identities,

$$\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha - \beta) - \cos(\alpha + \beta)\Big)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\Big(\cos(\alpha + \beta) + \cos(\alpha - \beta)\Big)$$

$$\sin\alpha\cos\beta = \frac{1}{2}\Big(\sin(\alpha + \beta) + \sin(\alpha - \beta)\Big)$$

$$\{\text{if, } \alpha = \beta, \text{ then square to double angle identities}\}$$

$$\sin^2\alpha = \frac{1 - \cos(2\alpha)}{2} \quad \therefore \quad |\sin\alpha| = \sqrt{1 - \cos^2\alpha}$$

$$\cos^2\alpha = \frac{1 + \cos(2\alpha)}{2} \quad \therefore \quad |\cos\alpha| = \sqrt{1 - \sin^2\alpha}$$

[a] See, for example, https://en.wikipedia.org/wiki/List_of_trigonometric_identities (as of: Sep. 01, 2022).

Analyze functions in P.3.90 to P.3.103 that include trigonometric terms,[1] and sketch their respective plots (it is sufficient to make plots of only one period).

3.90. $f(x) = \sin(x + \pi/2)$

3.91. $f(x) = \sin^2 x, \quad x \in [-\pi, \pi]$

3.92. $f(x) = \sin(\pi\sqrt{x}), \quad x \in [0, 2\pi]$

3.93. $f(x) = \sin(\pi x^2), \quad x \in \left[-\sqrt{2}, \sqrt{2}\right]$

3.94. $f(x) = e^x \sin(2\pi x), \quad x \in [-3, 3]$

3.95. $f(x) = \ln x \sin(4x), \quad x \in [0, 2\pi]$

3.96. $f(x) = \cos^2 x - \cos x, \quad x \in [0, 2\pi]$

3.97. $f(x) = \dfrac{\sin x}{x}$

3.98. $f(x) = \tan x$

3.99. $f(x) = \arctan x$

3.100. $f(x) = \dfrac{\pi}{2} + \arctan\left(-x^2 + 1\right)$

3.101. $f(x) = \dfrac{1}{2}\left(1 - \cos(2x)\right)$

3.102. $f(x) = 2\sin^2 x + \sin x, \quad x \in [-\pi, \pi]$

3.103. $f(x) = \sin(2x) - \sin\left(x - \dfrac{\pi}{2}\right)$

3.6 Composite Functions

Analyze examples of composite functions in P.3.104 to P.3.114, and sketch their respective plots.

3.104. $f(x) = x^4 - 2x^2$

3.105. $f(x) = e^{1/x}$

3.106. $f(x) = x^x$

3.107. $f(x) = (2 - x^2)\, e^x$

3.108. $f(x) = \ln^2\left(1/(e^x + 1)\right)$

3.109. $f(x) = x\, e^{x^2 - x}$

3.110. $f(x) = 1 + 2x^3 - x^4$

3.111. $f(x) = \left|x^2 - |x| - 2\right|$

3.112. $f(x) = \ln|x^2 - 2x|$

3.113. $f(x) = (x - 1/4)\ln(1 - 1/x)$

3.114. $f(x) = \sin\left(2\, e^{x/2}\right), \quad x \in [-2\pi, \pi]$

[1] Some authors use $\tan^{-1} x$ notification instead of $\arctan x$. Note that $\tan^{-1} x \neq 1/\tan x$, which is sometimes confusing to the readers.

3.7 Orthogonal Functions

Reminder: In general:

1. It is said that two arbitrary functions $f(x)$ and $g(x)$ **intersect** at a given point x_0 if they are equal at that point; in other words, they share point (x_0, y_0), i.e.,

$$y_1 = f(x_1) = g(x_1)$$

2. Two straight lines with slopes m_1 and m_2 are **parallel** if their slopes (i.e., derivatives) at any given point are equal, i.e.,

$$m_1 = m_2$$

3. Two straight lines with slopes m_1 and m_2 are **orthogonal** if their slopes (i.e., derivatives) at the given point have negative inverse relationship as

$$m_1 = -\frac{1}{m_2}$$

Analyze relationships between functions given in each of P.3.115 to P.3.121, and sketch their respective plots.

3.115. Given linear function $f(x) = -(1/2)\,x + 1$, derive equation of a linear function $g(x) = ax + b$ that intersects $f(x)$ at $x_1 = 0$ point.

3.116. Given linear function $f(x) = f(x) = -(1/2)\,x + 1$, derive equation of a linear function $g(x) = ax + b$ that intersects $f(x)$ at $x = 0$ point and, at the same time, crosses $\mathscr{B} = (-1, -1)$ point.

3.117. Given linear functions $f(x)$ and $g(x)$ in P.3.116, calculate the angle between the two straight lines.

3.118. Given linear function $f(x) = 4x + 3$

1. Derive equation of a linear function $N_1(x) = ax + b$ that is normal to $f(x)$ for $x = 0$.
2. Derive equation of a linear function $N_2(x) = ax + b$ that is parallel to $N_1(x)$ for $x = -1$.

3.119. Given linear function $f(x) = -x^2 + 2$

1. Derive equation of a linear function $N_1(x) = ax + b$ that is normal to $f(x)$ for $x = 1$.
2. Derive equation of a linear function $N_2(x) = ax + b$ that is normal to $f(x)$ for $x = -2$.
3. Calculate the intersect point $\mathscr{A} = (x, y)$ of $N_1(x)$ and $N_2(x)$.

3.120. Given function studied in P.3.45:

$$f(x) = 2 - \frac{2(1-x)}{x^2+1}$$

1. Derive equation of a tangent T_1 to this function $f(x)$ at point $x = 1$.
2. Determine coordinates of point B so that tangent T_2 to $f(x)$ crossing B is parallel to line $y = -x$. What is the relationship between T_1 and T_2?

3.121. Given cubic function $f(x) = x^3 + 2x^2 - 1$, calculate parameters "a" and "b" so that quadratic function $g(x) = ax^2 + bx + 1/2$ is normal to $f(x)$.

1. at point $x = -1$,
2. at point $x = 1$

Answers

3.1 Simple Polynomial Functions

3.1. Given polynomial function in P.3.1,

$$f(x) = x$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

Reminder: a function can be odd, even, or none.

$$\text{if: } f(x) = f(-x) \ \therefore \ \text{even, for example, } x^2$$
$$\text{if: } f(x) = -f(-x) \ \therefore \ \text{odd, for example, } x^3$$

$$\text{even: } f(-x) = -x \neq f(x) \Rightarrow \text{ not even}$$
$$\text{odd: } -f(-x) = -(-x) = x = f(x) \Rightarrow \text{ odd}$$

In conclusion, this function is odd, i.e., symmetric relative to the origin point.

3. *Function's sign:*

Reminder: Solve equation: $f(x) = 0$ to determine intervals of x where the function is *positive* $(+)$ (i.e., above the horizontal axis, $f(x) > 0$) and where the function is *negative* $(-)$ (i.e., below the horizontal axis, $f(x) < 0$).

Thus,

$$\underline{f(x) = 0:} \ \therefore \ x = 0 \ \therefore \ x_1 = 0$$

To note, there is a first-order zero of $f(x)$ at $x = 0$. The positive/negative intervals of a function are determined by its zeros and poles. What is more, function's behavior is very different depending on whether zeros/poles are even or odd. In this case, $x = 0$ is odd,

x		0	
x	$-$	0	$+$
$f(x)$	$-$	0	$+$

In conclusion, $f(x)$ is negative for $x < 0$, and $f(x)$ changes its sign by crossing its odd zero $x = 0$; then, it is positive for $x > 0$.

4. *Function's parity:*

$$\begin{aligned} \text{even:} \quad & f(-x) = (-x) = -x \neq f(x) \Rightarrow \quad \text{not even} \\ \text{odd:} \; - & f(-x) = -(-x) = x = f(x) \Rightarrow \quad \text{odd} \end{aligned}$$

In conclusion, this function is odd, i.e., symmetric relative to the origin point.

5. *Limits:* there are no vertical asymptotes; thus, there are only two far extreme limits at $\pm\infty$ to be evaluated

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x = +\infty$$

6. *Oblique asymptote:*

Reminder: if a function $f(x)$ limits to a linear function $y_{\texttt{aa}} = ax + b$, i.e., the difference between $f(x)$ and $y_{\texttt{aa}}$ tends to zero, as

$$\lim_{x\to\infty} \Big[f(x) - (ax + b) \Big] = 0$$

then $y_{\texttt{aa}}$ is referred to as its "oblique asymptote." Determine $y_{\texttt{aa}} = ax + b$ by finding limits

$$a = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and,} \quad b = \lim_{x\to\infty} \Big(f(x) - ax \Big) \quad (\text{if } a \neq 0)$$

Thus,

$$\begin{aligned} a &= \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x}{x} = \Big\{ \quad \text{division is valid because} \quad x \neq 0 \Big\} = \lim_{x\to\infty} 1 \\ &= 1 \\ b &= \lim_{x\to\infty} \Big(f(x) - 1\, x \Big) = \lim_{x\to\infty} (x - x) = \lim_{x\to\infty} 0 \\ &= 0 \end{aligned}$$

In conclusion, oblique asymptote is $y_{\texttt{aa}} = x$, i.e., the function $f(x)$ itself (trivial case).

7. *Critical points:*

Reminder: Extreme minimum/maxim points are found by solving $f'(x) = 0$ and convex/concave (aka inflection) points by solving $f''(x) = 0$ equation.

Thus,

$$\underline{f'(x) = 0:}\ \ f'(x) = (x)' = 1 \neq 0$$

$$\underline{f''(x) = 0:}\ \ f''(x) = (x)'' = 0 \quad \text{(i.e. const.)}$$

In conclusion, because $f'(x) \neq 0$, there are no extreme points to calculate. What is more, because the first derivative is constant, consequently, the function is linear. On the other hand, because $f''(x) = 0$ is constant and equal to zero, the function is neither convex $\cap$ nor concave $\cup$. These are trivial cases.

8. *Graphical representation:* results of the analysis for P.3.1 are summarized by $f(x)$ graph (see Fig. 3.1) and compared with linear functions in P.3.2 and P.3.3.

3.2. Given polynomial function in P.3.2,

$$f(x) = -x$$

and already done procedure in P.3.1, we repeat the same steps.

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e. $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = -(-x) = x \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \ -f(-x) = -(x) = -x = f(x) \Rightarrow \quad \text{odd}$$

In conclusion, this function is odd, i.e., symmetric relative to the origin point.

3. *Function's sign:*

$$\underline{f(x) = 0:}\ \ \therefore\ \ -x = 0\ \ \therefore\ \ x_1 = 0$$

To note, there is a first-order zero of $f(x)$ at $x = 0$. The positive/negative intervals of a function are determined by its zeros and poles. What is more, function's behavior is very different depending on whether zeros/poles are even or odd. In this case, $x = 0$ is odd,

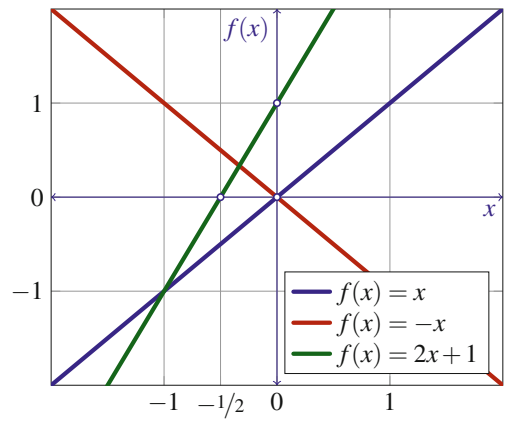


Fig. 3.1 Example P.3.1

x		0	
$-x$	$+$	0	$-$
$f(x)$	$+$	0	$-$

In conclusion, $f(x)$ is positive for $x < 0$ and changes its sign at $x = 0$, and it is negative for $x > 0$.

4. *Limits:* there are no vertical asymptotes, and thus at the far extremes $\pm\infty$, we find

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -x = -(-\infty) = +\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -x = -(+\infty) = -\infty$$

5. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{-x}{x} = \left\{ \text{division is valid because } x \neq 0 \right\} = \lim_{x\to\infty} -1$$
$$= -1$$
$$b = \lim_{x\to\infty} \Big(f(x) - ax \Big) = \lim_{x\to\infty} [-x - (-1)\,x] = \lim_{x\to\infty} (-x + x) = \lim_{x\to\infty} 0$$
$$= 0$$

In conclusion, oblique asymptote is: $y_{\mathtt{aa}} = -x$, i.e., the function $f(x)$ itself (trivial case).

6. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = (-x)' = -1 \neq 0$$
$$\underline{f''(x) = 0:}\ \ f''(x) = (-x)'' = 0 \quad \text{const.}$$

In conclusion, because $f'(x) \neq 0$, there are no extreme points to calculate. What is more, because the first derivative is constant, consequently, the function is linear. On the other hand, because $f''(x) = 0$ is constant and equal to zero, the function is neither convex ($\cap$) nor concave ($\cup$). These are trivial cases.

7. *Graphical representation:* results of the analysis for function in P.3.2 are summarized by $f(x)$ graph (see Fig. 3.1) and compared with linear functions in P.3.2 and P.3.2.

3.3. Given polynomial function

$$f(x) = 2x + 1$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = 2(-x) + 1 = -2x + 1 \neq f(x) \Rightarrow \quad \text{not even}$$
$$\text{odd:} \ -f(-x) = -(-2x + 1) = 2x - 1 \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is neither even nor odd, i.e., there is no symmetry.

3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; 2x + 1 = 0 \;\; \therefore \;\; x_1 = -\frac{1}{2}$$

To note, there is a first-order zero of $f(x)$ at $x = -1/2$. The positive/negative intervals of a function are determined by its zeros and poles; in this case, $x = -1/2$ as:

x		$-1/2$	
$2x + 1$	$-$	0	$+$
$f(x)$	$-$	0	$+$

In conclusion, $f(x)$ is negative for $x < -1/2$ and changes its sign at its odd zero $x = 0$, and it is positive for $x > -1/2$.

4. *y-axis crossing point:* Considering that the function does not cross the graph origin,

$$y_1 = f(0) = 2(0) + 1 = 1$$

In conclusion, $(x_1, y_1) = (0, 1)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus at there are only two far extremes limits $\pm\infty$ to evaluate

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (2x + 1) = 2(-\infty) + 1 = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (2x + 1) = 2(+\infty) + 1 = +\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2x+1}{x} = \lim_{x\to\infty} \left(2 + \cancelto{0}{1/x}\right) = \lim_{x\to\infty} 2$$
$$= 2$$
$$b = \lim_{x\to\infty} (f(x) - ax) = \lim_{x\to\infty} \left[\cancel{2x} + 1 - \cancel{2x}\right] = \lim_{x\to\infty} 1$$
$$= 1$$

In conclusion, oblique asymptote is: $y_{\mathtt{aa}} = 2x + 1$, i.e., the function $f(x)$ itself (trivial case). In general, a linear function is always its own oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \;\; f'(x) = (2x + 1)' = 2 > 0$$
$$\underline{f''(x) = 0:} \;\; f''(x) = (2x + 1)'' = 0 \quad \text{const.}$$

In conclusion, because $f'(x) \neq 0$, there are no extreme points to calculate. What is more, because the first derivative is constant, consequently, the function is linear. On the other hand, because

$f''(x) = 0$ is constant and equal to zero, the function is neither convex ($\cap$) nor concave ($\cup$). These are trivial cases.

8. *Graphical representation:* results of the analysis for function in P.3.3 are summarized by $f(x)$ graph (see Fig. 3.1) and compared with linear functions in P.3.1 and P.3.2.

3.4. Given polynomial function

$$f(x) = x^2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^2 = x^2 = f(x) \Rightarrow \quad \text{even}$$
$$\text{odd:} \quad -f(-x) = -(x^2) = -x^2 \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is even because $f(x) = f(-x)$, i.e., symmetrical relative to the vertical axis.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad x^2 = 0 \quad \therefore \quad x_1 = 0, x_2 = 0, \quad \text{(i.e., second-order zero} x_1 = x_2 = 0)$$

To note, there is a second-order zero of $f(x)$ at $x = 0$. The positive/negative intervals of a function are controlled by its zeros and poles; in this case, $x = 0$ as:

x		0	
x^2	+	0	+
$f(x)$	+	0	+

In conclusion, $f(x)$ is *always positive* and does not change its sign before and after its second (i.e., even)-order zero; instead, the function only touches the horizontal axis in one single point $x = 0$. This is a general property of even order zeros.

4. *y-axis crossing point:*

$$f(0) = (0)^2 = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which in this case is also function's zero.

5. *Limits:* there are no vertical asymptotes, and thus at the far extremes $\pm\infty$, we find

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2 = (-\infty)^2 = +\infty$$
$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^2 = (+\infty)^2 = +\infty$$

In conclusion, $f(x) = x^2$ is an even always positive function, and both left- and right-side limits tend to $+\infty$.

6. *Oblique asymptote:* note that $x \neq 0$, and by consequence, $x/x = 1$ is always determined

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(x^2\right)' = 2x \Rightarrow 2x = 0 \ \ \therefore\ \ x_1 = 0 \ \ \Big(\text{and, } f(x_1) = 0\Big)$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(x^2\right)'' = 2 > 0 \quad \text{const.}$$

Thus, there is one extreme point $(x, y) = (0, 0)$, whose nature (i.e., whether it is minimum or maximum point) is determined as:

x		0	
$2x$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

That is to say, there is minimum point at $(x, y) = (0, 0)$. On the other hand, because $f''(x)$ is constant and always positive, and there are no zeros, there are no points where $f''(x)$ could change its sign; thus, there are no inflection points, and by consequence, the function form is always concave (i.e., $f''(x) > 0 \Rightarrow \cup$).
In conclusion, this always concave function has one second-order extreme point at $(x, y) = (0, 0)$, where the function only touches the horizontal axis from the upper (positive) side of the graph.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{parity:}} &\quad \text{(even)} \\ \underline{\text{zeros:}} &\quad (x, y) = (0, 0) \\ \underline{\text{y-axis crossing point } f(0)\text{:}} &\quad y = 0 \\ \underline{\text{extremes:}} &\quad (x, y) = (0, 0) \quad \text{(min)} \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.4 are summarized by $f(x)$ graph (see Fig. 3.2) and compared with $f(x) = -x^2$ function in P.3.5.

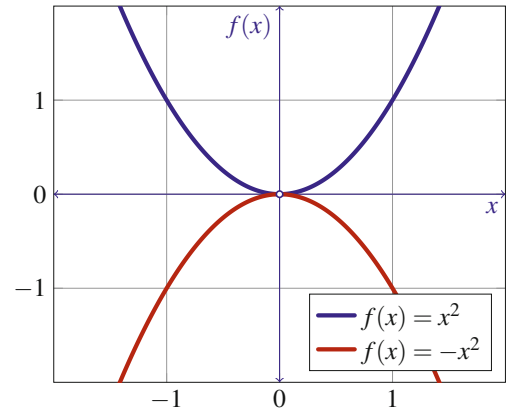
3.5. Given polynomial function (note that it is mirrored version of x^2)

$$f(x) = -x^2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = -(-x)^2 = -x^2 = f(x) \Rightarrow \quad \text{even}$$

Fig. 3.2 Example P.3.4



$$\text{odd: } -f(-x) = -(-x^2) = x^2 \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is even because $f(x) = f(-x)$, i.e., symmetrical relative to the vertical axis.

3. *Function's sign:*

$$\underline{f(x) = 0:} \ \therefore \ -x^2 = 0 \ \therefore \ x_1 = 0, \ \ x_2 = 0, \ \ \text{(i.e., second-order zero } \ x_1 = x_2\text{)}$$

To note, there is a second-order zero of $f(x)$ at $x = 0$. The positive/negative Intervals of a function are controlled by its zeros and poles; in this case, $x = 0$ as:

x		0	
$-x^2$	−	0	−
$f(x)$	−	0	−

In conclusion, $f(x)$ is *always negative* and does not change its sign before and after its second (i.e., even)-order zero; instead, the function only touches the horizontal axis in one single point $(x, y) = (0, 0)$.

4. *y-axis crossing point:*

$$f(0) = (0)^2 = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which in this case is also zero.

5. *Limits:* there are no vertical asymptotes, and thus there are only two extreme limits at $\pm\infty$ to evaluate

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -x^2 = -(-\infty)^2 = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -x^2 = -(+\infty)^2 = -\infty$$

In conclusion, $f(x) = -x^2$ is an even function that is always negative, and both left- and right-side limits tend to $-\infty$.

6. *Oblique asymptote:* note that for $x \neq 0$, division $x/x = 1$ is always defined, thus

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{-x^{\cancel{2}}}{\cancel{x}} = \lim_{x\to\infty} -x$$
$$= -\infty$$

In conclusion, because $a = -\infty$, thus oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(-x^2\right)' = -2x \Rightarrow -2x = 0 \ \ \therefore \ \ x = 0 \ \ \left(\text{and, } f(0) = 0\right)$$
$$\underline{f''(x) = 0:}\ \ f''(x) = \left(-x^2\right)'' = -2 < 0 \quad \text{(i.e. const.)}$$

Thus, there is one extreme point $(x, y) = (0, 0)$, whose nature (i.e., whether it is minimum or maximum point) is determined as:

x		0	
$-2x$	$+$	0	$-$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	0	$\searrow$

That is to say, there is maximum at $(x, y) = (0, 0)$. On the other hand, because $f''(x)$ is constant and always negative, there are no points where $f''(x)$ could change its sign; thus, there are no inflection points, and by consequence, the function form is always convex (i.e., $f''(x) < 0 \Rightarrow \cap$). In conclusion, this always convex function has one second-order zero at $(x, y) = (0, 0)$, where the function only touches the horizontal axis from the lower (negative) side of the graph.

8. *Summary of the important points:*

parity:	(even)
zeros:	$(x, y) = (0, 0)$
y-axis crossing point $f(0)$:	$y = 0$
extremes:	$(x, y) = (0, 0)$ (max)

9. *Graphical representation:* results of the analysis for function in P.3.5 are summarized by $f(x)$ graph (see Fig. 3.2) and compared with $f(x) = x^2$ in P.3.4. It should be noted that even though $f(x) = x^2$ and $f(x) = -x^2$ share same zero points, they are not identical – one is a mirrored (relative to the horizontal axis) version of the other.

3.6. Given polynomial function

$$f(x) = x^3$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^3 = -x^3 \neq f(x) \Rightarrow \quad \text{not even}$$
$$\text{odd:} \; -f(-x) = -(-x^3) = x^3 = f(x) \Rightarrow \quad \text{odd}$$

   This function is odd because $f(x) = -f(-x)$, i.e., symmetrical relative to the graph origin.
3. *Function's sign:*

$$f(x) = 0: \;\; \therefore \;\; x^3 = 0 \;\; \therefore \;\; x_{1,2,3} = 0, \quad \text{(i.e., third-order zero } \; x_1 = x_2 = x_3 \text{)}$$

   To note, there is a third (odd)-order zero of $f(x)$ at $x = 0$. The positive/negative Intervals of a function are controlled by its zeros and poles; in this case, $x = 0$ as:

x		0	
x^3	$-$	0	$+$
$f(x)$	$-$	0	$+$

   In conclusion, $f(x)$ is negative for $x < 0$, and then it crosses the horizontal axis at $x = 0$ and changes its sign after its third (i.e., odd)-order zero.
4. *y-axis crossing point:*

$$f(0) = (0)^3 = 0$$

   In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which in this case is also function's zero.
5. *Limits:* there are no vertical asymptotes; thus, there are only two extreme limits $\pm\infty$ to be evaluated

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^3 = (-\infty)^3 = -\infty$$
$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^3 = (+\infty)^3 = +\infty$$

   In conclusion, $f(x) = x^3$ is an odd function that changes its sign before and after its odd order zero, while left- and right-side limits tend to the opposite sides.
6. *Oblique asymptote:* note that $x \neq 0$, and by consequence, $x/x = 1$ is always determined

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3}{x} = \lim_{x \to \infty} x^2 = \infty$$

   In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left[x^3\right]' = 3x^2 \Rightarrow 3x^2 = 0\ \ \therefore\ \ x_1 = x_2 = 0\ \ \Big(\text{and, } f(0) = 0\Big)$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left[x^3\right]'' = 6x \Rightarrow 6x = 0\ \ \therefore\ \ x = 0\ \ \Big(\text{and, } f(0) = 0\Big)$$

Thus, there is one extreme second (even)-order point $(x, y) = (0, 0)$, whose nature (i.e., whether it is minimum or maximum point) is determined as:

x		0	
$3x^2$	+	0	+
$f'(x)$	+	0	+
$f(x)$	↗	0	↗

Note that second (even)-order root of $f'(x) = 0$ does *not* create extreme point, and the function constantly increases. There is a first (odd)-order root of $f''(x) = 0$ at $(x, y) = (0, 0)$, whose nature (i.e., convex and concave intervals) is determined as:

x		0	
$6x$	−	0	+
$f''(x)$	−	0	+
$f(x)$	∩	0	∪

This function has one third-order zero at $(x, y) = (0, 0)$ where the function changes its sign from negative on the left to positive on the right side of the graph. In addition, $(0, 0)$ is also inflection point where function changes from convex (∩) to concave (∪) form.

8. *Summary of the important points:*

parity:	(odd)
zeros:	$(x, y) = (0, 0)$
y-axis crossing point $f(0)$:	$y = 0$

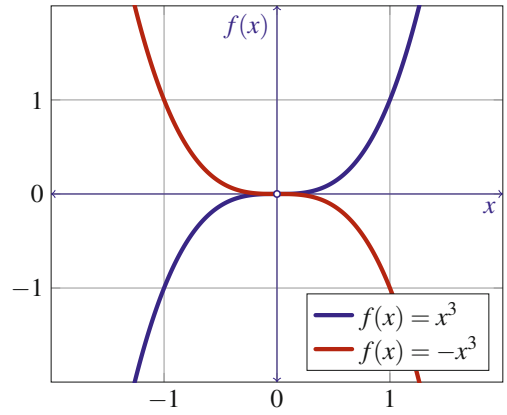
9. *Graphical representation:* results of the analysis for function in P.3.6 are summarized by $f(x)$ graph (see Fig. 3.3) and compared with $f(x) = -x^3$ function that is mirrored around the horizontal axis.

3.7. Given polynomial function

$$f(x) = -x^3$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.

Fig. 3.3 Example P.3.6



2. *Function's parity:*

$$\text{even:} \quad f(-x) = -(-x)^3 = x^3 \neq f(x) \Rightarrow \quad \text{not even}$$
$$\text{odd:} \quad -f(-x) = -(x^3) = -x^3 = f(x) \Rightarrow \quad \text{odd}$$

Tthis function is odd because $f(x) = -f(-x)$, i.e., symmetrical relative to the graph origin.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad -x^3 = 0 \quad \therefore \quad x_1 = x_2 = x_3 = 0, \quad \text{(i.e., third-order zero } \; x_1 = x_2 = x_3\text{)}$$

To note, there is a third (odd)-order zero of $f(x)$ at $x = 0$. The positive/negative intervals of a function are controlled by its zeros and poles; in this case, $x = 0$ as:

x		0	
$-x^3$	+	0	−
$f(x)$	+	0	−

In conclusion, $f(x)$ is positive for $x < 0$, crosses the horizontal axis at $x = 0$, and then changes its sign after its third-order zero.

4. *y-axis crossing point:*

$$f(0) = (0)^3 = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which in this case is also zero.

5. *Limits:* there are no vertical asymptotes, and thus, there are only two extreme limits at $\pm\infty$ to evaluate

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -x^3 = -(-\infty)^3 = +\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -x^3 = -(+\infty)^3 = -\infty$$

In conclusion, $f(x)$ is an odd function that changes its sign before and after its odd order zero, and left- and right-side limits tend to the opposite sides.

6. *Oblique asymptote:* note that $x \neq 0$, and by consequence, $x/x = 1$ is always determined

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{-x^3}{x} = \lim_{x\to\infty} -x^2 = -\infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(-x^3\right)' = -3x^2 \Rightarrow -3x^2 = 0 \ \ \therefore\ \ x_1 = x_2 = 0 \ \ \left(\text{and, } f(0) = 0\right)$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(-x^3\right)'' = -6x \Rightarrow -6x = 0 \ \ \therefore\ \ x = 0 \ \ \left(\text{and, } f(0) = 0\right)$$

Thus, there is one extreme second (even)-order point $(x, y) = (0, 0)$, whose nature (i.e., whether it is minimum or maximum point) is investigated as:

x		0	
$-3x^2$	$-$	0	$-$
$f'(x)$	$-$	0	$-$
$f(x)$	$\searrow$	0	$\searrow$

Note that second (even)-order root of $f'(x) = 0$ does *not* create extreme point. There is one first-order root of $f''(x) = 0$ at $(x, y) = (0, 0)$, whose nature (i.e., convex and concave intervals) is evaluated as

x		0	
$-6x$	$+$	0	$-$
$f''(x)$	$+$	0	$-$
$f(x)$	$\cup$	0	$\cap$

This function has one third-order zero at $(x, y) = (0, 0)$, where the function changes its sign from positive on the left to negative on the right side of the graph. In addition, $(0, 0)$ is also the inflection point where function changes from concave $\cup$ to convex $\cap$ form.

In general, it is important to note all similarities between symmetrical functions that share same zeros and to pay attention to the differences between them.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{parity:}} &\quad \text{(odd)} \\ \underline{\text{zeros:}} &\quad (x, y) = (0, 0) \\ \underline{\text{y-axis crossing point } f(0)\text{:}} &\quad y = 0 \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.7 are summarized by $f(x)$ graph (see Fig. 3.3) and compared with $f(x) = x^3$ function that is mirrored around the horizontal axis.

3.8. Given polynomial function

$$f(x) = x^4$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^4 = x^4 = f(x) \Rightarrow \quad \text{even}$$
$$\text{odd:} \; -f(-x) = -(x^4) = -x^4 \neq f(x) \Rightarrow \quad \text{not odd}$$

   In conclusion, this function is even because $f(x) = f(-x)$, i.e., symmetrical relative to the vertical axis.
3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; x^4 = 0 \;\; \therefore \;\; x_{1,2,3,4} = 0 \;\; \text{(i.e., fourth-order zero)}$$

   To note, there is a fourth (even)-order zero of $f(x)$ at $x = 0$. The positive/negative intervals of a function are controlled by its zeros and poles; in this case, $x = 0$ as

x		0	
x^4	+	0	+
$f(x)$	+	0	+

   In conclusion, $f(x)4$ is *always* positive, and thus it does not change its sign at zero; it merely touches the horizontal axis at $x = 0$ and stays in the upper (positive) side of the graph.
4. *y-axis crossing point:*

$$f(0) = (0)^4 = 0$$

   In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which in this case is also zero.
5. *Limits:* there are no vertical asymptotes, and thus, there are only two extreme limits at $\pm\infty$ to evaluate

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^4 = (-\infty)^4 = +\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^4 = (+\infty)^4 = +\infty$$

   In conclusion, $f(x)$ is an even function that does not change its sign before and after its even order zero, and left- and right-side limits tend to the same sides, here to positive infinity.
6. *Oblique asymptote:* note that $x \neq 0$, and by consequence, $x/x = 1$ is always determined

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^{4\,3}}{x} = \lim_{x\to\infty} x^3$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(x^4\right)' = 4x^3 \Rightarrow 4x^3 = 0\ \ \therefore\ \ x_{1,2,3} = 0\ \ \left(\text{and, } f(0) = 0\right)$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(x^4\right)'' = 12x^2 \Rightarrow 12x^2 = 0\ \ \therefore\ \ x_{1,2} = 0\ \ \left(\text{and, } f(0) = 0\right)$$

Thus, there is one extreme odd order point $(x, y) = (0, 0)$, whose nature (i.e., whether it is minimum or maximum point) is evaluated as

x		0	
$4x^3$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

There is one even order root of $f''(x) = 0$ at $(x, y) = (0, 0)$, whose nature (i.e., convex and concave intervals) is evaluated as

x		0	
$12x^2$	$+$	0	$+$
$f''(x)$	$+$	0	$+$
$f(x)$	$\cup$	0	$\cup$

In conclusion, this always positive function has one fourth-order zero at $x = 0$. Its second derivative $f''(x)$ has second-order root at $(0, 0)$; thus, because $f''(x) > 0$, this function is always in the concave $\cup$ form.

8. *Summary of the important points:*

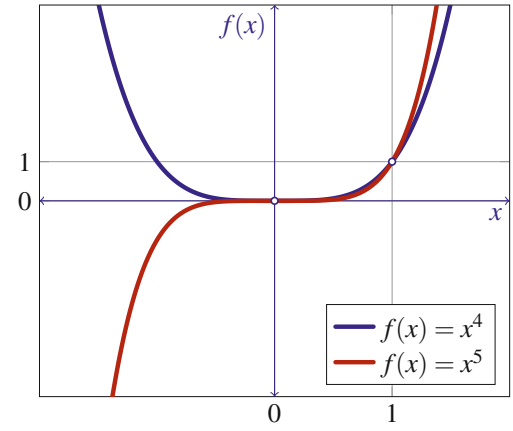
parity:	(even)
zeros:	$(x, y) = (0, 0)$
y-axis crossing point $f(0)$:	$y = 0$

9. *Graphical representation:* results of the analysis for function in P.3.8 are summarized by $f(x)$ graph (see Fig. 3.4) and compared with $f(x) = x^5$ function, P.3.9. We note that all simple x^n functions share two points: one zero at $(0, 0)$ and one point at $(1, 1)$ because of the algebraic identities $0^n = 0$ (where $n \neq 0$), and $1^n = 1$.

3.9. Given polynomial function

$$f(x) = x^5$$

Fig. 3.4 Example P.3.8



1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^5 = -x^5 \neq f(x) \Rightarrow \quad \text{not even}$$
$$\text{odd:} \; -f(-x) = -(-x^5) = x^5 = f(x) \Rightarrow \quad \text{odd}$$

   In conclusion, this function is odd because $f(x) = -f(-x)$, i.e., symmetrical relative to the graph origin.
3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; x^5 = 0 \;\; \therefore \;\; x_{1,2,3,4,5} = 0 \;\; \text{(i.e., fifth-order zero)}$$

   To note, there is a fifth (odd)-order zero of $f(x)$ at $x = 0$. The positive/negative intervals of a function are controlled by its zeros and poles; in this case, $x = 0$ as

x		0	
x^5	$-$	0	$+$
$f(x)$	$-$	0	$+$

   In conclusion, $f(x)$ is negative for $x < 0$, crosses its odd order zero $(0, 0)$, and then changes its sign for $x > 0$.
4. *y-axis crossing point:*

$$f(0) = (0)^5 = 0$$

   In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which in this case is also function's zero.
5. *Limits:* there are no vertical asymptotes, and thus two extreme limits $\pm\infty$ to evaluate

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^5 = (-\infty)^5 = -\infty$$
$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^5 = (+\infty)^5 = +\infty$$

In conclusion, $f(x)$ is an odd function, and left- and right-side limits tend to the opposite sides.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^5}{x} \quad (x \neq 0) = \lim_{x\to\infty} x^4 = \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(x^5\right)' = 5x^4 \Rightarrow 5x^4 = 0 \quad \therefore \quad x_{1,2,3,4} = 0 \quad \left(\text{and, } f(0) = 0\right)$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(x^5\right)'' = 20x^3 \Rightarrow 20x^3 = 0 \quad \therefore \quad x_{1,2,3} = 0 \quad \left(\text{and, } f(0) = 0\right)$$

Thus, there is an even extreme point $(x, y) = (0, 0)$, whose nature (i.e., whether it is minimum or maximum point) is evaluated as

x		0	
$5x^4$	+	0	+
$f'(x)$	+	0	+
$f(x)$	↗	0	↗

This function's first derivative does not change its sign at zero and $f'(x) > 0$; therefore, the function is always increasing. There is one odd root of $f''(x) = 0$ at $(x, y) = (0, 0)$, whose nature (i.e., convex and concave intervals) is evaluated as:

x		0	
$20x^3$	−	0	+
$f''(x)$	−	0	+
$f(x)$	∩	0	∪

In conclusion, this function has one fifth-order zero at $x = 0$ where it changes its sign from being negative to being positive. Its second derivative $f''(x)$ has third-order root at $(0, 0)$, where function changes from being convex ∩ to being concave ∪.

8. *Summary of the important points:*

parity:	(odd)
zeros:	$(x, y) = (0, 0)$
y-axis crossing point $f(0)$:	$y = 0$

9. *Graphical representation:* results of the analysis for function in P.3.9 are summarized by $f(x)$ graph (see Fig. 3.4) and compared with $f(x) = x^4$ function, P.3.8. We note that all simple x^n functions share two points: one zero at $(0, 0)$ and one point at $(1, 1)$ because of the algebraic identities $0^n = 0$ (where $n \neq 0$) and $1^n = 1$.

3.10. Given polynomial function

$$f(x) = x^2 - x - 2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because each of its three terms is in the simple $a_n x^n$ monomial form whose domains are not limited.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^2 - (-x) - 2 = x^2 + x - 2 \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -(x^2 + x - 2) = -x^2 - x + 2 \neq f(x) \Rightarrow \quad \text{not odd}$$

   In conclusion, when looking at its graph, this function may look even, but as it is shifted from the vertical axis, it is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.
3. *Function's sign:* (note: it is useful to learn multiple techniques to factorize a quadratic polynomial.)

$$\begin{aligned} f(x) = 0 \quad \therefore \; & x^2 - x - 2 = 0 \\ & x^2 - 2x + x - 2 = 0 \\ & x(x-2) + (x-2) = 0 \\ & (x-2)(x+1) = 0 \\ \therefore \; & x_1 = 2, \quad x_2 = -1 \end{aligned}$$

   There are two *first-order zeros* of $f(x)$ at $x_1 = 2$ and $x_2 = -1$.

Reminder: Recall that if $A \cdot B = 0$, then it must be $A = 0$ or $B = 0$. Similarly, if $A/B = 0$, then it must be that $A = 0$ and $B \neq 0$. Therefore, in order to determine sign of an expression, first the expression (or both numerator and denominator in the case of rational expressions) is factorized, and then finding positive/negative sign of products/ratios is straightforward, for example,

$$(-) \times (-) = (+) \quad \text{or,} \quad \frac{(-)}{(+)} = (-) \quad \text{etc.}$$

Reminder: Sign of powers of a binomial form $(a - b)^n$ depends upon the relative relationship between the two terms $(a, b > 0)$, as well as upon the parity of "n," that is to say

$$\text{if } a < b: \; \Rightarrow (a-b) < 0 \Rightarrow \begin{cases} (a-b)^n > 0 & \text{if 'n' even} \\ (a-b)^n < 0 & \text{if 'n' odd} \end{cases}$$

(continued)

$$\underline{\text{if } a = b:} \Rightarrow (a-b) = 0 \Rightarrow (a-b)^n = 0$$
$$\underline{\text{if } a > b:} \Rightarrow (a-b) > 0 \Rightarrow (a-b)^n > 0$$

Instead of trying to memorize the obvious, which you already know, simply try a few combinations of positive/negative numbers, for example,

$$(3-7)^2 = (-4)^2 = (-4)(-4) = +16 \text{ i.e., positive, '+', but,}$$
$$(3-7)^3 = (-4)^3 = (-4)(-4)(-4) = -64 \text{ i.e., negative, '-', etc.}$$

or, in the general form, (x_0 is a constant)

$$(x-x_0)^n = 0 \quad \text{if,} \quad x = x_0,$$
$$(x-x_0)^n > 0 \quad \text{if,} \quad x > x_0, \quad \text{etc.}$$

In order to determine positive/negative intervals of a function, it is advantageous to examine its factorized polynomial form and determine signs of each factor separately. In this notation, if $A = x + 1$ and $B = x - 2$, then their product is:

x		-1		2	
$x+1$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$+$

In conclusion, this function changes its sign after crossing through each its odd zeros.

4. *y-axis crossing point:*

$$f(0) = (0)^2 - (0) - 2 = -2$$

In conclusion, $(x, y) = (0, -2)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus only two extreme limits at $\pm\infty$ are to be evaluated

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x-2)(x+1) = (-\infty)(-\infty) = +\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x-2)(x+1) = (+\infty)(+\infty) = +\infty$$

In conclusion, $f(x)$ tends to $+\infty$ on both sides.

6. *Oblique asymptote:* recall that $\lim_{x\to\infty} {}^{n}/_{x} = 0$, so that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2 - x - 2}{x} \quad \{x \to \infty \;\; \therefore \;\; x \neq 0 \;\; \therefore \;\; x/x = 1\}$$

$$= \lim_{x\to\infty}\left(x - 1 - \cancel{\frac{2}{x}}^{0}\right) = \lim_{x\to\infty}(x-1)$$

$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\quad f'(x) = \left(x^2 - x - 2\right)' = 2x - 1 \Rightarrow 2x - 1 = 0 \quad \therefore \quad x_1 = \frac{1}{2}$$

$$\text{thus: } f(x_1) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2 = -\frac{9}{4}$$

$$\text{extreme point: } \left(\frac{1}{2}, -\frac{9}{4}\right)$$

$$\underline{f''(x) = 0:}\quad f''(x) = \left(x^2 - x - 2\right)'' = 2 > 0 \quad \text{(const.)}$$

Thus, there is one odd extreme point $(x, y) = (1/2, -9/4)$, whose nature (i.e., whether it is minimum or maximum point) is determined as:

x		$1/2$	
$2x-1$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	$-9/4$	$\nearrow$

That is to say, there is minimum at $x_1 = -1$ and $x_2 = 2$. Second derivative of this function is constant and positive; thus, $f(x)$ form is always concave $\cup$.

8. *Summary of the important points:*

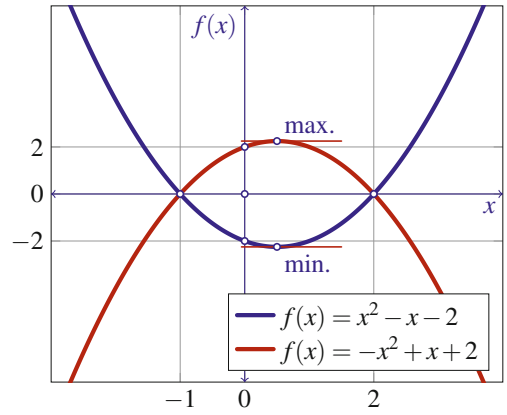
parity:	(none)
zeros:	$(x, y) = (-1, 0)$
	$(x, y) = (2, 0)$
y-axis crossing point $f(0)$:	$y = -2$
extremes:	$(x, y) = (1/2, -9/4)$ (min)

9. *Graphical representation:* results of the analysis for function in P.3.10 are summarized by $f(x)$ graph (see Fig. 3.5) and compared with $f(x) = -x^2 + x + 2$ function, P.3.11. This is an example of symmetric functions that share same zeros.

3.11. Given polynomial function

$$f(x) = -x^2 + x + 2 = -(x^2 - x - 2)$$

Fig. 3.5 Example P.3.10



1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because each of its three terms is in the simple $a_n x^n$ monomial form whose domains are not limited.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = -(-x)^2 + (-x) + 2 = -x^2 - x + 2 \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -(-x^2 - x + 2) = x^2 + x - 2 \neq f(x) \Rightarrow \quad \text{not odd}$$

Even though by looking at its graph this function may look even, it is shifted from the vertical axis, and therefore it is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad -x^2 + x + 2 = 0$$
$$-x^2 + 2x - x + 2 = 0$$
$$-x(x-2) - (x-2) = 0$$
$$-1\,(x-2)(x+1) = 0$$
$$\therefore \; x_1 = 2, x_2 = -1$$

There are two *first-order zeros* of $f(x)$ at $x_1 = 2$ and $x_2 = -1$. In order to determine positive/negative intervals of a function, it is advantageous to examine its factorized polynomial form. Note that there is "-1" factor as well, therefore:

x		-1		2	
-1	$-$	$-$	$-$	$-$	$-$
$x+1$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$

This function changes its sign after crossing through each of its odd zeros.

4. *y-axis crossing point:*

$$f(0) = -(0)^2 + (0) + 2 = 2$$

In conclusion, $(x, y) = (0, 2)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus two extreme limits at $\pm\infty$ are to be examined.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -(x-2)(x+1) = -(-\infty)(-\infty) = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -(x-2)(x+1) = -(+\infty)(+\infty) = -\infty$$

In conclusion, $f(x)$ tends to $-\infty$ on both sides.

6. *Oblique asymptote:* recall that $\lim_{x\to\infty} {}^{n}/_{x} = 0$, so that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{-x^2+x+2}{x} \quad \{x \to \infty \;\; \therefore \;\; x \neq 0 \;\; \therefore \;\; x/x = 1\}$$
$$= \lim_{x\to\infty} \left(-x + 1 + \cancel{{}^{2}/_{x}}^{\,0}\right) = \lim_{x\to\infty} (-x+1)$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:} \;\; f'(x) = \left(-x^2+x+2\right)' = -2x+1 \Rightarrow -2x+1 = 0 \;\; \therefore \;\; x_1 = \frac{1}{2}$$
$$\text{thus: } f(x_1) = -\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) + 2 = \frac{9}{4}$$
$$\text{extreme point: } \left(\frac{1}{2}, \frac{9}{4}\right)$$
$$\underline{f''(x) = 0:} \;\; f''(x) = \left(-x^2+x+2\right)'' = -2 < 0 \;\; \text{(const.)}$$

Thus, there is one odd extreme point $(x, y) = ({}^{1}/_{2}, {}^{9}/_{4})$, whose nature (i.e., whether it is minimum or maximum point) is examined as:

x		${}^{1}/_{2}$	
$-2x+1$	$+$	0	$-$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	${}^{9}/_{4}$	$\searrow$

That is to say, there is maximum point at $(x, y) = ({}^{1}/_{2}, {}^{9}/_{4})$. Second derivative of this function is constant and negative; thus its form is always convex $\cap$.

8. *Summary of the important points:*

parity: (none)

$$\begin{aligned} \underline{\text{zeros:}} \quad & (x, y) = (-1, 0) \\ & (x, y) = (2, 0) \\ \underline{\text{y-axis crossing point } f(0)\text{:}} \quad & y = 2 \\ \underline{\text{extremes:}} \quad & (x, y) = (1/2, 9/4) \ \ (\text{max}) \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.11 are summarized by $f(x)$ graph (see Fig. 3.5) and compared with $f(x) = x^2 - x - 2$ function, P.3.10. Even though they share same zeros, these two functions are symmetric but not same.

3.12. Given polynomial function

$$f(x) = x^2 - 2x + 1$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because each of its three terms is in the simple $a_n x^n$ monomial form whose domains are not limited.
2. *Function's parity:*

$$\begin{aligned} \text{even:} \quad & f(-x) = (-x)^2 - 2(-x) + 1 = x^2 + 2x + 1 \neq f(x) \Rightarrow \quad \text{not even} \\ \text{odd:} \quad & -f(-x) = -(x^2 + 2x + 1) = -x^2 - 2x - 1 \neq f(x) \Rightarrow \quad \text{not odd} \end{aligned}$$

This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

3. *Function's sign:*

$$\begin{aligned} \underline{f(x) = 0:} \quad \therefore \ & x^2 - 2x + 1 = 0 \\ & \{\text{recall } (a+b)^2 = a^2 + 2ab + b^2 \ \text{ or, simply } \} \\ & x^2 - x - x + 1 = 0 \\ & x(x-1) - (x-1) = 0 \\ & (x-1)(x-1) = 0 \\ & (x-1)^2 = 0 \\ \therefore \ & x_1 = 1, \quad x_2 = 1 \end{aligned}$$

This quadratic function $f(x)$ is in the special form known as "square of a binomial," as listed among other recognizable algebraic identities (patterns). Consequently, this form always has *second-order zero*, here at $x = 1$. In addition, a square function whose quadratic term is positive is also *always positive*:

$$f(x) = (x-1)^2 \geq 0$$

4. *y-axis crossing point:*

$$f(0) = (0)^2 - 2(0) + 1 = 1$$

In conclusion, $(x, y) = (0, 1)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes; thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (x-1)^2 = (-\infty)^2 = +\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (x+1)^2 = (+\infty)^2 = +\infty$$

In conclusion, being always positive $f(x) = (x+1)^2$ tends to $+\infty$ on both sides.

6. *Oblique asymptote:* recall that $\lim_{x\to\infty} {}^{n}/_{x} = 0$ and $x \neq 0$, so that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2 - 2x + 1}{x} = \lim_{x\to\infty} \left(x - 2 + \cancel{1/x}^{\,0} \right) = \lim_{x\to\infty} (x-2)$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (x^2 - 2x + 1)' = 2x - 2 \Rightarrow 2x - 2 = 0$$
$$\therefore \quad 2(x-1) = 0 \quad \therefore \quad x_1 = 1$$
$$\text{thus: } f(x_1) = (1)^2 - 2(1) + 1 = 0$$
$$\text{extreme point: } (1, 0)$$
$$\underline{f''(x) = 0:} \quad f''(x) = (x^2 - 2x + 1)'' = 2 > 0 \quad \text{(const.)}$$

Thus, there is one odd extreme point $(x, y) = (1, 0)$, whose nature (i.e., whether it is minimum or maximum point) is evaluated as:

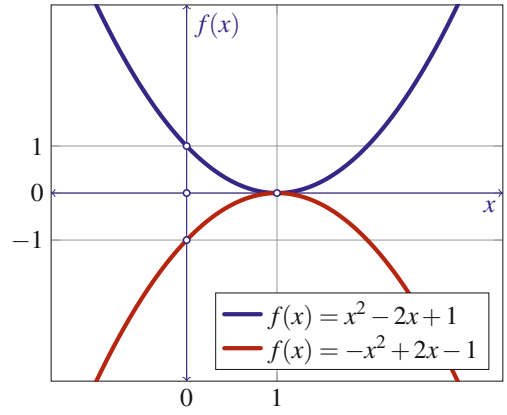
x		1	
$2(x-1)$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	0	$\nearrow$

That is to say, there is minimum point at $(x, y) = (1, 0)$. Second derivative of this function is constant and positive; thus, its form is always concave $\cup$. In general, by their nature, quadratic functions are always either concave or convex, that is to say, they have no inflection points.

8. *Summary of the important points:*

$$\underline{\text{parity:}} \quad \text{(none)}$$
$$\underline{\text{zeros:}} \quad (x, y) = (1, 0)$$

Fig. 3.6 Example P.3.12



y-axis crossing point $f(0)$: $y = 1$

extremes: $(x, y) = (1, 0)$ (min)

9. *Graphical representation:* results of the analysis for function in P.3.12 are summarized by $f(x)$ graph (see Fig. 3.6) and compared with $f(x) = -x^2 + 2x - 1$ function, P.3.13. This is another example of symmetric functions that share same zeros.

3.13. Given polynomial function

$$f(x) = -x^2 + 2x - 1 = -(x^2 - 2x + 1) = -(x - 1)^2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because each of its three terms is in the simple $a_n x^n$ monomial form whose domains are not limited.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = -(-x)^2 + 2(-x) - 1 = -x^2 - 2x - 1 \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -(-x^2 - 2x - 1) = x^2 + 2x + 1 \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

3. *Function's sign:*

$$f(x) = 0: \quad \therefore \quad -x^2 + 2x - 1 = 0 \quad \therefore \quad -(x^2 - 2x + 1) = 0$$

$$\left\{(a+b)^2 = a^2 + 2ab + b^2\right\}$$

$$-(x-1)^2 = 0 \quad \therefore \quad x_1 = 1, \quad x_2 = 1$$

In conclusion, this square of a binomial is multiplied by "-1"; by consequence, this function is *always negative*:

$$f(x) = -(x-1)^2 \leq 0$$

4. *y-axis crossing point:*

$$f(0) = -(0)^2 + 2(0) - 1 = -1$$

In conclusion, $(x, y) = (0, -1)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes; thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -(x-1)^2 = -(-\infty)^2 = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -(x+1)^2 = -(+\infty)^2 = -\infty$$

In conclusion, $f(x) = -(x+1)^2$ tends to $-\infty$ on both sides.

6. *Oblique asymptote:* recall that $\lim_{x\to\infty} {}^{n}/_{x} = 0$ and $x \neq 0$, so that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{-x^2+2x-1}{x} = \lim_{x\to\infty} \left(-x + 2 - \cancel{1/x}^{\,0}\right) = \lim_{x\to\infty} (-x+2)$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\quad f'(x) = (-x^2+2x-1)' = -2x+2 \Rightarrow -2x+2 = 0 \quad \therefore \quad -2(x-1) = 0$$
$$\therefore \quad -2(x-1) = 0 \quad \therefore \quad x_1 = 1$$
$$\text{thus: } f(x_1) = -(1)^2 + 2(1) - 1 = 0$$
$$\text{extreme point: } (1, 0)$$
$$\underline{f''(x) = 0:}\quad f''(x) = (-x^2+2x-1)'' = -2 < 0 \quad \text{(const.)}$$

Thus, there is one odd extreme point $(x, y) = (1, 0)$, whose nature is

x		1	
-2	$-$	$-$	$-$
$x-1$	$-$	0	$+$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	0	$\searrow$

That is to say, there is maximum point at $(x, y) = (1, 0)$. Second derivative of this function is constant and negative; thus $f(x)$ form is always convex $\cap$. In general, by their nature quadratic functions are always either concave or convex; in other words, there are no inflection points.

8. *Summary of the important points:*

$$\underline{\text{parity:}} \quad \text{(none)}$$

$$\begin{aligned} \underline{\text{zeros:}} \quad & (x, y) = (1, 0) \\ \underline{\text{y-axis crossing point } f(0)\text{:}} \quad & y = -1 \\ \underline{\text{extremes:}} \quad & (x, y) = (1, 0) \ \ (\text{max}) \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.13 are summarized by $f(x)$ graph (see Fig. 3.6) and compared with $f(x) = x^2 - 2x + 1$ function, P.3.12. Even though they share same zeros, these two functions are symmetric but not same.

3.14. Given polynomial function

$$f(x) = x^2 - 2x + 2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because each of its three terms is in the simple $a_n x^n$ monomial form whose domains are not limited.
2. *Function's parity:*

$$\begin{aligned} \text{even:} \quad & f(-x) = (-x)^2 - 2(-x) + 2 = x^2 + 2x + 2 \neq f(x) \Rightarrow \quad \text{not even} \\ \text{odd:} \quad & -f(-x) = -(x^2 + 2x + 2) = -x^2 - 2x - 2 \neq f(x) \Rightarrow \quad \text{not odd} \end{aligned}$$

   This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.
3. *Function's sign:*

$$\begin{aligned} \underline{f(x) = 0:} \quad & \therefore \ x^2 - 2x + 2 = 0 \\ & \{\text{complex roots}\} \\ & x_{1,2} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \\ & \therefore \ x_1 = 1 + I, \quad x_2 = 1 - i \end{aligned}$$

   In conclusion, this quadratic function $f(x)$ has two distinct *complex* zeros and thus *never* crosses the horizontal axis. By consequence, this square function is *always* strictly positive (note sign of the quadratic term "$+x^2$"):

$$f(x) = x^2 - 2x + 2 > 0$$

4. *y-axis crossing point:*

$$f(0) = (0)^2 - 2(0) + 2 = 2$$

   In conclusion, $(x, y) = (0, 2)$ is the vertical axis crossing point.
5. *Limits:* there are no vertical asymptotes, and thus

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2 - 2x + 2 = \lim_{x \to -\infty} x(x - 2) + 2 = (-\infty)(-\infty) + 2 = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^2 - 2x + 2 = \lim_{x\to+\infty} x(x-2) + 2 = (+\infty)(+\infty) + 2 = +\infty$$

In conclusion, $f(x) = x^2 - 2x + 2$ tends to $+\infty$ on both sides.

6. *Oblique asymptote:* recall that $\lim_{x\to\infty} {}^{n}/_{x} = 0$ and $x \neq 0$, so that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2 - 2x + 2}{x} = \lim_{x\to\infty} \left(x - 2 + \cancel{{}^{2}/_{x}}^{\,0} \right) = \lim_{x\to\infty} (x-2)$$

$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (x^2 - 2x + 2)' = 2x - 2 \Rightarrow 2x - 2 = 0$$

$$\therefore \quad 2(x-1) = 0 \quad \therefore \quad x_1 = 1$$

$$\text{thus: } f(x_1) = (1)^2 - 2(1) + 2 = 1$$

$$\text{extreme point: } (1, 1)$$

$$\underline{f''(x) = 0:} \quad f''(x) = (x^2 - 2x + 2)'' = 2 > 0 \quad \text{(const.)}$$

Thus, there is one odd extreme point $(x, y) = (1, 1)$, as

x		1	
2	+	+	+
$x-1$	−	0	+
$f'(x)$	−	0	+
$f(x)$	↘	1	↗

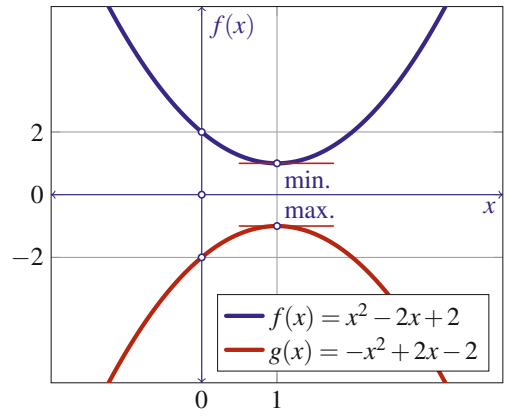
That is to say, there is minimum point at $(x, y) = (1, 1)$. Second derivative of this function is constant and positive, and thus $f(x)$ form is always concave ∪.

8. *Summary of the important points:*

parity:	(none)
zeros:	(none)
y-axis crossing point $f(0)$:	$y = 2$
extremes:	$(x, y) = (1, 1)$ (min)

9. *Graphical representation:* results of the analysis for function in P.3.14 are summarized by $f(x)$ graph, (see Fig. 3.7) and compared with $f(x) = -x^2 + 2x - 2$ function, P.3.15. This is another example of symmetric functions that share same zeros.

Fig. 3.7 Example P.3.14



3.15. Given polynomial function

$$f(x) = -x^2 + 2x - 2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because each of its three terms is in the simple $a_n x^n$ monomial form whose domains are not limited.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^2 - 2(-x) + 2 = x^2 + 2x + 2 \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -(x^2 + 2x + 2) = -x^2 - 2x - 2 \neq f(x) \Rightarrow \quad \text{not odd}$$

   This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; -x^2 + 2x - 2 = 0$$

$$\{\text{complex roots, quadratic equation}\}$$

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-1) \cdot (-2)}}{2 \cdot (-1)}$$

$$\therefore \; x_1 = 1 - i, \quad x_2 = 1 + i$$

   In conclusion, this quadratic function $f(x)$ has two distinct complex zeros and thus *never* crosses the horizontal axis. By consequence, this square function is *always* strictly negative (note sign of the quadratic term "$-x^2$"):

$$f(x) = -x^2 + 2x - 2 < 0$$

4. *y-axis crossing point:*

$$f(0) = -(0)^2 + 2(0) - 2 = -2$$

In conclusion, $(x, y) = (0, 2)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -x^2 + 2x - 2 = \lim_{x\to-\infty} -x^2\left(1 - \cancelto{0}{2/x} + \cancelto{0}{2/x^2}\right)$$
$$= \lim_{x\to-\infty} -x^2 = -(-\infty)^2 = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -x^2 + 2x - 2 = \lim_{x\to+\infty} -x^2\left(1 - \cancelto{0}{2/x} + \cancelto{0}{2/x^2}\right)$$
$$= \lim_{x\to+\infty} -x^2 = -(+\infty)^2 = -\infty$$

In conclusion, $f(x)$ tends to $+\infty$ on both sides.

6. *Oblique asymptote:* recall that $\lim_{x\to\infty} {}^{n}/_{x} = 0$ and $x \neq 0$, so that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{-x^2 + 2x - 2}{x} = \lim_{x\to\infty}\left(-x + 2 - \cancelto{0}{2/x}\right) = \lim_{x\to\infty}(-x + 2)$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\quad f'(x) = (-x^2 + 2x - 2)' = -2x + 2 \Rightarrow -2x + 2 = 0$$
$$\therefore\ -2(x - 1) = 0\ \ \therefore\ x_1 = 1$$
$$\text{thus: } f(x_1) = -(1)^2 + 2(1) - 2 = -1$$
$$\text{extreme point: } (1, -1)$$
$$\underline{f''(x) = 0:}\quad f''(x) = (-x^2 + 2x - 2)'' = -2 < 0\ \ \text{(const.)}$$

Thus, there is one odd extreme point $(x, y) = (1, -1)$, as

x		1	
-2	$-$	$-$	$-$
$x - 1$	$-$	0	$+$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	-1	$\searrow$

That is to say, there is maximum point at $(x, y) = (1, -1)$. Second derivative of this function is constant and negative; thus $f(x)$ form is always convex $\cap$.

8. *Summary of the important points:*

parity: (none)

zeros: (none)

$$\underline{\text{y-axis crossing point } f(0)}: \quad y = -2$$

$$\underline{\text{extremes:}} \quad (x, y) = (1, -1) \ \ (\text{max})$$

9. *Graphical representation:* results of the analysis for function in P.3.14 are summarized by $f(x)$ graph (see Fig. 3.7) and compared with $f(x) = -x^2 + 2x - 2$ function, P.3.15. Even though they share same zeros, these two functions are symmetric but not same.

3.16. Given polynomial function

$$f(x) = x^3 + 3x^2 + 2x = x(x^2 + 3x + 2)$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because all of the polynomial terms have simple monomial form $a_n x^n$, thus defined for all real arguments.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^3 + 3(-x)^2 + 2(-x) = -x^3 + 3x^2 - 2x \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \ -f(-x) = -(-x^3 + 3x^2 - 2x) = x^3 - 3x^2 + 2x \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

3. *Function's sign:*

$$\begin{aligned} \underline{f(x) = 0:} \ \therefore \ x^3 + 3x^2 + 2x &= \{\text{factorise}\} \\ &= x(x^2 + 3x + 2) \\ &= x(x^2 + 2x + x + 2) \\ &= x\left(x(x+2) + (x+2)\right) \\ &= x(x+1)(x+2) \\ \therefore \ x_1 = 0, \quad x_2 &= -1, \quad x_3 = -2 \end{aligned}$$

This cubic function $f(x)$ has three distinct odd order zeros; thus its sign intervals are,

x		-2		-1		0	
x	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

That is to say, this cubic function changes its sign after crossing through any of its odd zeros.

4. *y-axis crossing point:*

$$f(0) = (0)^3 + 3(0)^2 + 2(0) = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus there are only two far extreme limits at $\pm\infty$ to evaluate

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 + 3x^2 + 2x = \lim_{x\to-\infty} x^3 \left(1 + \cancelto{0}{\tfrac{3}{x}} + \cancelto{0}{\tfrac{2}{x^2}}\right)$$
$$= \lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 + 3x^2 + 2x = \lim_{x\to+\infty} x^3 \left(1 + \cancelto{0}{\tfrac{3}{x}} + \cancelto{0}{\tfrac{2}{x^2}}\right)$$
$$= \lim_{x\to+\infty} x^3 = (+\infty)^3 = +\infty$$

In conclusion, $f(x)$ tends to $-\infty$ for negative numbers and to $+\infty$ for positive numbers, because $x^3 > 0$, if $x > 0$, and $x^3 < 0$, if $x < 0$.

6. *Oblique asymptote:* note that for $x \neq 0$, division x/x is always defined; thus

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3 + 3x^2 + 2x}{x} = \lim_{x\to\infty} \left(x^2 + 3x + 2\right)$$
$$= \lim_{x\to\infty} x^2 \left(1 + \cancelto{0}{\tfrac{3}{x}} + \cancelto{0}{\tfrac{2}{x^2}}\right) = \lim_{x\to\infty} x^2$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (x^3 + 3x^2 + 2x)' = 3x^2 + 6x + 2$$
$$\therefore \; 3x^2 + 6x + 2 = 0 \; \{\text{quadratic formula}\}$$
$$\therefore \; x_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}$$
$$\therefore \; x_1 = -1 + \frac{\sqrt{3}}{3}, \quad x_2 = -1 - \frac{\sqrt{3}}{3}$$
$$(\text{note that } x_2 < x_1)$$

$$\underline{f''(x) = 0:} \quad f''(x) = (x^3 + 3x^2 + 2x)'' = 6x + 6$$
$$\therefore \; 6x + 6 = 0 \quad \therefore \; x_3 = -1$$
$$\text{so, } f(x_3) = (-1)^3 + 3(-1)^2 + 2(-1) = 0$$

And, y–coordinates of two extreme points $x_{1,2}$ are

$$f(x_1) = x_1(x_1+1)(x_1+2) = \left(-1+\frac{\sqrt{3}}{3}\right)\left(\cancel{-1}+\frac{\sqrt{3}}{3}\cancel{+1}\right)\left(\cancel{-1}+\frac{\sqrt{3}}{3}+\cancel{2}+1\right)$$

$$= \left(\frac{\sqrt{3}}{3}+1\right)\left(\frac{\sqrt{3}}{3}-1\right)\left(\frac{\sqrt{3}}{3}\right)$$

$$\left\{(a+b)(a-b) = a^2-b^2\right\}$$

$$= \left(\frac{3}{9}-1\right)\frac{\sqrt{3}}{3} = -\frac{2\sqrt{3}}{9} \quad \therefore \quad (x_1\,,\,y_1) = \left(-1+\frac{\sqrt{3}}{3},\ -\frac{2\sqrt{3}}{9}\right)$$

and,

$$f(x_2) = x_2(x_2+1)(x_2+2) = \left(-1-\frac{\sqrt{3}}{3}\right)\left(\cancel{-1}-\frac{\sqrt{3}}{3}\cancel{+1}\right)\left(\cancel{-1}-\frac{\sqrt{3}}{3}+\cancel{2}+1\right)$$

$$= -\left(1+\frac{\sqrt{3}}{3}\right)\left(1-\frac{\sqrt{3}}{3}\right)\left(-\frac{\sqrt{3}}{3}\right)$$

$$\left\{(a+b)(a-b) = a^2-b^2\right\}$$

$$= \left(1-\frac{3}{9}\right)\frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{9} \quad \therefore \quad (x_2\,,\,y_2) = \left(-1-\frac{\sqrt{3}}{3},\ \frac{2\sqrt{3}}{9}\right)$$

Two extreme points in $f'(x) = (x-x_1)(x-x_2)$ are determined as

x		x_2		x_1	
$x-x_2$	$-$	0	$+$	$+$	$+$
$x-x_1$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	y_2	$\searrow$	y_1	$\nearrow$

That is to say, there is maximum point at x_2, y_2 and minimum point at x_1, y_1. Second derivative $f''(x) = 6(x+1) = 0$ points are examined as

x		-1	
6	$+$	$+$	$+$
$x+1$	$-$	0	$+$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$	0	$\cup$

In conclusion, this cubic function has its inflection point at $(x, y) = (-1, 0)$, where the function changes from convex $\cap$ to concave $(\cup)$ form.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{y-axis crossing point } f(0)}: &\quad y = 0 \\
\underline{\text{zeros}}: &\quad (x, y) = (-2, 0) \\
&\quad (x, y) = (-1, 0) \\
&\quad (x, y) = (0, 0) \\
\underline{\text{extremes}}: &\quad (x, y) = \left(-1 - \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right) \quad \text{(max)} \\
&\quad (x, y) = \left(-1 + \frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right) \quad \text{(min)} \\
\underline{\text{inflections}}: &\quad (x, y) = (-1, 0)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.16 are summarized by $f(x)$ graph (see Fig. 3.8.

3.17. Given polynomial function

$$f(x) = x^3 + 2x^2 + x = x(x^2 + 2x + 1)$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because all of the polynomial terms have simple monomial form $a_n x^n$, thus defined for all real arguments.
2. *Function's parity:*

$$\begin{aligned}
\text{even:} &\quad f(-x) = (-x)^3 + 2(-x)^2 + (-x) = -x^3 + 2x^2 - x \neq f(x) \Rightarrow \quad \text{not even} \\
\text{odd:} &\quad -f(-x) = -(-x^3 + 2x^2 - x) = x^3 - 2x^2 + x \neq f(x) \Rightarrow \quad \text{not odd}
\end{aligned}$$

Tthis function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

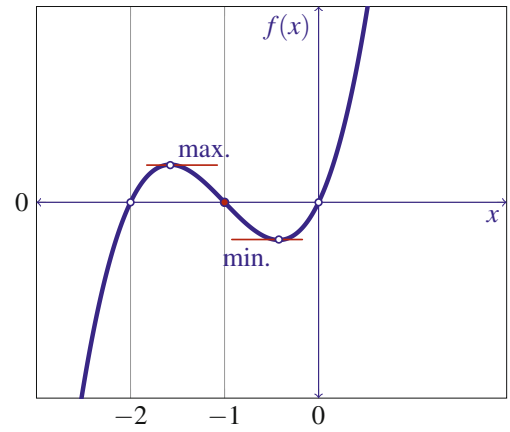


Fig. 3.8 Example P.3.16

3. *Function's sign:*

$$\begin{aligned} \underline{f(x)=0:} \quad \therefore\ x^3+2x^2+x &= \{\text{factorise}\} \\ &= x(x^2+2x+1) = \left\{(a+b)^2 = a^2+2ab+b^2\right\} \\ &= x(x+1)^2 \\ \therefore\ x_1 = 0, \quad x_2 &= -1, \quad x_3 = -1 \end{aligned}$$

This cubic function $f(x)$ has one first-order zero at $(0, 0)$ and one second-order zero at $(-1, 0)$; thus, its sign intervals are,

x		-1		0	
x	$-$	$-$	$-$	0	$+$
$(x+1)^2$	$+$	0	$+$	1	$+$
$f(x)$	$-$	0	$-$	0	$+$

In conclusion, this cubic function does not change its sign at the second (even) order zero and thus stays negative in $[-\infty, 0)$ interval and becomes positive after passing through the first (odd)-order zero into $(0, +\infty]$ interval.

4. *y-axis crossing point:*

$$f(0) = (0)(0+1)^2 = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus there are only two extreme limits to evaluate

$$\begin{aligned} \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} x^3+2x^2+x = \lim_{x\to-\infty} x^3\left(1+\cancel{\tfrac{2}{x}}^{\,0}+\cancel{\tfrac{1}{x^2}}^{\,0}\right) \\ &= \lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty \\ \lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} x^3+2x^2+x = \lim_{x\to+\infty} x^3\left(1+\cancel{\tfrac{2}{x}}^{\,0}+\cancel{\tfrac{1}{x^2}}^{\,0}\right) \\ &= \lim_{x\to+\infty} x^3 = (+\infty)^3 = +\infty \end{aligned}$$

In conclusion, $f(x)$ tends to $-\infty$ for negative numbers and to $+\infty$ for positive numbers, because $x^3 > 0$, if $x > 0$, and $x^3 < 0$, if $x < 0$.

6. *Oblique asymptote:* note that for $x \neq 0$, division x/x is always defined; thus

$$\begin{aligned} a &= \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3+2x^2+x}{x} = \lim_{x\to\infty} \left(x^2+2x+1\right) \\ &= \lim_{x\to\infty} x^2\left(1+\cancel{\tfrac{2}{x}}^{\,0}+\cancel{\tfrac{1}{x^2}}^{\,0}\right) = \lim_{x\to\infty} x^2 \\ &= \infty \end{aligned}$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = (x^3 + 2x^2 + x)' = 3x^2 + 4x + 1$$

$$\therefore\ \ 3x^2 + 4x + 1 = 0\ \{\text{factorise}\}$$

$$3x^2 + 3x + x + 1 = 0$$

$$3x(x+1) + (x+1) = 0$$

$$(x+1)(3x+1) = 0$$

$$\therefore\ \ x_1 = -1,\ \ x_2 = -\frac{1}{3}$$

so, $f(x_1) = 0$

and, $f(x_2) = (-1/3)^3 + 2(-1/3)^2 + (-1/3) = -\dfrac{4}{27}$

$$\underline{f''(x) = 0:}\ \ f''(x) = (x^3 + 2x^2 + x)'' = 6x + 4$$

$$\therefore\ \ 6x + 4 = 0\ \ \therefore\ \ 2(3x+2) = 0\ \ \therefore\ \ x_3 = -\frac{2}{3}$$

so, $f(x_3) = (-2/3)^3 + 2(-2/3)^2 + (-2/3) = -\dfrac{2}{27}$

That is to say, the two possible extreme points at $(-1, 0)$ and $(-1/3, -4/27)$ are determined as follows:

x		-1		$-1/3$	
$x+1$	$-$	0	$+$	$+$	$+$
$3x+1$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	0	$\searrow$	$-4/27$	$\nearrow$

That is to say, there is maximum at $(-1, 0)$ and minimum at $(-1/3, -4/27)$. Intervals of function's second derivative $f''(x) = 2(3x+2)$ are examined as:

x		$-2/3$	
2	$+$	$+$	$+$
$3x+2$	$-$	0	$+$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$	$-2/27$	$\cup$

In conclusion, this cubic function has nflection point at $(x, y) = (-2/3, -2/27)$, where the function changes from convex $\cap$ to concave $(\cup)$ form.

8. *Summary of the important points:*

$$\begin{aligned} \text{y-axis crossing point } f(0): &\quad y = 0 \\ \text{zeros:} &\quad (x, y) = (-1, 0) \\ &\quad (x, y) = (0, 0) \\ \text{extremes:} &\quad (x, y) = (-1, 0) \quad \text{(max)} \\ &\quad (x, y) = (-1/3, -4/27) \quad \text{(min)} \\ \text{inflections:} &\quad (x, y) = (-2/3, -2/27) \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.17 are summarized by $f(x)$ graph (see Fig. 3.9).

3.18. Given polynomial function

$$f(x) = x^3 + x^2 = x^2(x + 1)$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because all of the polynomial terms have simple monomial form $a_n x^n$, thus defined for all real arguments.
2. *Function's parity:*

$$\begin{aligned} \text{even:} &\quad f(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2 \neq f(x) \Rightarrow \quad \text{not even} \\ \text{odd:} &\quad -f(-x) = -(-x^3 + x^2) = x^3 - x^2 \neq f(x) \Rightarrow \quad \text{not odd} \end{aligned}$$

This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

3. *Function's sign:*

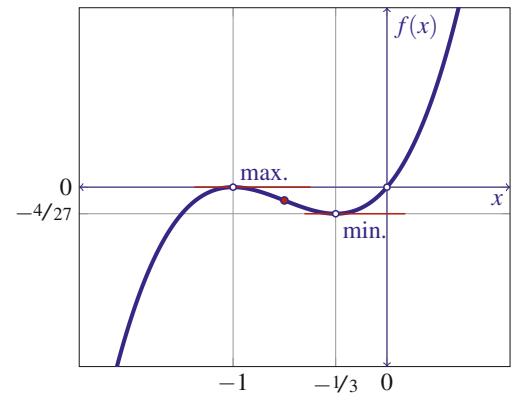


Fig. 3.9 Example P.3.17

$$\underline{f(x) = 0:} \quad \therefore\ x^3 + x^2 = \{\text{factorize}\}$$
$$= x^2(x+1)$$
$$\therefore\ x_1 = 0, \quad x_2 = 0, \quad x_3 = -1$$

This cubic function $f(x)$ has one first-order zero at $(-1, 0)$ and one second-order zero at $(0, 0)$; thus, its sign intervals are,

x		-1		0	
x^2	$+$	$+$	$+$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$+$

In conclusion, once it crosses its first-order zero at $x = -1$, this cubic function does not change its sign at the second-order zero and thus stays positive in $(-1, +\infty]$ interval.

4. *y-axis crossing point:*

$$f(0) = (0)^2(0+1) = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus there are only two extreme limits to evaluate

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 + x^2 = \lim_{x\to-\infty} x^3\left(1 + \cancelto{0}{1/x}\right)$$
$$= \lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 + x^2 = \lim_{x\to+\infty} x^3\left(1 + \cancelto{0}{1/x}\right)$$
$$= \lim_{x\to+\infty} x^3 = (+\infty)^3 = +\infty$$

In conclusion, $f(x)$ tends to $-\infty$ for negative numbers and to $+\infty$ for positive numbers, because $x^3 > 0$, if $x > 0$, and $x^3 < 0$, if $x < 0$.

6. *Oblique asymptote:* note that for $x \neq 0$, division x/x is always defined; thus

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3 + x^2}{x} = \lim_{x\to\infty} \left(x^2 + x\right)$$
$$= \lim_{x\to\infty} x^2\left(1 + \cancelto{0}{1/x}\right) = \lim_{x\to\infty} x^2$$
$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:}\quad f'(x) = (x^3 + x^2)' = 3x^2 + 2x$$

$$\therefore\quad x(3x + 2) = 0$$

$$\therefore\quad x_1 = 0,\quad x_2 = -\frac{2}{3}$$

$$\text{so, } f(0) = 0$$

$$\text{and, } f(-2/3) = (-2/3)^3 + (-2/3)^2 = \frac{4}{27}$$

$$\underline{f''(x) = 0:}\quad f''(x) = (x^3 + x^2)'' = 6x + 2$$

$$\therefore\quad 2(3x + 1) = 0\quad \therefore\quad x_3 = -\frac{1}{3}$$

$$\text{so, } f(x_3) = (-1/3)^3 + (-1/3)^2 = \frac{2}{27}$$

There are two first-order roots of $f'(x) = 0$, at $x_1 = 0$ and $x_2 = -2/3$, so

x		$-2/3$		0	
x	$-$	$-$	$-$	0	$+$
$3x + 2$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	$4/27$	$\searrow$	0	$\nearrow$

That is to say, there is one maximum at $(x, y) = (-2/3, 4/27)$ and one minimum point at $(0, 0)$. In addition, there is one first-order root of $f''(x) = 0$ at $x = -1/3$,

x		$-1/3$	
2	$+$	$+$	$+$
$3x + 1$	$-$	0	$+$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$	$2/27$	$\cup$

In conclusion, this cubic function has inflection point at $(x, y) = (-1/3, 2/27)$, where the function changes from convex $\cap$ to concave $\cup$ form.

8. *Summary of the important points:*

$$\underline{\text{zeros:}}\quad (x, y) = (-1, 0)$$

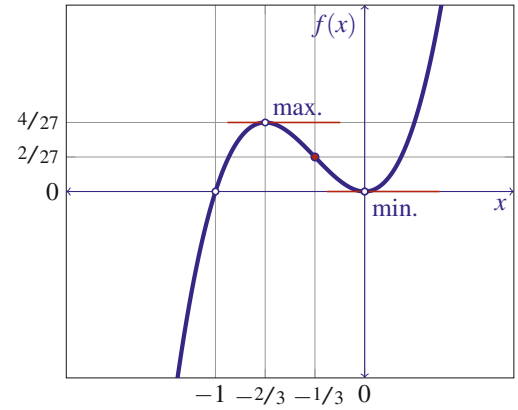
$$(x, y) = (0, 0)$$

$$\underline{\text{y-axis crossing point } f(0)\text{:}}\quad y = 0$$

$$\underline{\text{extremes:}}\quad (x, y) = (-2/3, 4/27)\quad (\text{max})$$

$$(x, y) = (0, 0)\quad (\text{min})$$

Fig. 3.10 Example P.3.18



$$\underline{\text{inflections:}} \quad (x, y) = (-1/3, 2/27)$$

9. *Graphical representation:* results of the analysis for function in P.3.18 are summarized by $f(x)$ graph (see Fig. 3.10).

3.19. Given polynomial function

$$f(x) = x^3 + 2x^2 - x - 2$$

1. *Domain of definition* $\mathbb{D}$: $x \in \{\mathbb{R}\}$, i.e., $(-\infty, +\infty)$, because all of the polynomial terms have simple monomial form $a_n x^n$, thus defined for all real arguments.
2. *Function's parity:*

$$\begin{aligned} \text{even:} \quad f(-x) &= (-x)^3 + 2(-x)^2 - (-x) - 2 = -x^3 + 2x^2 + x - 2 \neq f(x) \\ &\Rightarrow \quad \text{not even} \\ \text{odd:} \quad -f(-x) &= -(-x^3 + 2x^2 + x - 2) = x^3 - 2x^2 - x + 2 \neq f(x) \\ &\Rightarrow \quad \text{not odd} \end{aligned}$$

This function is not even because $f(x) \neq f(-x)$, and it is not odd because $f(x) \neq -f(-x)$.

3. *Function's sign:*

$$\begin{aligned} \underline{f(x) = 0:} \quad \therefore\ & x^3 + 2x^2 - x - 2 \quad \{\text{factorise}\} \\ &= x^2(x+2) - (x+2) \\ &= (x+2)(x^2 - 1) \quad \{(a^2 - b^2) = (a+b)(a-b)\} \\ &= (x+2)(x+1)(x-1) \\ \therefore\ & x_1 = -2, \quad x_2 = -1, \quad x_3 = 1 \end{aligned}$$

This cubic function $f(x) = (x+2)(x+1)(x-1)$ has three first-order zeros; thus, its sign intervals are

x		-2		-1		1	
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

In conclusion, every time a function crosses one of its first-order zeros, function changes its sign.

4. *y-axis crossing point:*

$$f(0) = (0)^3 + 2(0)^2 - (0) - 2 = -2$$

In conclusion, $(x, y) = (0, -2)$ is the vertical axis crossing point.

5. *Limits:* there are no vertical asymptotes, and thus there are only two extreme limits to be examined at $\pm\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 + 2x^2 - x - 2 = \lim_{x\to-\infty} x^3 \left(1 + \cancelto{0}{2/x} - \cancelto{0}{1/x^2} - \cancelto{0}{2/x^3}\right)$$

$$= \lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 + 2x^2 - x - 2 = \lim_{x\to+\infty} x^3 \left(1 + \cancelto{0}{2/x} - \cancelto{0}{1/x^2} - \cancelto{0}{2/x^3}\right)$$

$$= \lim_{x\to+\infty} x^3 = (+\infty)^3 = +\infty$$

In conclusion, $f(x) = x^3 + 2x^2 - x - 2$ tends to $-\infty$ for negative numbers and to $+\infty$ for positive numbers, because $x^3 > 0$, if $x > 0$, and $x^3 < 0$, if $x < 0$.

6. *Oblique asymptote:* note that for $x \neq 0$, division x/x is always defined; thus

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^3 + 2x^2 - x - 2}{x} = \lim_{x\to\infty} \frac{\cancel{x}\left(x^2 + 2x - 1 - 2/x\right)}{\cancel{x}}$$

$$= \lim_{x\to\infty} x^2 \left(1 + \cancelto{0}{2/x} - \cancelto{0}{1/x^2} - \cancelto{0}{2/x^3}\right) = \lim_{x\to\infty} x^2$$

$$= \infty$$

In conclusion, oblique asymptote does not exist.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (x^3 + 2x^2 - x - 2)'$$

$$\therefore \quad 3x^2 + 4x - 1 = 0 \quad \{\text{quadratic formula}\}$$

$$\therefore\ x_{1,2} = \frac{-4 \pm \sqrt{4^2 + 4 \cdot 3 \cdot 1}}{2 \cdot 3}$$

$$= \frac{-4 \pm \sqrt{4 \cdot 7}}{2 \cdot 3} = \frac{-4 \pm 2\sqrt{7}}{2 \cdot 3}$$

$$\therefore\ x_1 = \frac{-2 + \sqrt{7}}{3},\quad x_2 = \frac{-2 - \sqrt{7}}{3}$$

(note that $x_2 < x_1$)

so,

$$f(x_1) = (x_1 - 2)(x_1 - 1)(x_1 + 1)$$

$$= \left(\frac{-2 + \sqrt{7}}{3} - 2\right)\left(\frac{-2 + \sqrt{7}}{3} - 1\right)\left(\frac{-2 + \sqrt{7}}{3} + 1\right)$$

$$\{(a - b)(a + b) = a^2 - b^2\}$$

$$= \left(\frac{4 + \sqrt{7}}{3}\right)\left(\frac{(-2 + \sqrt{7})^2}{9} - 1\right)$$

$$= \frac{4 + \sqrt{7}}{3}\ \frac{4 - 4\sqrt{7} + 7 - 9}{9} = \frac{4 + \sqrt{7}}{3}\ \frac{2 - 4\sqrt{7}}{9}\quad \therefore$$

$$y_1 = -\frac{20 + 14\sqrt{7}}{27}\quad \text{and similarly,}$$

$$f(x_2) = y_2 = \frac{-20 + 14\sqrt{7}}{27}$$

$\underline{f''(x) = 0:}$

$$f''(x) = (x^3 + 2x^2 - x - 2)'' = 6x + 4$$

$$\therefore\ 2(3x + 2) = 0\quad \therefore\quad \frac{x}{3} = -\frac{2}{3}$$

$$\text{so, } f(x_3) = y_3 = (-2/3)^3 + 2(-2/3)^2 - (-2/3) - 2 = -\frac{20}{27}$$

That is to say, there are two first-order roots of $f'(x) = (x - x_1)(x - x_2)$. Keep in mind that $x_2 < x_1$; in other words $x_1 - x_2 > 0$ or $x_2 - x_1 < 0$, so

x		x_2		x_1	
$x - x_2$	$-$	0	$+$	$+$	$+$
$x - x_1$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	y_2	$\searrow$	y_1	$\nearrow$

That is to say, there is one maximum at $(x, y) = (x_2, y_2)$ and one minimum extreme point at $(x, y) = (x_1, y_1)$. In addition, there is one first-order root of $f''(x) = 2(3x + 2)$ at $x_3 = -2/3$, thus:

x		$-2/3$	
2	$+$	$+$	$+$
$3x + 2$	$-$	0	$+$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$	$-20/27$	$\cup$

In conclusion, there is one inflection point at $(x_3, y_3) = (-2/3, -20/27)$, where the function changes from convex $\cap$ to concave $\cup$ form.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{y-axis crossing point } f(0)}: &\quad y = -2 \\
\underline{\text{zeros:}} &\quad (x, y) = (-2, 0) \\
&\quad (x, y) = (-1, 0) \\
&\quad (x, y) = (1, 0) \\
\underline{\text{extremes:}} &\quad (x, y) = \left(\frac{-2 - \sqrt{7}}{3}, \frac{-20 + 14\sqrt{7}}{27}\right) \quad \text{(max)} \\
&\quad (x, y) = \left(\frac{-2 + \sqrt{7}}{3}, -\frac{20 + 14\sqrt{7}}{27}\right) \quad \text{(min)} \\
\underline{\text{inflections:}} &\quad (x, y) = (-2/3, -20/27)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.19 are summarized by $f(x)$ graph (see Fig. 3.11).

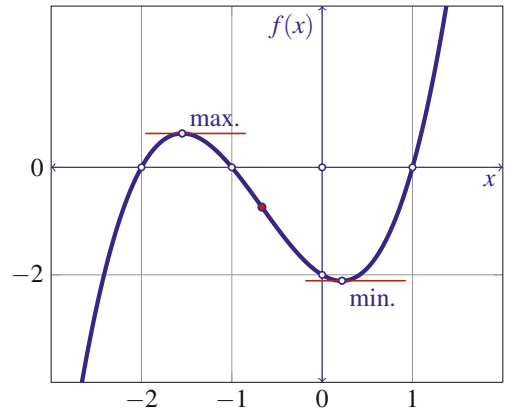


Fig. 3.11 Example P.3.19

3.2 Rational Functions

3.20. Given rational function

$$f(x) = \frac{1}{x} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 1 \quad \text{and} \quad Q(x) = x$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x \neq 0$$

   In conclusion, there is one *first-order pole* of $f(x)$ at $x = 0$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 0\}$. That is to say, there is one vertical asymptote at $x = 0$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{1}{-x} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \quad -f(-x) = -\left(\frac{1}{-x}\right) = \frac{1}{x} = f(x) \Rightarrow \quad \text{odd}$$

   This function is odd because $f(x) = -f(-x)$, i.e., symmetrical relative to the graph origin.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad P(x) = 1 \neq 0 \quad \text{therefore, there is } \textit{no zero}.$$

   Positive/negative intervals (keep in mind pole at $x = 0$) are examined as

x		0	
1	+	+	+
x	−	0	+
$f(x)$	−	n.d.	+

   In conclusion, $f(x)$ is an odd function, whose sign changes relative to its pole at $x = 0$ so that it is negative for $x < 0$ and positive for $x > 0$.
4. *y-axis crossing point:* This function does *not* cross the vertical axis because $\{\mathbb{R}|x \neq 0\}$, (vertical asymptote).
5. *Limits:* there is one vertical asymptote at $x = 0$, and thus it is necessary to resolve four limits in total,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0\rfloor$$

   i.e. as $(x \to -\infty)$, $y \to 0$ from the negative (lower) side

$$\lim_{x\to 0\atop\rightarrow} f(x) = \lim_{x\to 0\atop\rightarrow} \frac{1}{x} = \frac{1}{0\atop\rightarrow} = -\infty \quad \text{i.e. as } (x \to 0\atop\rightarrow),\ y \to -\infty$$

$$\lim_{x\to 0\atop\leftarrow} f(x) = \lim_{x\to 0\atop\leftarrow} \frac{1}{x} = \frac{1}{0\atop\leftarrow} = +\infty \quad \text{i.e. as } (x \to 0\atop\leftarrow),\ y \to +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x} = \frac{1}{+\infty} = 0\downarrow$$

i.e. as $(x \to +\infty)$, $y \to 0$ from the positive (upper) side

In conclusion, $f(x)$ is an odd function that has one *vertical* asymptote at $\lim_{x\to 0} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$. These limits, asymptotes, and negative/positive regions are illustrated in Fig. 3.12 (left). In the following graphs, the excluded values of x are indicated by "⊗" symbol.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1/x}{x} = \lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{\infty^2} = 0$$

In conclusion, oblique asymptote $y_{\mathtt{aa}} = ax + b$ does not exist because $a = 0$, which leaves only the constant "b" that would be a horizontal asymptote already found in the previous calculations above.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0 \quad \text{that is to say,} \quad f'(x) \neq 0$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(\frac{1}{x}\right)'' = \frac{2}{x^3} \neq 0$$

Thus, there are no extreme points (because $f'(x) \neq 0$), sign of first derivative is examined as

x		0	
-1	$-$	$-$	$-$
x^2	$+$	0	$+$
$f'(x)$	$-$	n.d.	$+$
$f(x)$	↘	n.d.	↘

That is to say, $f(x)$ always descends ↘. Second derivative $f''(x)$ is examined as

x		0	
2	$+$	$+$	$+$
x^3	$-$	0	$+$
$f''(x)$	$-$	n.d.	$+$
$f(x)$	∩	n.d.	∩

In conclusion, vertical asymptotes must be included in the analysis of sign, extremes, and inflection points because a function can change its form, as illustrated here, at $x = 0$, even though this function is not defined at that single point.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{defined:}} \quad & x \neq 0 \\ \underline{\text{parity:}} \quad & (\text{odd}) \\ \underline{\text{v. asymptote:}} \quad & x = 0 \\ \underline{\text{h. asymptote:}} \quad & y = 0 \end{aligned}$$

9. *Graphical representation:* the analysis of $f(x)$ in P.3.20 is summarized by its graph (see Fig. 3.12): sketch of various limits, asymptotes, and the negative/positive regions (left) and function's complete plot (right).

3.21. Given rational function

$$f(x) = \frac{1}{x-1} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 1 \quad \text{and} \quad Q(x) = x - 1$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x - 1 \neq 0 \quad \therefore \quad x_1 \neq 1$$

In conclusion, there is one *first-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R} | x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.

2. *Function's parity:*

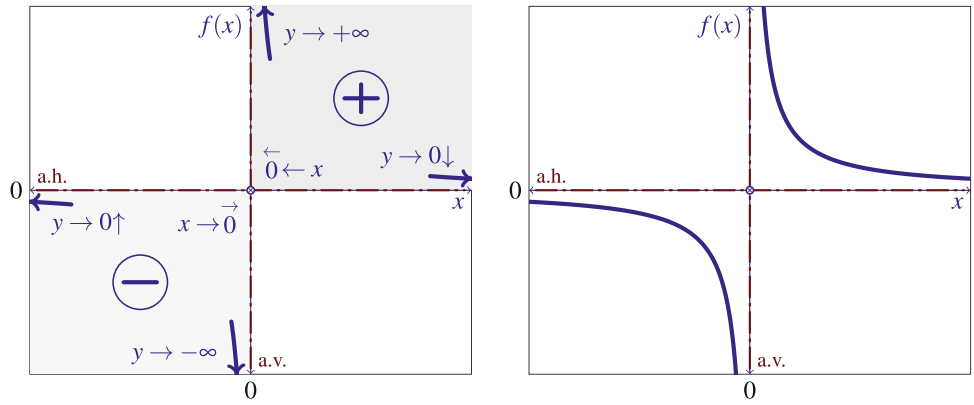


Fig. 3.12 Example P.3.20

$$\text{even:} \quad f(-x) = \frac{1}{(-x)-1} = -\frac{1}{x+1} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = \frac{1}{x+1} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$ and not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; P(x) = 0 \quad \text{however,} \quad P(x) = 1 \neq 0 \quad \text{therefore, there is } \textit{no zero}.$$

Positive/negative intervals (mind pole at $x = 1$) are examined as

x		1	
1	+	+	+
$x-1$	−	0	+
$f(x)$	−	n.d.	+

In conclusion, $f(x)$ changes relative to its pole at $x = 1$ so that it is negative for $x < 1$ and positive for $x > 1$.

4. *y-axis crossing point:*

$$f(0) = \frac{1}{(0)-1} = -1$$

In conclusion, $(x, y) = (0, -1)$ is the vertical axis crossing point.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x-1} = \frac{1}{-\infty} = 0\uparrow$$

i.e., $f(x)$ tends to zero from the lower (negative) side

$$\lim_{x\to 1\atop\rightarrow} f(x) = \lim_{x\to 1\atop\rightarrow} \frac{1}{x-1} = \frac{1}{0\atop\rightarrow} = -\infty \quad \text{i.e. } f(x) \text{ tends to} \quad -\infty$$

$$\lim_{x\to 1\atop\leftarrow} f(x) = \lim_{x\to 1\atop\leftarrow} \frac{1}{x-1} = \frac{1}{0\atop\leftarrow} = +\infty \quad \text{i.e. } f(x) \text{ tends to} \quad +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x-1} = \frac{1}{+\infty} = 0\downarrow$$

i.e., $f(x)$ tends to zero from the upper (positive) side

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to 1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{(x-1)\,x} = \lim_{x\to\infty} \frac{1}{\infty^2}$$
$$= 0$$

In conclusion, oblique asymptote $y_{\mathtt{aa}} = ax + b$ does not exist because $a = 0$, which leaves only the constant "b" that would be a horizontal asymptote already found in the previous calculations above.

7. *Critical points:*

$$\underline{f'(x) = 0:}\quad f'(x) = \left(\frac{1}{x-1}\right)' = -\frac{1}{(x-1)^2} < 0 \quad \text{that is to say,} \quad f'(x) \neq 0$$
$$\underline{f''(x) = 0:}\quad f''(x) = \left(\frac{1}{x-1}\right)'' = \frac{2}{(x-1)^3} \neq 0$$

There are no extreme points; first derivative $f'(x)$ is examined as

x		1	
-1	$-$	$-$	$-$
$(x-1)^2$	$+$	0	$+$
$f'(x)$	$-$	n.d.	$-$
$f(x)$	$\searrow$	n.d.	$\searrow$

That is to say, function always descends $\searrow$. Second derivative $f''(x)$ is examined as

x		1	
2	$+$	$+$	$+$
$(x-1)^3$	$-$	0	$+$
$f''(x)$	$-$	n.d.	$+$
$f(x)$	$\cap$	n.d.	$\cap$

In conclusion, vertical asymptotes must be included in the analysis of sign, extremes, and inflection points because a function can change its form, as illustrated here at $x = 1$, even though this function is not defined at that single point.

8. *Summary of the important points:*

$$\underline{\text{defined:}} \quad x \neq 1$$
$$\underline{\text{y-axis crossing point } f(0):} \quad y = -1$$
$$\underline{\text{v. asymptote:}} \quad x = 1$$
$$\underline{\text{h. asymptote:}} \quad y = 0$$

9. *Graphical representation:* results of the analysis for function in P.3.21 are summarized by $f(x)$ graph (see Fig. 3.13).

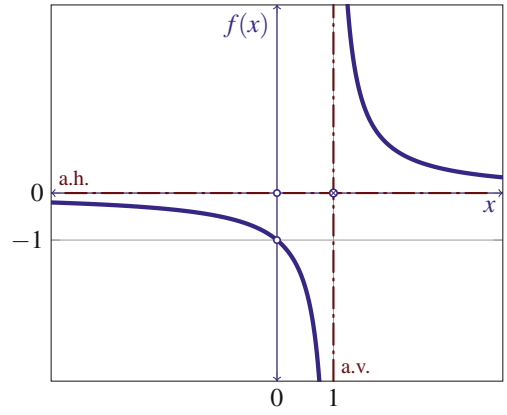


Fig. 3.13 Example P.3.21

3.22. Given rational function

$$f(x) = \frac{x}{x-1} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x \quad \text{and} \quad Q(x) = x - 1$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x - 1 \neq 0 \quad \therefore \quad x_1 \neq 1$$

   In conclusion, there is one *first-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)}{(-x)-1} = \frac{x}{x+1} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \quad -f(-x) = -\left(\frac{x}{x+1}\right) = -\frac{x}{x+1} \neq f(x) \Rightarrow \quad \text{not odd}$$

   In conclusion, this function is not odd because $f(x) \neq -f(-x)$ and not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x = 0 \quad \text{therefore, there is zero at} \quad x_1 = 0$$

   Positive/negative intervals (keep in mind pole at $x = 1$) are examined as

x		0		1	
x	−	0	+	1	+
$x-1$	−	−	−	0	+
$f(x)$	+	0	−	n.d.	+

In conclusion, $f(x)$ is a function whose sign changes relative to its zero at $(x_1 = 0)$ as well as its pole at $x = 1$, so the positive/negative intervals alternate.

4. *y-axis crossing point:*

$$f(0) = \frac{(0)}{(0) - 1} = 0$$

In conclusion, $(x, y) = (0, 0)$ is the vertical axis crossing point, which is where function's zero is located as well.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total. Multiple techniques could be used to resolve limits of the (∞/∞), etc. (i.e., not defined) form, for example, by L'Hôpital's rule.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x-1} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{x'}{(x-1)'} = \lim_{x\to-\infty} \frac{1}{\underset{\to}{1}} = 1\uparrow$$

i.e., $f(x)$ tends to $y = 1$ from the negative (lower) side

$$\lim_{x\to\underset{\to}{1}} f(x) = \lim_{x\to\underset{\to}{1}} \frac{x}{x-1} = \frac{1}{\underset{\to}{0}} = -\infty \quad \text{i.e. } f(x) \text{ tends to } \quad -\infty$$

$$\lim_{x\to\underset{\leftarrow}{1}} f(x) = \lim_{x\to\underset{\leftarrow}{1}} \frac{x}{x-1} = \frac{1}{\underset{\leftarrow}{0}} = +\infty \quad \text{i.e. } f(x) \text{ tends to } \quad +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{x-1} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{x'}{(x-1)'} = \lim_{x\to+\infty} \frac{1}{\underset{\leftarrow}{1}} = 1\downarrow$$

i.e., $f(x)$ tends to $y = 1$ from the positive (upper) side

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to1} f(x)$ and one *horizontal* asymptote at $y = 1$ because $\lim_{x\to\pm\infty} f(x) = 1$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\cancel{x}\,1}{(x-1)\cancel{x}} = \lim_{x\to\infty} \frac{1}{\infty} = 0$$

In conclusion, oblique asymptote $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{x}{x-1}\right)' = -\frac{1}{(x-1)^2} < 0 \quad \text{that is to say,} \quad f'(x) \neq 0$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{x}{x-1}\right)'' = \frac{2}{(x-1)^3} \neq 0$$

Thus, there are no extreme points, and first derivative $f'(x)$ is examined as

x		0		1	
-1	$-$	$-$	$-$	$-$	$-$
$(x-1)^2$	$+$	$+$	$+$	0	$+$
$f'(x)$	$-$	$-$	$-$	n.d.	$-$
$f(x)$	$\searrow$	$\searrow$	$\searrow$	n.d.	$\searrow$

Second derivative $f''(x)$ is examined (mind $x = 1$) as:

x		0		1	
2	$+$	$+$	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	n.d.	$+$
$f(x)$	$\cap$	$\cap$	$\cap$	n.d.	$\cup$

In conclusion, this function changes its concavity/convexity around its vertical asymptote.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{defined:}} \quad & x \neq 1 \\ \underline{\text{zero:}} \quad & (x, y) = (0, 0) \\ \underline{\text{y-axis crossing point } f(0)\text{:}} \quad & y = 0 \\ \underline{\text{v. asymptote:}} \quad & x = 1 \\ \underline{\text{h. asymptote:}} \quad & y = 1 \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.22 are summarized by $f(x)$ graph (see Fig. 3.14).

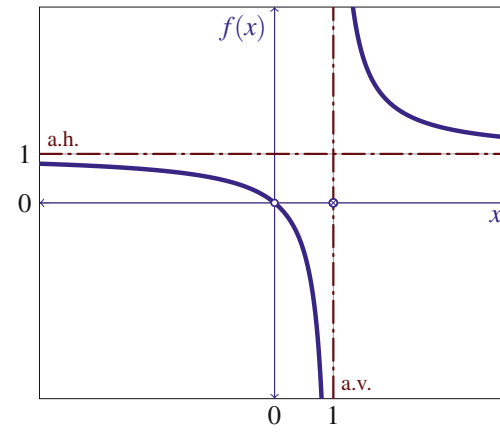


Fig. 3.14 Example P.3.22

3.23. Given rational function

$$f(x) = \frac{1}{x^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 1 \quad \text{and} \quad Q(x) = x^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x^2 \neq 0 \quad \therefore \quad x_1 \neq 0, \quad x_2 \neq 0$$

   In conclusion, there is one *second-order pole* of $f(x)$ at $x = 0$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 0\}$. That is to say, there is one vertical asymptote at $x = 0$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = f(x) \Rightarrow \quad \text{even}$$

$$\text{odd:} \quad -f(-x) = -\left(\frac{1}{x^2}\right) = -\frac{1}{x^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

   This function is not odd because $f(x) \neq -f(-x)$, and it is even because $f(x) = f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 1 \quad \text{however,} \quad 1 \neq 0 \quad \text{(there are no zeros as} \quad f(x) > 0)$$

   Positive/negative intervals are determined (keep in mind pole at $x = 0$) as

x		0	
1	+	+	+
x^2	+	0	+
$f(x)$	+	n.d.	+

   In conclusion, $f(x)$ does not change its sign before and after its even order pole.
4. *y-axis crossing point:* this function does not have the vertical axis crossing point; instead, the vertical axis is its vertical asymptote.
5. *Limits:* there is one vertical asymptote at $x = 0$, and thus there are four limits in total to resolve.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x^2} = \lim_{x \to -\infty} \frac{1}{(-\infty)^2} = 0\downarrow$$

i.e., as $(x \to -\infty)$, $y \to 0$ from the positive (upper) side

$$\lim_{x \to \underset{\twoheadrightarrow}{0}} f(x) = \lim_{x \to \underset{\twoheadrightarrow}{0}} \frac{1}{(\underset{\twoheadrightarrow}{0})^2} = +\infty \quad \text{i.e. as } (x \to \underset{\twoheadrightarrow}{0}),\ y \to +\infty$$

$$\lim_{x \to \underset{\twoheadleftarrow}{0}} f(x) = \lim_{x \to \underset{\twoheadleftarrow}{0}} \frac{1}{(\underset{\twoheadleftarrow}{0})^2} = +\infty \quad \text{i.e. as } (x \to \underset{\twoheadleftarrow}{0}),\ y \to +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{(+\infty)^2} = 0\rfloor$$

i.e., as $(x \to +\infty)$, $y \to 0$ from the positive (upper) side

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to 0} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,x^2} = \lim_{x\to\infty} \frac{1}{(\infty)^3}$$
$$= 0$$

In conclusion, oblique asymptote $y_{\texttt{aa}} = ax + b$ does not exist because $a = 0$, which leaves only the constant "b" that would be a horizontal asymptote already found in the previous calculations above.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(\frac{1}{x^2}\right)' = -\frac{2}{x^3} \neq 0$$
$$\underline{f''(x) = 0:}\ \ f''(x) = \left(\frac{1}{x^2}\right)'' = \frac{6}{x^4} > 0$$

Thus, there are no extreme points, and first derivative $f'(x)$ is examined as

x		0	
-2	$-$	$-$	$-$
x^3	$-$	0	$+$
$f'(x)$	$+$	n.d.	$-$
$f(x)$	$\nearrow$	nd.d	$\searrow$

That is to say, function ascends $\nearrow$ before its vertical asymptote and descends $\searrow$ after $x = 0$. Second derivative $f''(x)$ is examined (keep in mind pole at $x = 0$), as:

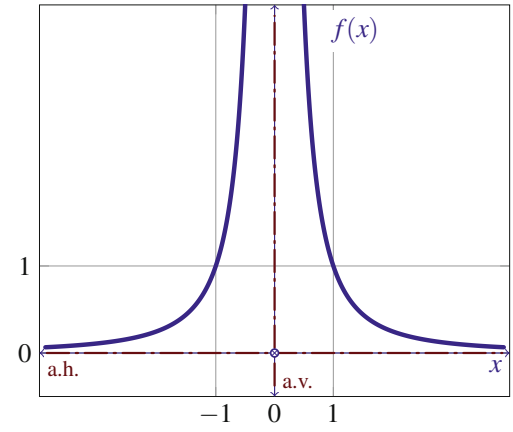
x		0	
6	$+$	$+$	$+$
x^4	$+$	0	$+$
$f''(x)$	$+$	n.d.	$+$
$f(x)$	$\cup$	n.d.	$\cup$

This function does not change it concavity/convexity before and after its even order pole.

8. *Summary of the important points:*

$$\underline{\text{defined:}} \quad x \neq 0$$
$$\underline{\text{parity:}} \quad \text{(even)}$$

Fig. 3.15 Example P.3.23



v. asymptote: $x = 0$

h. asymptote: $y = 0$

9. *Graphical representation:* results of the analysis for function in P.3.23 are summarized by $f(x)$ graph (see Fig. 3.15).

3.24. Given rational function

$$f(x) = \frac{1}{(x-1)^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 1 \quad \text{and} \quad Q(x) = (x-1)^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x-1)^2 \neq 0 \quad \therefore \quad x_1 \neq 1 \quad x_2 \neq 1$$

   In conclusion, there is one *second-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R} | x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{1}{((-x)-1)^2} = \frac{1}{(-1)^2(x+1)^2} \neq f(x) = \frac{1}{(x+1)^2} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\text{odd:} \quad -f(-x) = -\left(\frac{1}{(x+1)^2}\right) = -\frac{1}{(x+1)^2} \neq f(x)$$

$$\Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 1 \quad \text{however,} \quad 1 \neq 0 \quad \text{(there are no zeros as} \quad f(x) > 0)$$

Positive/negative intervals (keep in mind pole at $x = 1$) are examined as

x		1	
1	+	+	+
$(x-1)^2$	+	0	+
$f(x)$	+	n.d.	+

In conclusion, $f(x)$ is always positive.

4. *y-axis crossing point:*

$$f(0) = \frac{1}{((0)-1)^2} = 1$$

In conclusion, $(x, y) = (0, 1)$ is the vertical axis crossing point.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{(x-1)^2} = \lim_{x\to-\infty} \frac{1}{(-\infty)^2} = 0\downarrow$$

i.e., as $(x \to -\infty)$, $y \to 0$ from the positive (upper) side

$$\lim_{x\to\underset{\twoheadrightarrow}{1}} f(x) = \lim_{x\to\underset{\twoheadrightarrow}{1}} \frac{1}{(x-1)^2} = \frac{1}{(\underset{\twoheadrightarrow}{0})^2} = +\infty \quad \text{i.e., as } (x \to \underset{\twoheadrightarrow}{1}),\ y \to +\infty$$

$$\lim_{x\to\underset{\twoheadleftarrow}{1}} f(x) = \lim_{x\to\underset{\twoheadleftarrow}{1}} \frac{1}{(x-1)^2} = \frac{1}{(\underset{\twoheadleftarrow}{0})^2} = +\infty \quad \text{i.e. as } (x \to \underset{\twoheadleftarrow}{1}),\ y \to +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{(x-1)^2} = \frac{1}{(+\infty)^2} = 0\downarrow$$

i.e., as $(x \to +\infty)$, $y \to 0$ from the positive (upper) side

In conclusion, $f(x) = 1/(x-1)^2$ has one *vertical* asymptote at $\lim_{x\to1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,(x-1)^2} = \lim_{x\to\infty} \frac{1}{(\infty)^3} = 0$$

In conclusion, oblique asymptote $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{1}{(x-1)^2}\right)' = -\frac{2}{(x-1)^3} \neq 0$$

$$f''(x) = 0: \quad f''(x) = \left(\frac{1}{(x-1)^2}\right)'' = \frac{6}{(x-1)^4} > 0$$

Thus, there are no extreme points, while the second derivative $f''(x)$ is examined as (keep in mind pole at $x = 1$):

x		1	
6	+	+	+
$(x-1)^4$	+	0	+
$f''(x)$	+	n.d.	+
$f(x)$	∪	n.d.	∪

In conclusion, this function has one vertical and one horizontal asymptote, no zeros, and has no inflection points.

8. *Summary of the important points:*

$$\begin{aligned} \text{defined:} \quad & x \neq 1 \\ \text{y-axis crossing point } f(0): \quad & y = 1 \\ \text{v. asymptote:} \quad & x = 1 \\ \text{h. asymptote:} \quad & y = 0 \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.24 are summarized by $f(x)$ graph (see Fig. 3.16).

3.25. Given rational function

$$f(x) = \frac{x}{(x-1)^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x \quad \text{and} \quad Q(x) = (x-1)^2$$

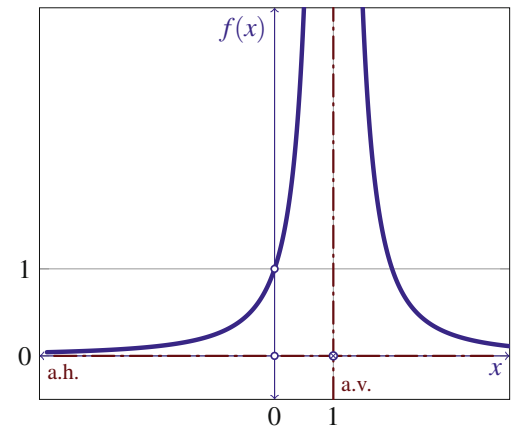


Fig. 3.16 Example P.3.24

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \;\; \therefore \;\; (x-1)^2 \neq 0 \;\; \therefore \;\; x_1 \neq 1 \;\; x_2 \neq 1$$

In conclusion, there is one *second-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.

2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)}{((-x)-1)^2} = -\frac{x}{(x+1)^2} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -\left(-\frac{x}{(x+1)^2}\right) = \frac{1}{(x+1)^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; P(x) = x \;\; \therefore \;\; x = 0 \Rightarrow x_1 = 0$$

Positive/negative intervals (keep in mind zero at $x_1 = 0$) of function are determined as

x		0		1	
x	$-$	0	$+$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$	0	$+$
$f(x)$	$-$	0	$+$	n.d.	$+$

In conclusion, $f(x)$ is negative for $x < 0$ and changes its sign after crossing first-order zero at $x = 0$, and it does not change sign around second-order pole.

4. *y-axis crossing point:*

$$f(0) = \frac{0}{(0-1)^2} = 0$$

Therefore, $f(x)$ crosses the vertical axis at 0, 0, which is also the function's zero.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus it is necessary to resolve four limits in total. Multiple techniques can be used to resolve limits of the (∞/∞), etc. (i.e., not determined) form, for example, by L'Hôpital's rule,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{x'}{\left((x-1)^2\right)'} = \lim_{x\to-\infty} \frac{1}{2(x-1)} = \frac{1}{2(-\infty)} = 0\rceil$$

i.e., $f(x)$ tends to $y = 0$ from the negative (lower) side

$$\lim_{x\to 1_{\to}} f(x) = \lim_{x\to 1_{\to}} \frac{x}{(x-1)^2} = \left\{(x-1)^2 > 0 \quad \text{for} \quad x \neq 1\right\} = \frac{1}{(0)^2_{\to}} = +\infty$$

$$\lim_{x\to 1\atop\leftarrow} f(x) = \lim_{x\to 1\atop\leftarrow} \frac{x}{(x-1)^2} = \frac{1}{(0)^2} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{x'}{\left((x-1)^2\right)'} = \lim_{x\to+\infty} \frac{1}{2(x-1)} = 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to 1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,(x-1)^2} = \lim_{x\to\infty} \frac{1}{(\infty)^3} = 0$$

In conclusion, oblique asymptote $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:}\quad f'(x) = \left(\frac{x}{(x-1)^2}\right)' = \left\{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}\right\}$$

$$= \frac{(x-1)^{\cancel{2}} - 2x\cancel{(x-1)}}{(x-1)^{\cancel{4}3}} = -\frac{x+1}{(x-1)^3}$$

$$\therefore\ x+1=0\ \ \therefore\ x_1 = -1 \quad \text{and} \quad y = f(x_1) = \frac{-1}{((-1)-1)^2} = -\frac{1}{4}$$

$$\underline{f''(x) = 0:}\quad f''(x) = \left(-\frac{x+1}{(x-1)^3}\right)' = -\frac{(x-1)^{\cancel{2}} - 3(x+1)\cancel{(x+1)^2}}{(x-1)^{\cancel{6}4}}$$

$$= 2\frac{x+2}{(x-1)^4}$$

$$\therefore\ x+2=0\ \ \therefore\ x_2 = -2 \quad \text{and} \quad y = f(x_2) = \frac{-2}{\left((-2)-1\right)^2} = -\frac{2}{9}$$

Thus, the extreme point $f'(x) = 0$ is examined as,

x		-1		1	
-1	$-$	$-$	$-$	$-$	$-$
$x+1$	$-$	0	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	n.d.	$-$
$f(x)$	$\searrow$	$-1/4$	$\nearrow$	n.d.	$\searrow$

There is minimum at $(x, y) = (-1, -1/4)$. Second derivative $f''(x) = 0$ examined is

x		-2		1	
2	+	+	+	+	+
$x+2$	−	0	+	+	+
$(x-1)^4$	+	+	+	0	+
$f''(x)$	−	0	+	n.d.	+
$f(x)$	∩	$-2/9$	∪	n.d.	∪

In conclusion, this function has one inflection point at $(x, y) = (-2, -2/9)$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} &\quad x \neq 1\\
\underline{\text{y-axis crossing point } f(0)\text{:}} &\quad y = 0\\
\underline{\text{v. asymptote:}} &\quad x = 1\\
\underline{\text{h. asymptote:}} &\quad y = 0\\
\underline{\text{zero:}} &\quad (x, y) = (0, 0)\\
\underline{\text{extremes:}} &\quad (x, y) = (-1, -1/4) \ \text{(min)}\\
\underline{\text{inflections:}} &\quad (x, y) = (-2, -2/9)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.25 are summarized by $f(x)$ graph (see Fig. 3.17).

3.26. Given rational function

$$f(x) = \frac{1}{x^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 1 \quad \text{and} \quad Q(x) = x^3$$

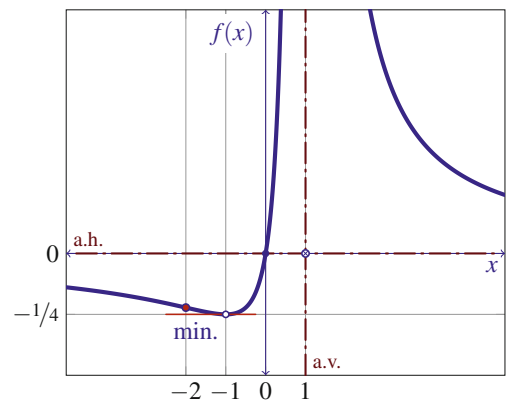


Fig. 3.17 Example P.3.25

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x^3 \neq 0 \quad \therefore \quad x_{1,2,3} \neq 0$$

   In conclusion, there is one *third-order pole* of $f(x)$ at $x = 0$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 0\}$. That is to say, there is one vertical asymptote at $x = 0$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{1}{(-x)^3} = -\frac{1}{x^3} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -\left(-\frac{1}{x^3}\right) = \frac{1}{x^3} = f(x) \Rightarrow \quad \text{odd}$$

   In conclusion, this function is odd because $f(x) = -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \text{however,} \quad 1 \neq 0 \quad \therefore \quad \text{there are no zeros}$$

   Positive/negative intervals (keep in mind pole at $x = 0$) are

x		0	
1	+	+	+
x^3	−	0	+
$f(x)$	−	n.d.	+

   In conclusion, $f(x)$ is negative for $x < 0$ and then changes its sign on the other side of the third (odd)-order pole for $x > 0$.
4. *y-axis crossing point:* this function is not defined at $x = 0$, and thus, there is no y-axis crossing point.
5. *Limits:* there is one vertical asymptote at $x = 0$, and thus it is necessary to resolve four limits in total, as

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x^3} = \frac{1}{(-\infty)^3} = 0\downarrow$$

   i.e., $f(x)$ tends to $y = 0$ from the negative (lower) side

$$\lim_{x \to \underset{\rightarrow}{0}} f(x) = \lim_{x \to \underset{\rightarrow}{0}} \frac{1}{(0)^3} = -\infty$$

$$\lim_{x \to \underset{\leftarrow}{0}} f(x) = \lim_{x \to \underset{\leftarrow}{0}} \frac{1}{x^3} = \frac{1}{(0)^3} = +\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x^3} = \frac{1}{(+\infty)^3} = 0\downarrow$$

   i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x)$ is a function that has one *vertical* asymptote at $\lim_{x\to 0} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x^4} = 0$$

In conclusion, oblique asymptote $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(\frac{1}{x^3}\right)' = -\frac{3}{x^4} < 0$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(\frac{1}{x^3}\right)'' = \frac{12}{x^5} \neq 0$$

Thus, the extreme point $f'(x) = 0$ is examined as,

x		0	
-3	$-$	$-$	$-$
x^4	$+$	0	$+$
$f'(x)$	$-$	n.d.	$-$
$f(x)$	$\searrow$	n.d.	$\searrow$

while the second derivative $f''(x) = 0$ is

x		0	
12	$+$	$+$	$+$
x^5	$-$	0	$+$
$f''(x)$	$-$	n.d.	$+$
$f(x)$	$\cap$	n.d.	$\cup$

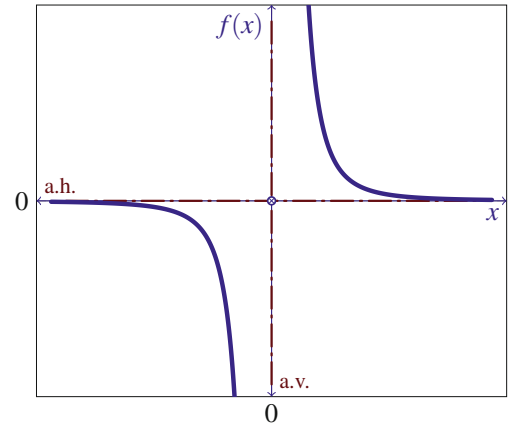
In conclusion, this function changes its convexity into concavity around the vertical asymptote.

8. *Summary of the important points:*

defined:	$x \neq 0$
parity:	(odd)
v. asymptote:	$x = 0$
h. asymptote:	$y = 0$

9. *Graphical representation:* results of the analysis for function in P.3.26 are summarized by $f(x)$ graph (see Fig. 3.18).

Fig. 3.18 Example P.3.26



3.27. Given rational function

$$f(x) = \frac{1}{(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 1 \quad \text{and} \quad Q(x) = (x+1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x+1)^3 \neq 0 \quad \therefore \quad x_{1,2,3} \neq -1$$

   In conclusion, there is one *third-order pole* of $f(x)$ at $x = -1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq -1\}$. That is to say, there is one vertical asymptote at $x = -1$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{1}{((-x)+1)^3} = \frac{1}{(-x+1)^3} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -\left(\frac{1}{(-x+1)^3}\right) = -\frac{1}{(-x+1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

   This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \text{however,} \quad 1 \neq 0 \quad \therefore \quad \text{there are no zeros}$$

   In this case, the positive/negative intervals of this function are determined by its pole at $x = -1$, as:

x		-1	
1	1	1	1
$(x+1)^3$	$-$	0	$+$
$f(x)$	$-$	∞	$+$

In conclusion, $f(x)$ is negative for $x < -1$ and then changes its sign on the other side of its third-order pole for $x > 0$.

4. *y-axis crossing point:* this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{1}{((0)+1)^3} = 1$$

5. *Limits:* there is one vertical asymptote at $x = -1$, and there are four limits in total to resolve, as

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{(x+1)^3} = \frac{1}{(-\infty)^3} = 0\uparrow$$

i.e., $f(x)$ tends to $y = 0$ from the negative (lower) side

$$\lim_{\substack{x\to-1\\ \to}} f(x) = \lim_{\substack{x\to-1\\ \to}} \frac{1}{(x+1)^3} = \frac{1}{(0)^3} = -\infty$$

$$\lim_{\substack{x\to-1\\ \leftarrow}} f(x) = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{1}{(x+1)^3} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{1}{(0)^3} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x^3} = \frac{1}{(+\infty)^3} = 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to-1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,(x+1)^3} = \lim_{x\to\infty} \frac{1}{(\infty)^4}$$
$$= 0$$

In conclusion, oblique asymptote $y_{\mathtt{aa}} = ax + b$ does not exist because $a = 0$, which leaves only the constant "b" that would be a horizontal asymptote already found in the previous calculations above.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(\frac{1}{(x+1)^3}\right)' = -\frac{3}{(x+1)^4} < 0$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(\frac{1}{(x+1)^3}\right)'' = \frac{12}{(x+1)^5} \neq 0$$

Thus, there are no extreme points because $f'(x) \neq 0$ and $f'(x) < 0$ for all x; in other words, this function is always descending ($\searrow$) as

x		-1	
-3	$-$	$-$	$-$
$(x+1)^4$	$+$	0	$+$
$f'(x)$	$-$	n.d	$-$
$f(x)$	$\searrow$	n.d.	$\searrow$

At the same time, the second derivative $f''(x) \neq 0$; however, there is a possibility of changing its sign around its vertical asymptote, as

x		-1	
12	$+$	$+$	$+$
$(x+1)^5$	$-$	0	$+$
$f''(x)$	$-$	n.d.	$+$
$f(x)$	$\cap$	n.d.	$\cup$

That is to say, $f(x)$ changes its convexity into concavity around its vertical asymptote.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{defined:}} \quad & x \neq -1 \\ \underline{\text{y-axis crossing point } f(0):} \quad & y = 1 \\ \underline{\text{v. asymptote:}} \quad & x = -1 \\ \underline{\text{h. asymptote:}} \quad & y = 0 \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.27 are summarized by $f(x)$ graph (see Fig. 3.19).

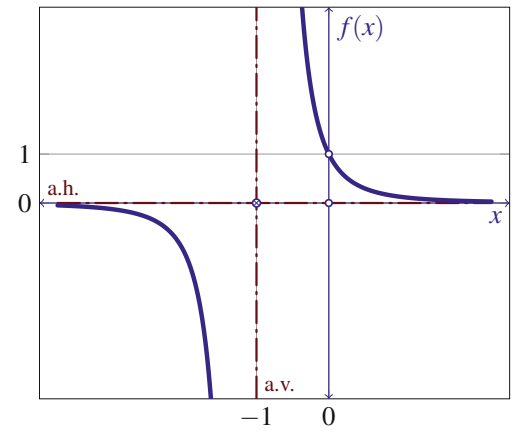


Fig. 3.19 Example P.3.27

3.28. Given rational function

$$f(x) = \frac{x}{(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x \quad \text{and} \quad Q(x) = (x+1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x+1)^3 \neq 0 \quad \therefore \quad x_{1,2,3} \neq -1$$

   In conclusion, there is one *third-order pole* of $f(x)$ at $x = -1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq -1\}$. That is to say, there is one vertical asymptote at $x = -1$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)}{((-x)+1)^3} = \frac{-x}{(-1)^3(x-1)^3} = \frac{x}{(x-1)^3} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -\left(\frac{x}{(x-1)^3}\right) = -\frac{x}{(x-1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

   Tthis function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x = 0 \quad \therefore \qquad \text{there is first-order zero}$$

   Positive/negative intervals are determined (mind zero at $x = 0$) as

x		-1		0	
x	$-$	$-$	$-$	0	$+$
$(x+1)^3$	$-$	0	$+$	$+$	$+$
$f(x)$	$+$	n.d.	$-$	0	$+$

   In conclusion, $f(x)$ alternates its sign every time it crosses either its zero or its pole.
4. *y-axis crossing point:* this function crosses the y-axis at $(0, 1)$ as,

$$f(0) = \frac{0}{((0)+1)^3} = 0$$

   In conclusion, this function crosses the vertical axis at $(0, 0)$.
5. *Limits:* there is one vertical asymptote at $x = 0$, and thus, it is necessary to resolve four limits in total. If the limit's form is ∞/∞, etc. (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest order polynomial term while keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as illustrated here

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{(x+1)^3}$$

$$\left\{\text{Pascal's triangle: } (a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3\right\}$$

$$= \lim_{x\to-\infty} \frac{x}{x^3+3x^2+3x+1}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^3}}{\cancel{x^3}} \frac{\cancel{x}\,1/\cancel{x^3}^{2}}{1+\cancel{3/x}^{0}+\cancel{3/x^2}^{0}+\cancel{1/x^3}^{0}} \qquad \text{note:} \quad (x \neq 0)$$

$$= \lim_{x\to-\infty} \frac{1}{x^2} = \frac{1}{(-\infty)^2} = 0\rfloor$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

$$\lim_{\substack{x\to-1\\ \rightarrow}} f(x) = \lim_{\substack{x\to-1\\ \rightarrow}} \frac{x}{(x+1)^3} = \lim_{\substack{x\to-1\\ \rightarrow}} \frac{-1}{(0)^3} = +\infty$$

$$\lim_{\substack{x\to-1\\ \leftarrow}} f(x) = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{x}{(x+1)^3} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{-1}{(0)} = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{(x+1)^3} = \lim_{x\to+\infty} \frac{x}{x^3+3x^2+3x+1}$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^3}}{\cancel{x^3}} \frac{\cancel{x}\,1/\cancel{x^3}^{2}}{1+\cancel{3/x}^{0}+\cancel{3/x^2}^{0}+\cancel{1/x^3}^{0}} \quad \left\{\text{here: } (x \neq 0)\right\}$$

$$= \lim_{x\to+\infty} \frac{1}{x^2} = \frac{1}{(+\infty)^2} = 0\rfloor$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to-1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\,(x+1)^3} = \lim_{x\to\infty} \frac{1}{(\infty)^3}$$

$$= 0$$

In conclusion, oblique asymptote $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{x}{(x+1)^3}\right)' = \frac{(x+1)^{\cancel{3}} - 3x\cancel{(x+1)^2}}{(x+1)^{\cancel{6}4}} = \frac{-2x+1}{(x+1)^4}$$

$$\therefore \quad -2x+1 = 0 \quad \therefore \quad x_1 = \frac{1}{2} \quad \text{and,}$$

$$f(x_1) = \frac{1/2}{(1/2+1)^3} = \frac{1/\cancel{2}}{27/\cancel{8}4} = \frac{4}{27}$$

$$f''(x)=0:\ f''(x)=\left(\frac{-2x+1}{(x+1)^4}\right)'=\frac{-2(x+1)^4-4(-2x+1)(x+1)^3}{(x+1)^8}$$

$$=\frac{6(x-1)}{(x+1)^5}\ \ \therefore\ \ x-1=0\ \ \therefore\ \ x_2=1\quad\text{and}\quad f(x_2)=\frac{1}{(1+1)^3}=\frac{1}{8}$$

There is extreme point at $(1/2, 4/27)$ and possible inflection point at $(1, 1/8)$, determined as

x		-1		$1/2$	
$-2x+1$	$+$	$+$	$+$	0	$-$
$(x+1)^4$	$+$	0	$+$	$+$	$+$
$f'(x)$	$+$	n.d.	$+$	0	$-$
$f(x)$	$\nearrow$	n.d.	$\nearrow$	$4/27$	$\searrow$

That is to say, there is maximum point at $(x, y) = (1/2, 4/27)$. At the same time, the second derivative $f''(x) = 0$ at $(1, 1/8)$, therefore

x		-1		1	
6	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
$(x+1)^5$	$-$	0	$+$	$+$	$+$
$f''(x)$	$+$	n.d.	$-$	0	$+$
$f(x)$	$\cup$	n.d.	$\cap$	$1/8$	$\cup$

In conclusion, this function changes its convexity/concavity around the vertical asymptote $x = -1$ as well as at $(x, y) = (1, 1/8)$ inflection point.

8. *Summary of the important points:*

$$\begin{aligned}
\text{defined:} &\quad x\neq -1\\
\text{y-axis crossing point } f(0)\text{:} &\quad y=0\\
\text{v. asymptote:} &\quad x=-1\\
\text{h. asymptote:} &\quad y=0\\
\text{zero:} &\quad (x,y)=(0,0)\\
\text{extremes:} &\quad (x,y)=(1/2,4/27)\ \ (\text{max})\\
\text{inflections:} &\quad (x,y)=(1,1/8)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.28 are summarized by $f(x)$ graph (see Fig. 3.20).

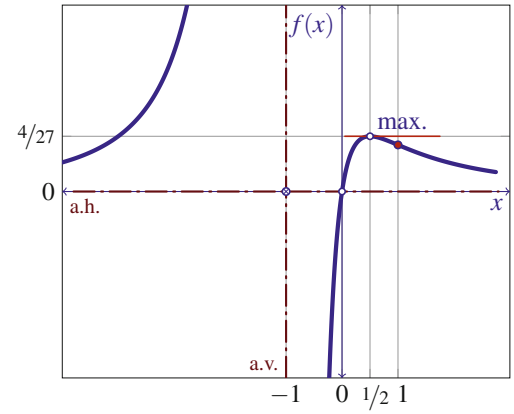


Fig. 3.20 Example P.3.28

3.29. Given rational function

$$f(x) = \frac{x+1}{x-1} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x+1 \quad \text{and} \quad Q(x) = x-1$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x-1 \neq 0 \quad \therefore \quad x \neq 1$$

   In conclusion, there is one *first-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R} | x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)+1}{(-x)-1} = -\frac{-x+1}{x+1} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \quad -f(-x) = -\left(-\frac{-x+1}{x+1}\right) = \frac{-x+1}{x+1} \neq f(x) \Rightarrow \quad \text{not odd}$$

   Tthis function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x+1 = 0 \quad \therefore \quad x_1 = -1 \quad \text{there is first-order zero}$$

   Positivity/negativity intervals are determined (mind pole at $x = 1$) as

x		-1		1	
$x+1$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	n.d.	$+$

   In conclusion, $f(x)$ alternates its sign every time it crosses either its first-order (odd) zero or its first-order (odd) pole.

4. *y-axis crossing point:* this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{(0)+1}{(0)-1} = -1$$

In conclusion, this function crosses the vertical axis at $(0, -1)$.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total. If the limit's form is ∞/∞, etc. (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest order polynomial term while keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x+1}{x-1} = \left(\frac{\infty}{\infty}\right) \{\text{factorize } x\}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x}}{\cancel{x}} \frac{1+\cancel{1/x}^{\,0}}{1-\cancel{1/x}^{\,0}} = \lim_{x\to-\infty} \frac{1}{1} \left\{ \text{if } x \to -\infty \Rightarrow \frac{1+1/(-x)}{1-1/(-x)} = \frac{1-1/x}{1+1/x} < 1 \right\}$$

$$= 1\uparrow$$

i.e., $f(x)$ tends to $y = 1$ from the negative (bottom) side

$$\lim_{x\to 1_{\to}} f(x) = \lim_{x\to 1_{\to}} \frac{x+1}{x-1} = \lim_{x\to 1_{\to}} \frac{2}{x-1} = \frac{2}{0_{\to}} = -\infty$$

$$\lim_{x\to 1_{\leftarrow}} f(x) = \lim_{x\to 1_{\leftarrow}} \frac{x+1}{x-1} = \lim_{x\to 1_{\leftarrow}} \frac{2}{x-1} = \frac{2}{0_{\leftarrow}} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x+1}{x-1} = \left(\frac{\infty}{\infty}\right) \{\text{factorize } x\}$$

$$= \lim_{x\to+\infty} \frac{\cancel{x}}{\cancel{x}} \frac{1+\cancel{1/x}^{\,0}}{1-\cancel{1/x}^{\,0}} = \lim_{x\to+\infty} \frac{1}{1}$$

$$\left\{ \text{note: if } x \to +\infty \Rightarrow \frac{1+1/(+x)}{1-1/(+x)} = \frac{1+1/x}{1-1/x} > 1 \right\}$$

$$= 1\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

Reminder: Recall these useful relationships:

$$\text{if,} \quad a > 0, \quad \text{then,} \quad 1+a > 1-a, \quad \text{in other words:} \quad \frac{1+a}{1-a} > 1$$

$$\text{or, equivalently:} \quad \frac{1-a}{1+a} < 1$$

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to 1} f(x)$ and one *horizontal* asymptote at $y = 1$ because $\lim_{x\to\pm\infty} f(x) = 1$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x+1}{x(x-1)} = \lim_{x\to\infty} \frac{x\left(1+1/x\right)}{x\,(x-1)} = \lim_{x\to\infty} \frac{1}{x-1}$$
$$= 0$$

In conclusion, oblique asymptote $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:}\ \ f'(x) = \left(\frac{x+1}{x-1}\right)' = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2} < 0$$

$$\underline{f''(x) = 0:}\ \ f''(x) = \left(\frac{-2}{(x-1)^2}\right)' = \frac{4(x-1)}{(x-1)^4} = \frac{4}{(x-1)^3} \neq 0$$

Thus, there are no extreme points and no inflection points; however, there is vertical asymptote at $x = 1$, so

x		-1		1	
-2	$-$	$-$	$-$	$-$	$-$
$(x-1)^2$	$+$	0	$+$	0	$+$
$f'(x)$	$-$	$-$	$-$	n.d.	$-$
$f(x)$	$\searrow$	0	$\searrow$	n.d.	$\searrow$

That is to say, $f(x)$ always descends $\searrow$. Also, the second derivative $f''(x) \neq 0$, so that

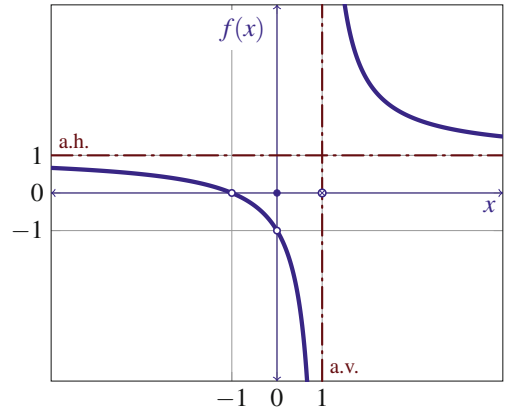
x		1	
4	$+$	$+$	$+$
$(x-1)^3$	$-$	0	$+$
$f''(x)$	$-$	n.d.	$+$
$f(x)$	$\cap$	n.d.	$\cup$

In conclusion, this function has one vertical at $x = 1$ and one horizontal asymptote at $y = 1$, and it changes its convexity/concavity around the vertical asymptote $x = 1$.

8. *Summary of the important points:*

defined:	$x \neq 1$
y-axis crossing point $f(0)$:	$y = -1$
v. asymptote:	$x = 1$
h. asymptote:	$y = 1$
zero:	$(x, y) = (-1, 0)$

Fig. 3.21 Example P.3.29



9. *Graphical representation:* results of the analysis for function in P.3.29 are summarized by $f(x)$ graph (see Fig. 3.21).

3.30. Given rational function

$$f(x) = \frac{x+1}{(x-1)^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x+1 \quad \text{and} \quad Q(x) = (x-1)^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x-1)^2 \neq 0 \quad \therefore \quad x_{1,2} \neq 1$$

   In conclusion, there is one *second-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{(-x)+1}{\big((-x)-1\big)^2} = \frac{(-x)+1}{(-1)^2\,(x+1)^2} = -\frac{x-1}{(x+1)^2} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{x-1}{(x+1)^2}\right) = \frac{x-1}{(x+1)^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

   In conclusion, this function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x+1 = 0 \quad \therefore \quad x_1 = -1 \quad \text{there is first-order zero}$$

Positivity/negativity intervals of this function are determined (keep in mind zero at $x = -1$) as:

x		-1		1	
$x+1$	$-$	0	$+$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$	0	$+$
$f(x)$	$-$	0	$+$	n.d.	$+$

In conclusion, $f(x)$ changes its sign after crossing its first-order zero, but not after passing its second-order pole.

4. *y-axis crossing point:* this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{(0)+1}{\big((0)-1\big)^2} = 1$$

In conclusion, this function crosses the vertical axis at (0, 1).

5. *Limits:* there is one vertical asymptote at $x = 1$; thus, it is necessary to resolve four limits in total. If the limit's form is ∞/∞, etc. (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest-order polynomial term while keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as illustrated here

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x+1}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital's rule}\} \\
&= \lim_{x\to-\infty} \frac{(x+1)'}{\big((x-1)^2\big)'} = \lim_{x\to-\infty} \frac{1}{2(x-1)} = \lim_{x\to-\infty} \frac{1}{-\infty} \\
&= 0\uparrow \\
&\text{i.e., } f(x) \text{ tends to } y = 0 \text{ from the negative (bottom) side}
\end{aligned}$$

$$\lim_{x\to\underset{\to}{1}} f(x) = \lim_{x\to\underset{\to}{1}} \frac{x+1}{(x-1)^2} = \lim_{x\to\underset{\to}{1}} \frac{2}{(x-1)^2} = \frac{2}{(\underset{\to}{0})^2} = +\infty$$

$$\lim_{x\to\underset{\leftarrow}{1}} f(x) = \lim_{x\to\underset{\leftarrow}{1}} \frac{x+1}{(x-1)^2} = \lim_{x\to\underset{\leftarrow}{1}} \frac{2}{(x-1)^2} = \frac{2}{(\underset{\leftarrow}{0})^2} = +\infty$$

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x+1}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital's rule}\} \\
&= \lim_{x\to+\infty} \frac{(x+1)'}{\big((x-1)^2\big)'} = \lim_{x\to+\infty} \frac{1}{2(x-1)} = \lim_{x\to+\infty} \frac{1}{+\infty} \\
&= 0\downarrow \\
&\text{i.e., } f(x) \text{ tends to } y = 0 \text{ from the positive (upper) side}
\end{aligned}$$

In conclusion, $f(x)$ has one *vertical* asymptote at $\lim_{x\to1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x+1}{x(x-1)^2} = \lim_{x\to\infty} \frac{x\left(1+{}^{1}\!/\!_{x}\right)}{x(x-1)^2} = \frac{1}{\infty}$$
$$= 0$$

In conclusion, oblique asymptote does not exist because $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{x+1}{(x-1)^2}\right)' = \frac{(x-1)^2 - 2(x+1)(x-1)}{(x-1)^4} = -\frac{x+3}{(x-1)^3}$$
$$\therefore\ -(x+3) = 0 \ \therefore\ x_1 = -3$$
$$\text{and, } f(x_1) = f(-3) = \frac{(-3)+1}{((-3)-1)^2} = y_1 = -\frac{1}{8}$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-\frac{x+3}{(x-1)^3}\right)' = -\frac{(x-1)^3 - 3(x+3)(x-1)^2}{(x-1)^6}$$
$$= -\frac{-2x-10}{(x-1)^4} = \frac{2(x+5)}{(x-1)^4}$$
$$\therefore\ 2(x+5) = 0 \ \therefore\ x+5 = 0 \ \therefore\ x_2 = -5$$
$$\text{and, } f(x_2) = f(-5) = \frac{(-5)+1}{((-5)-1)^2} = y_2 = -\frac{1}{9}$$

Extreme point is determined (mind pole $x = 1$) as

x		-3		1	
-1	$-$	$-$	$-$	$-$	$-$
$x+3$	$-$	0	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	n.d.	$-$
$f(x)$	$\searrow$	$-1/8$	$\nearrow$	n.d.	$\searrow$

That is to say, there is minimum at $(x, y) = (-3, -1/8)$. Inflection point is determined (pole is at $x = 1$) as

x		-5		1	
2	$+$	$+$	$+$	$+$	$+$
$x+5$	$-$	0	$+$	$+$	$+$
$(x-1)^4$	$+$	$+$	$+$	0	$+$
$f''(x)$	$-$	0	$+$	n.d.	$+$
$f(x)$	$\cap$	$-1/9$	$\cup$	n.d.	$\cup$

In conclusion, there is one inflection point at $(x_2\,,\,y_2\,) = (-5,\,-1/9)$ as well one addition change of its convexity/concavity around the vertical asymptote $x = 1$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} &\quad x \neq 1\\
\underline{\text{zero:}} &\quad (x,\,y) = (-1,\,0)\\
\underline{\text{y-axis crossing point } f(0)\text{:}} &\quad y = 1\\
\underline{\text{v. asymptote:}} &\quad x = 1\\
\underline{\text{h. asymptote:}} &\quad y = 0\\
\underline{\text{extremes:}} &\quad (x,\,y) = (-3,\,-1/8)\ \ \text{(min)}\\
\underline{\text{inflections:}} &\quad (x,\,y) = (-5,\,-1/9)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.30 are summarized by $f(x)$ graph (see Fig. 3.22): complete function plot (left) and zoom-in around extreme and inflection points (right).

3.31. Given rational function

$$f(x) = \frac{27x + 27}{2(x-1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 27(x+1) \quad \text{and} \quad Q(x) = 2(x-1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad 2(x-1)^3 \neq 0 \quad \therefore \quad x_{1,2,3} \neq 1$$

There is *third-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 1\}$, i.e., there is vertical asymptote at $x = 1$.

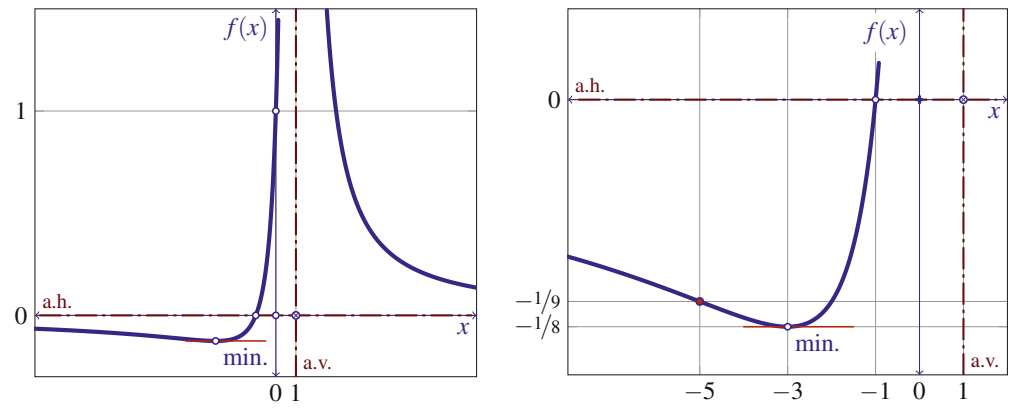


Fig. 3.22 Example P.3.30

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{27}{2}\frac{(-x)+1}{((-x)-1)^3} = \frac{27}{2}\frac{-x+1}{(-1)^3(x+1)^3} = \frac{27}{2}\frac{x-1}{(x+1)^3} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\frac{27}{2}\frac{x-1}{(x+1)^3}\right) = -\frac{27}{2}\frac{x-1}{(x+1)^3} \neq f(x)$$

$$\Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad 27(x+1) = 0 \quad \therefore \quad x_1 = -1 \quad \text{(first-order zero)}$$

Positivity/negativity intervals are determined (mind pole at $x = 1$) as

x		-1		1	
$^{27}/_{2}$	$+$	$+$	$+$	$+$	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	n.d	$+$

In conclusion, $f(x)$ changes its sign after crossing its first (odd)-order zero, as well after passing its third (odd)-order pole.

4. *y-axis crossing point:* this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{27}{2}\frac{(0)+1}{((0)-1)^3} = -\frac{27}{2}$$

This function crosses the vertical axis at $y = -^{27}/_{2}$.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total. If the limit's form is ∞/∞, etc. (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest-order polynomial term while keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{27}{2}\frac{x+1}{(x-1)^3} = \left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \frac{27}{2}\lim_{x\to-\infty}\frac{(x+1)'}{\left((x-1)^3\right)'} = \frac{27}{2}\lim_{x\to-\infty}\frac{1}{3(x-1)^2} = \frac{27}{2}\lim_{x\to-\infty}\frac{1}{+\infty}$$

$$= 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

$$\lim_{x\to 1^{\rightarrow}} f(x) = \frac{27}{2}\lim_{x\to 1^{\rightarrow}} \frac{x+1}{(x-1)^3} = \frac{27}{2}\lim_{x\to 1^{\rightarrow}} \frac{2}{(x-1)^3} = \frac{27}{2}\frac{2}{(0)^3} = -\infty$$

$$\lim_{x\to 1^{\leftarrow}} f(x) = \frac{27}{2}\lim_{x\to 1^{\leftarrow}} \frac{x+1}{(x-1)^3} = \frac{27}{2}\lim_{x\to 1^{\leftarrow}} \frac{2}{(x-1)^3} = \frac{27}{2}\frac{2}{(0)^3} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \frac{27}{2}\lim_{x\to+\infty} \frac{x+1}{(x-1)^3} = \frac{27}{2}\left(\frac{\infty}{\infty}\right) \{\text{L'Hôpital's rule}\}$$

$$= \frac{27}{2}\lim_{x\to+\infty} \frac{(x+1)'}{\left((x-1)^3\right)'} = \frac{27}{2}\lim_{x\to+\infty} \frac{1}{3(x-1)^2} = \frac{27}{2}\lim_{x\to+\infty} \frac{1}{+\infty}$$

$$= 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x) =$ is a function that has one *vertical* asymptote at $\lim_{x\to 1} f(x)$, and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \frac{27}{2}\lim_{x\to\infty} \frac{x+1}{x(x-1)^3} = \frac{27}{2}\lim_{x\to\infty} \frac{x\left(1+1/x^{\to 0}\right)}{x(x-1)^3} = \frac{27}{2}\frac{1}{\infty}$$

$$= 0$$

Thus, oblique asymptote does not exist because $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{27}{2}\frac{x+1}{(x-1)^3}\right)' = \frac{27}{2}\frac{(x-1)^{\not 3} - 3(x+1)(x-1)^2}{(x-1)^{\not 6 4}}$$

$$= -\frac{27}{\not 2}\frac{\not 2(x+2)}{(x-1)^4} \quad \therefore \quad -27(x+2) = 0 \quad \therefore \quad x_1 = -2$$

$$\text{and, } f(x_1) = f(-2) = \frac{27}{2}\frac{(-2)+1}{((-2)-1)^3} = y_1 = \frac{1}{2}$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-27\frac{x+2}{(x-1)^4}\right)' = -27\frac{(x-1)^{\not 4} - 4(x+2)(x-1)^3}{(x-1)^{\not 8 5}}$$

$$= 27\frac{3(x+3)}{(x-1)^5} = \frac{81(x+3)}{(x-1)^5}$$

$$\therefore \quad 81(x+3) = 0 \quad \therefore \quad x+3 = 0 \quad \therefore \quad x_2 = -3$$

$$\text{and, } f(x_2) = f(-3) = \frac{(-3)+1}{((-3)-1)^2} = y_2 = \frac{27}{64}$$

Extreme point is examined (keep in mind pole $x = 1$), as

x		-2		1	
-27	$-$	$-$	$-$	$-$	$-$
$x+2$	$-$	0	$+$	$+$	$+$
$(x-1)^4$	$+$	$+$	$+$	0	$+$
$f'(x)$	$+$	0	$-$	n.d.	$-$
$f(x)$	$\nearrow$	$-1/2$	$\searrow$	n.d.	$\searrow$

There is one maximum point at $(x, y) = (-2, -1/2)$. Inflection point is pole $x = 1$ as

x		-3		1	
81	$+$	$+$	$+$	$+$	$+$
$x+3$	$-$	0	$+$	$+$	$+$
$(x-1)^5$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	n.d.	$+$
$f(x)$	$\cup$	$27/64$	$\cap$	n.d.	$\cup$

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} &\quad x \neq 1 \\
\underline{\text{y-axis crossing point } f(0)\text{:}} &\quad y = -27/2 \\
\underline{\text{v. asymptote:}} &\quad x = 1 \\
\underline{\text{h. asymptote:}} &\quad y = 0 \\
\underline{\text{zero:}} &\quad (x, y) = (-1, 0) \\
\underline{\text{extremes:}} &\quad (x, y) = (-2, 1/2) \ \ (\text{max}) \\
\underline{\text{inflections:}} &\quad (x, y) = (-3, 27/64)
\end{aligned}$$

9. *Graphical representation:* see Fig. 3.23: the complete (left) and zoom–in (right) plots.

3.32. Given rational function

$$f(x) = \left(\frac{x+1}{x-1}\right)^2 = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = (x+1)^2 \quad \text{and} \quad Q(x) = (x-1)^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x-1)^2 \neq 0 \quad \therefore \quad x_{1,2} \neq 1$$

There is *second-order pole* of $f(x)$ at $x = 1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R} | x \neq 1\}$, i.e., there is vertical asymptote at $x = 1$.

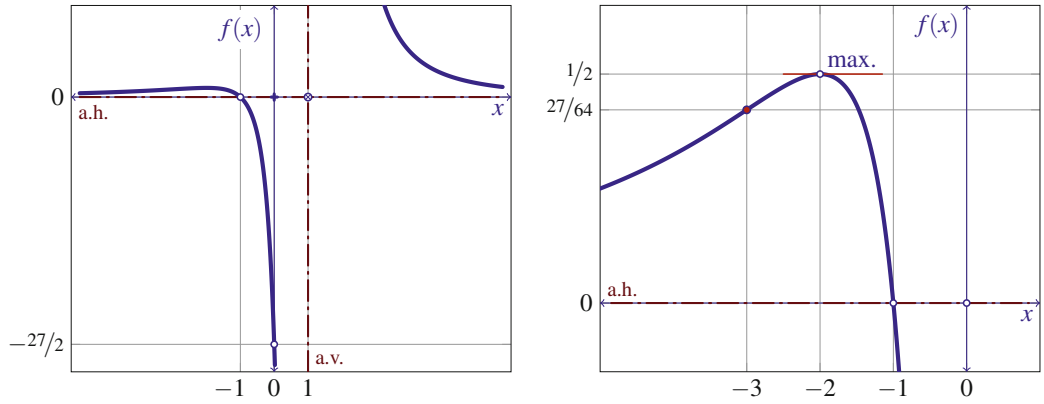


Fig. 3.23 Example P.3.31

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\big((-x)+1\big)^2}{\big((-x)-1\big)^2} = \frac{(-x+1)^2}{(-x-1)^2} = \frac{\cancel{(-1)^2}(x-1)^2}{\cancel{(-1)^2}(x+1)^2} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\frac{(x-1)^2}{(x+1)^2}\right) = -\frac{(x-1)^2}{(x+1)^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad (x+1)^2 = 0 \quad \therefore \quad x_{1,2} = -1 \quad \text{(second-order zero)}$$

The positivity/negativity intervals of this function are determined (keep in mind its pole at $x = 1$ as well) as,

x		-1		1	
$(x+1)^2$	$+$	0	$+$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$	0	$+$
$f(x)$	$+$	0	$+$	n.d.	$+$

In conclusion, $f(x) = (x+1)^2/(x-1)^2$ is always either zero or positive (the consequence of its even order pole and zero).

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{\big((0)+1\big)^2}{\big((0)-1\big)^2} = 1$$

In conclusion, this function crosses the vertical axis at $y = 1$.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus it is necessary to resolve four limits in total. If the limit's form is ∞/∞, etc. (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest-order polynomial term while keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as illustrated here

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x+1)^2}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{develop and factorize}\}$$

$$= \lim_{x\to-\infty} \frac{x^2+2x+1}{x^2-2x+1} = \lim_{x\to-\infty} \frac{\cancel{x^2}\left(1+\cancel{\frac{2}{x}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}\right)}{\cancel{x^2}\left(1-\cancel{\frac{2}{x}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}\right)}$$

$$= 1\uparrow$$

i.e., $f(x)$ tends to $y = 1$ from the negative (bottom) side

$$\lim_{x\to 1_{\to}} f(x) = \lim_{x\to 1_{\to}} \frac{(x+1)^2}{(x-1)^2} = \frac{4}{(0)^2_{\to}} = +\infty$$

$$\lim_{x\to 1_{\leftarrow}} f(x) = \lim_{x\to 1_{\leftarrow}} \frac{(x+1)^2}{(x-1)^2} = \lim_{x\to 1_+} \frac{4}{(0)^2_{\leftarrow}} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x+1)^2}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{develop and factorise}\}$$

$$= \lim_{x\to+\infty} \frac{x^2+2x+1}{x^2-2x+1} = \lim_{x\to+\infty} \frac{\cancel{x^2}\left(1+\cancel{\frac{2}{x}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}\right)}{\cancel{x^2}\left(1-\cancel{\frac{2}{x}}^{\,0}+\cancel{\frac{1}{x^2}}^{\,0}\right)}$$

$$= 1\downarrow$$

i.e., $f(x)$ tends to $y = 1$ from the positive (upper) side

Reminder: Recall these useful relationships:

$$\text{if,} \quad a > 0, \quad \text{then,} \quad 1+a > 1-a, \quad \text{in other words:} \quad \frac{1+a}{1-a} > 1$$

$$\text{or, equivalently:} \quad \frac{1-a}{1+a} < 1$$

In conclusion, $f(x) = (x+1)^2/(x-1)^2$ is a function that has one *vertical* asymptote at $\lim_{x\to 1} f(x)$ and one *horizontal* asymptote at $y = 1$ because $\lim_{x\to\pm\infty} f(x) = 1$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(x+1)^2}{x(x-1)^2} = \lim_{x\to\infty} \frac{x^2+2x+1}{x\left(x^2-2x+1\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(1 + \cancel{\frac{2}{x}}^{0} + \cancel{\frac{1}{x^2}}^{0}\right)}{x\ \cancel{x^2}\left(1 - \cancel{\frac{2}{x}}^{0} + \cancel{\frac{1}{x^2}}^{0}\right)} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty}$$

$$= 0$$

Oblique asymptote does not exist because $a - 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{(x+1)^2}{(x-1)^2}\right)' = \frac{2(x+1)\cancel{(x-1)^1} - 2(x+1)^2\cancel{(x-1)}}{(x-1)^{\cancel{4}3}}$$

$$\left\{(a-b)(a+b) = a^2 - b^2 \quad \text{and,} \quad (a+b)^2 = a^2 + 2ab + b^2\right\}$$

$$= \frac{2(x^2-1) - 2(x^2+2x+1)}{(x-1)^3}$$

$$= \frac{\cancel{2x^2} - \cancel{2x^2} - 2 - 4x - 2}{(x-1)^3} = -4\frac{x+1}{(x-1)^3}$$

$$\therefore\ -4(x+1) = 0 \ \therefore\ x_1 = -1$$

$$\text{and, } f(x_1) = f(-1) = \frac{\left((-1)+1\right)^2}{\left((-1)-1\right)^2} = y_1 = 0$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-4\frac{x+1}{(x-1)^3}\right)' = -4\,\frac{\cancel{(x-1)^3} - 3(x+1)\cancel{(x-1)^2}}{(x-1)^{\cancel{6}4}}$$

$$= -4\,\frac{-2x-4}{(x-1)^4} = \frac{8(x+2)}{(x-1)^4}$$

$$\therefore\ 8(x+2) = 0 \ \therefore\ x+2 = 0 \ \therefore\ x_2 = -2$$

$$\text{and, } f(x_2) = f(-2) = \frac{(-2)+1}{((-2)-1)^2} = y_2 = \frac{1}{9}$$

Extreme point is determined (mind pole $x = 1$) as

x		-1		1	
-4	$-$	$-$	$-$	$-$	$-$
$x+1$	$-$	0	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	n.d.	$-$
$f(x)$	↘	0	↗	n.d.	↘

There is one minimum point at $(x, y) = (-1, 0)$. Inflection point is (mind pole $x = 1$)

x		-2		1	
8	+	+	+	+	+
$x+2$	−	0	+	+	+
$(x-1)^4$	+	+	+	0	+
$f''(x)$	−	0	+	n.d.	+
$f(x)$	∩	1/9	∪	n.d.	∪

That is to say, there is one inflection point at $(x_2\,,\,y_2\,) = (-2,\,1/9)$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} &\quad x \neq 1\\
\underline{\text{y-axis crossing point } f(0)\text{:}} &\quad y = 1\\
\underline{\text{v. asymptote:}} &\quad x = 1\\
\underline{\text{h. asymptote:}} &\quad y = 1\\
\underline{\text{zero:}} &\quad (x,\,y) = (-1,\,0)\\
\underline{\text{extremes:}} &\quad (x,\,y) = (-1,\,0)\ \ \text{(min)}\\
\underline{\text{inflections:}} &\quad (x,\,y) = (-2,\,1/9)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.32 are summarized by $f(x)$ graph (see Fig. 3.24): complete function plot (left) and zoom-in around extreme and inflection points (right).

3.33. Given rational function

$$f(x) = \frac{(x+1)^2}{(x-1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = (x+1)^2 \quad \text{and} \quad Q(x) = (x-1)^3$$

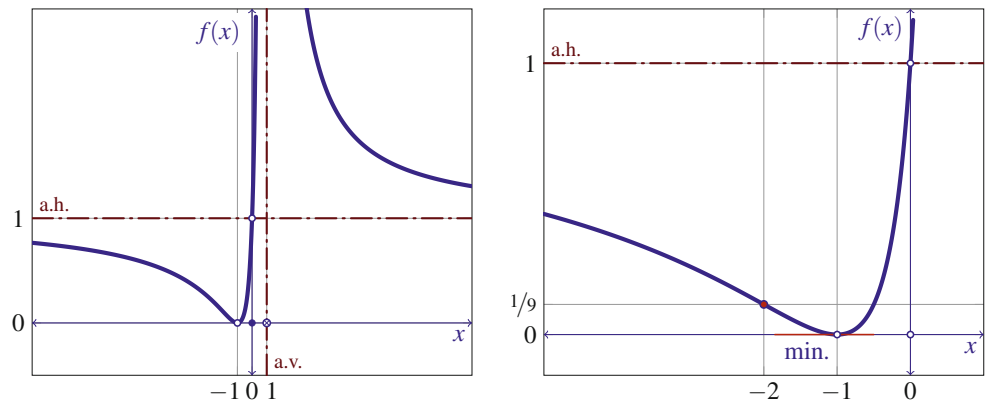


Fig. 3.24 Example P.3.32

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \;\; \therefore \;\; (x-1)^3 \neq 0 \;\; \therefore \;\; x_{1,2,3} \neq 1$$

In conclusion, there is one *third order pole* of $f(x)$ at $x = 1$, and thus $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R}|x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\big((-x)+1\big)^2}{\big((-x)-1\big)^3} = \frac{(-x+1)^2}{(-x-1)^3} = \frac{\cancel{(-1)^2}(x-1)^2}{\cancel{(-1)^3}(x+1)^3} = -\frac{(x-1)^2}{(x+1)^3}$$

$$\neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \;\; -f(-x) = -\left(-\frac{(x-1)^2}{(x+1)^3}\right) = \frac{(x-1)^2}{(x+1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

In conclusion, this function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; P(x) = 0 \;\; \therefore \;\; (x+1)^2 = 0 \;\; \therefore \;\; x_{1,2} = -1 \quad \text{(second-order zero)}$$

The positivity/negativity intervals of this function are determined (keep in mind its pole at $x = 1$ as well) as,

x		-1		1	
$(x+1)^2$	$+$	0	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$-$	n.d.	$+$

In conclusion, $f(x) = (x+1)^2/(x-1)^3$ is either zero or negative for $x < 1$ and positive after passing its pole.

4. *y-axis crossing point* $f(0)$*:* this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{\big((0)+1\big)^2}{\big((0)-1\big)^3} = -1$$

This function crosses the vertical axis at $y = -1$.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total. If the limit's form is ∞/∞, etc. (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest-order polynomial term and keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x+1)^2}{(x-1)^3} = \left(\frac{\infty}{\infty}\right) \{\text{develop and factorize}\}$$

$$\{\text{Pascal's triangle: } (a \pm b)^3 = 1a^3b^0 \pm 3a^2b^1 + 3a^1b^2 \pm 1a^0b^3\}$$

$$= \lim_{x\to-\infty} \frac{x^2+2x+1}{x^3-3x^2+3x-1} = \lim_{x\to-\infty} \frac{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}{x^3\left(1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}\right)}$$

$$= \lim_{x\to-\infty} \frac{1}{x} = \frac{1}{-\infty} = 0\uparrow$$

i.e., $f(x)$ tends to $y = 0$ from the negative (bottom) side

$$\lim_{x\to 1_{\rightarrow}} f(x) = \lim_{x\to 1_{\rightarrow}} \frac{(x+1)^2}{(x-1)^3} = \lim_{x\to 1_{\rightarrow}} \frac{4}{(0)^3} = -\infty$$

$$\lim_{x\to 1_{\leftarrow}} f(x) = \lim_{x\to 1_{\leftarrow}} \frac{(x+1)^2}{(x-1)^3} = \lim_{x\to 1_{\leftarrow}} \frac{4}{(0)^3} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x+1)^2}{(x-1)^3} = \left(\frac{\infty}{\infty}\right) \{\text{develop and factorise}\}$$

$$= \lim_{x\to+\infty} \frac{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}{x^3\left(1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}\right)}$$

$$= \lim_{x\to+\infty} \frac{1}{x} = \frac{1}{+\infty} = 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x) = (x+1)^2/(x-1)^3$ is a function that has one *vertical* asymptote at $\lim_{x\to1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(x+1)^2}{x(x-1)^3} = \lim_{x\to\infty} \frac{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}{x\,x^3\left(1-\frac{3}{x}+\frac{3}{x^2}-\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{1}{x^2} = \frac{1}{\infty} = 0$$

Oblique asymptote does not exist because $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{(x+1)^2}{(x-1)^3}\right)' = \frac{2(x+1)(x-1)^3 - 3(x+1)^2(x-1)^2}{(x-1)^6}$$

$$\{(a-b)(a+b) = a^2 - b^2 \quad \text{and,} \quad (a \pm b)^2 = a^2b^0 \pm 2ab + a^0b^2\}$$

$$= \frac{2(x^2-1) - 3(x^2+2x+1)}{(x-1)^4} = \frac{-x^2-2-6x-3}{(x-1)^4}$$

$$= -\frac{x^2+6x+5}{(x-1)^4} = -\frac{x^2+x+5x+5}{(x-1)^4} = -\frac{(x+1)(x+5)}{(x-1)^4}$$

$$\therefore \quad -(x+1)(x+5) = 0 \quad \therefore \quad x_1 = -1 \quad \text{and,} \quad x_2 = -5$$

$$\text{and, } f(x_1) = f(-1) = \frac{\big((-1)+1\big)^2}{\big((-1)-1\big)^3} = y_1 = 0$$

$$f(x_2) = f(-5) = \frac{\big((-5)+1\big)^2}{\big((-5)-1\big)^3} = y_2 = -\frac{2}{27}$$

$$f''(x) = 0: \quad f''(x) = \left(-\frac{x^2+6x+5}{(x-1)^4}\right)' = -\frac{(2x+6)(x-1)^4 - 4(x^2+6x+5)(x-1)^3}{(x-1)^8}$$

$$= -\frac{(2x+6)(x-1)^4 - 4(x+1)(x+5)(x-1)^3}{(x-1)^8}$$

$$= 2\,\frac{x^2+10x+13}{(x-1)^5}$$

where the the quadratic equation roots are,

$$x^2+10x+13 = 0 \quad \therefore \quad x_{3,4} = \frac{-10 \pm \sqrt{10^2 - 4\cdot 13}}{2} = -5 \pm 2\sqrt{3}$$

I bit of work to deliver,

$$f(x_3) = \frac{\Big((-5+2\sqrt{3})+1\Big)^2}{\Big((-5+2\sqrt{3})-1\Big)^3} = \frac{\Big(-4+2\sqrt{3}\Big)^2}{\Big(-6+2\sqrt{3}\Big)^3} = \frac{4\Big(7-4\sqrt{3}\Big)}{\Big(-6+2\sqrt{3}\Big)\Big(-6+2\sqrt{3}\Big)^2}$$

$$= \frac{4\Big(7-4\sqrt{3}\Big)}{24\Big(-6+2\sqrt{3}\Big)\Big(2-\sqrt{3}\Big)} = \frac{4\Big(7-4\sqrt{3}\Big)}{48\,12\Big(-9+5\sqrt{3}\Big)}\,\frac{7+4\sqrt{3}}{7+4\sqrt{3}}$$

$$= \frac{1}{12\Big(-3-\sqrt{3}\Big)} \Rightarrow y_3 = -\frac{1}{12\Big(3+\sqrt{3}\Big)}$$

and in the same manner,

$$f(x_4) = \frac{\Big((-5-2\sqrt{3})+1\Big)^2}{\Big((-5-2\sqrt{3})-1\Big)^3} = \frac{\Big(-4-2\sqrt{3}\Big)^2}{\Big(-6-2\sqrt{3}\Big)^3} \Rightarrow y_4 = -\frac{1}{12\Big(3-\sqrt{3}\Big)}$$

Two extreme points $(x_1\,,\,y_1\,) = (-1,\,0)$ and $(x_2\,,\,y_2\,) = (-5,\,-2/27)$ are examined as

x		-5		-1		1	
-1	-1	-1	-1	-1	-1	-1	-1
$x+5$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-1)^4$	$+$	$+$	$+$	$+$	$+$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$	n.d.	$-$
$f(x)$	$\searrow$	$-2/27$	$\nearrow$	0	$\searrow$	n.d	$\searrow$

There is one minimum point at $(x, y) = (-5, -2/27)$ and one maximum point at $(x, y) = (-1, 0)$.

Inflection points $(x_3\,,\, y_3\,)$ and $(x_4\,,\, y_4\,)$, pole $x = 1$, and $x^2 + 10x + 13 = (x - x_3)(x - x_4)$, are determined as

x		x_4		x_3		1	
2	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x - x_4$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-1)^5$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	0	$+$	0	$-$	n.d.	$+$
$f(x)$	$\cap$	y_4	$\cup$	y_3	$\cap$	n.d.	$\cup$

In conclusion, this function has one vertical at $x = 1$ and one horizontal asymptote at $y = 0$, and it has two extremes, maximum point at $(x_1\,,\, y_1\,) = (-1, 0)$ and minimum at $(x_2, y_2) = (-5, -2/27)$. Also, there are two inflection points at $(x_3\,,\, y_3\,)$ and $(x_4\,,\, y_4\,)$ as calculated above.

8. *Summary of the important points:*

$$
\begin{aligned}
\underline{\text{defined:}} \quad & x \neq 1 \\
\underline{\text{v. asymptote:}} \quad & x = 1 \\
\underline{\text{h. asymptote:}} \quad & y = 0 \\
\underline{\text{y-axis crossing point } f(0)\text{:}} \quad & y = -1 \\
\underline{\text{zeros}} \quad & (x, y) = (-1, 0) \\
& (x, y) = (-1, 0) \\
\underline{\text{extremes:}} \quad & (x, y) = (-5, -2/27) \ \ \text{(min)} \\
& (x, y) = (-1, 0) \ \ \text{(max)} \\
\underline{\text{inflections:}} \quad & (x, y) = \left((-5 - 2\sqrt{63}), -\frac{1}{12(3 - \sqrt{3})}\right) \\
& (x, y) = \left((-5 + 2\sqrt{63}), -\frac{1}{12(3 + \sqrt{3})}\right)
\end{aligned}
$$

9. *Graphical representation:* see Fig. 3.25, complete plot (left) and zoom-in (right).

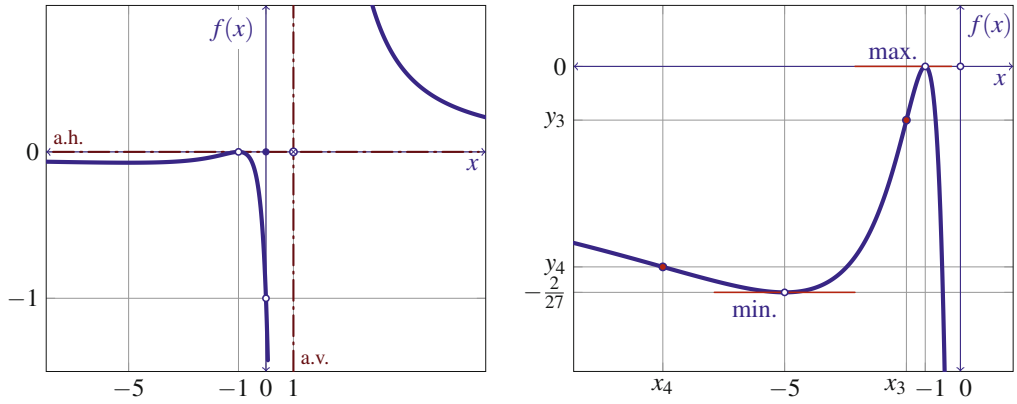


Fig. 3.25 Example P.3.33

3.34. Given rational function

$$f(x) = \frac{(x+1)^2}{(x-1)^4} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = (x+1)^2 \quad \text{and} \quad Q(x) = (x-1)^4$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x-1)^4 \neq 0 \quad \therefore \quad x_{1,2,3,4} \neq 1$$

   In conclusion, there is one *fourth-order pole* of $f(x)$ at $x = 1$, and thus $f(x)$ is defined for all real numbers x except where the function becomes infinite, i.e., $\{\mathbb{R} | x \neq 1\}$. That is to say, there is one vertical asymptote at $x = 1$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\big((-x)+1\big)^2}{\big((-x)-1\big)^4} = \frac{(-x+1)^2}{(-x-1)^4} = \frac{\cancel{(-1)^2}(x-1)^2}{\cancel{(-1)^4}(x+1)^4} = \frac{(x-1)^2}{(x+1)^4}$$

$$\neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\frac{(x-1)^2}{(x+1)^4}\right) = -\frac{(x-1)^2}{(x+1)^4} \neq f(x) \Rightarrow \quad \text{not odd}$$

   In conclusion, this function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad (x+1)^2 = 0 \quad \therefore \quad x_{1,2} = -1 \quad \text{(second-order zero)}$$

   In this case, the positivity/negativity intervals of this function are bound by its pole at $x = 1$ and its even order zero at $x = -1$, as:

x		-1		1	
$(x+1)^2$	$+$	0	$+$	$+$	$+$
$(x-1)^4$	$+$	$+$	$+$	0	$+$
$f(x)$	$+$	0	$+$	n.d.	$+$

In conclusion, $f(x) = (x+1)^2/(x-1)^4$ is always either zero or positive.

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at $(0, 1)$ as,

$$f(0) = \frac{\big((0)+1\big)^2}{\big((0)-1\big)^4} = 1$$

In conclusion, this function crosses the vertical axis at $y = 1$.

5. *Limits:* there is one vertical asymptote at $x = 1$, and thus, it is necessary to resolve four limits in total. If the limit's form is ∞/∞ (i.e., undetermined), then there are multiple techniques that can be used: L'Hôpital's rule or, for example, by factoring the highest-order polynomial term while keeping in mind that $\lim_{x\to\infty} n/x = 0$ (n is a real number), as

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x+1)^2}{(x-1)^4} = \left(\frac{\infty}{\infty}\right) \{\text{develop and factorise}\}$$

$$\{\text{Pascal's triangle: } (a\pm b)^4 = 1a^4b^0 \pm 4a^3b^1 + 6a^2b^2 \pm 4a^1b^3 + 1a^0b^4\}$$

$$= \lim_{x\to-\infty} \frac{x^2+2x+1}{x^4-4x^3+6x^2-4x+1}$$

$$= \lim_{x\to-\infty} \frac{x^2\big(1+\cancel{2/x}^{\,0}+\cancel{1/x^2}^{\,0}\big)}{x^{\cancel{4}^{\,2}}\big(1-\cancel{4/x}^{\,0}+\cancel{6/x^2}^{\,0}-\cancel{4/x^3}^{\,0}+\cancel{1/x^4}^{\,0}\big)} = \lim_{x\to-\infty} \frac{1}{x^2} = \frac{1}{(-\infty)^2}$$

$$= 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

$$\lim_{x\to 1_{\rightarrow}} f(x) = \lim_{x\to 1_{\rightarrow}} \frac{(x+1)^2}{(x-1)^4} = \lim_{x\to 1_{\rightarrow}} \frac{4}{(0)^4_{\rightarrow}} = \frac{4}{0} = +\infty$$

$$\lim_{x\to 1_{\leftarrow}} f(x) = \lim_{x\to 1_{\leftarrow}} \frac{(x+1)^2}{(x-1)^4} = \lim_{x\to 1_{\leftarrow}} \frac{4}{(0)^4_{\leftarrow}} = \frac{4}{0} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x+1)^2}{(x-1)^4} = \lim_{x\to+\infty} \frac{x^2\big(1+\cancel{2/x}^{\,0}+\cancel{1/x^2}^{\,0}\big)}{x^{\cancel{4}^{\,2}}\big(1-\cancel{4/x}^{\,0}+\cancel{6/x^2}^{\,0}-\cancel{4/x^3}^{\,0}+\cancel{1/x^4}^{\,0}\big)}$$

$$= \lim_{x\to+\infty} \frac{1}{x^2} = \frac{1}{(+\infty)^2}$$

$$= 0\downarrow$$

i.e., $f(x)$ tends to $y = 0$ from the positive (upper) side

In conclusion, $f(x) = (x+1)^2/(x-1)^4$ is a function that has one *vertical* asymptote at $\lim_{x\to 1} f(x)$ and one *horizontal* asymptote at $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2\left(1 + \frac{2}{x} + \frac{1}{x^2}\right)}{x\, x^2\left(1 - \frac{4}{x} + \frac{6}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^3} = \frac{1}{\infty} = 0$$

Oblique asymptote does not exist because $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{(x+1)^2}{(x-1)^4}\right)' = \frac{2(x+1)(x-1)^4 - 4(x+1)^2(x-1)^3}{(x-1)^8}$$

$$\left\{(a-b)(a+b) = a^2 - b^2 \quad \text{and,} \quad (a+b)^2 = a^2 + 2ab + b^2\right\}$$

$$= \frac{2(x^2-1) - 4(x^2+2x+1)}{(x-1)^5} = \frac{-2x^2 - 2 - 8x - 4}{(x-1)^5}$$

$$= -2\,\frac{x^2+4x+3}{(x-1)^5} = -2\,\frac{x^2+x+3x+3}{(x-1)^5} = -2\,\frac{(x+1)(x+3)}{(x-1)^5}$$

$$\therefore \quad -2\,(x+1)(x+3) = 0 \quad \therefore \quad x_1 = -1 \quad \text{and,} \quad x_2 = -3$$

$$\text{where, } f(x_1) = f(-1) = \frac{\big((-1)+1\big)^2}{\big((-1)-1\big)^5} = y_1 = 0$$

$$f(x_2) = f(-3) = \frac{\big((-3)+1\big)^2}{\big((-3)-1\big)^5} = y_2 = \frac{1}{64}$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-2\,\frac{x^2+4x+3}{(x-1)^5}\right)' = -2\,\frac{(2x+4)(x-1)^5 - 5(x^2+4x+3)(x-1)^4}{(x-1)^{10}}$$

$$= -2\,\frac{(2x+4)(x-1) - 5(x^2+4x+3)}{(x-1)^6}$$

$$= -2\,\frac{(2x^2+2x-4) - 5(x^2+4x+3)}{(x-1)^6}$$

$$= 2\,\frac{3x^2+18x+19}{(x-1)^6}$$

where the quadratic equation's roots are

$$3x^2 + 18x + 19 = 0 \quad \therefore \quad x_{3,4} = \frac{-18 \pm \sqrt{18^2 - 4\cdot 3\cdot 19}}{2\cdot 3} = \frac{-9 \pm 2\sqrt{6}}{3}$$

I bit of work to deliver,

$$(x_3+1)^2=\left(\frac{-9+2\sqrt{6}}{3}+1\right)^2=\left(-\frac{2}{3}\,(3+\sqrt{6})\right)^2=\frac{4}{3}\left(5+2\sqrt{6}\right)$$

$$(x_3-1)^4=\left(\frac{-9+2\sqrt{6}}{3}-1\right)^4=\left(-\frac{2}{3}\,(6+\sqrt{6})\right)^2\left(\frac{2}{3}\,(6+\sqrt{6})\right)^2$$

$$=\left(\frac{8}{3}\,(7+2\sqrt{6})\right)^2=\frac{64}{9}\,(73+28\sqrt{6})$$

therefore,

$$f(x_3)=\frac{\cancel{4}/\cancel{3}\left(5+2\sqrt{6}\right)}{\cancel{64}16/\cancel{9}3\,(73+28\sqrt{6})}\;\frac{73-28\sqrt{6}}{73-28\sqrt{6}}=\frac{3}{16}\;\frac{\left(5+2\sqrt{6}\right)\left(73-28\sqrt{6}\right)}{73^2-28^2\cdot 6}$$

$$=\frac{3}{16}\,\frac{29+6\sqrt{6}}{625}\Rightarrow y_3=\frac{3}{10\,000}\left(29+6\sqrt{6}\right)$$

and in the same manner,

$$f(x_4)=\frac{\cancel{4}/\cancel{3}\left(5-2\sqrt{6}\right)}{\cancel{64}16/\cancel{9}3\,(73-28\sqrt{6})}\;\frac{73+28\sqrt{6}}{73+28\sqrt{6}}=\frac{3}{16}\;\frac{\left(5-2\sqrt{6}\right)\left(73+28\sqrt{6}\right)}{73^2-28^2\cdot 6}$$

$$=\frac{3}{16}\,\frac{29-6\sqrt{6}}{625}\Rightarrow y_4=\frac{3}{10\,000}\left(29-6\sqrt{6}\right)$$

Thus, the function extremes $(x_1\,,\,y_1)=(-1,0)$ and $(x_2\,,\,y_2)=(-3,\,1/64)$ are determined as

x		-3		-1		1	
-2	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-1)^5$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$	n.d.	$-$
$f(x)$	$\nearrow$	$1/64$	$\searrow$	0	$\nearrow$	n.d.	$\searrow$

There is one maximum point at $(x,\,y)=(-3,\,1/64)$ and one minimum point at $(x,\,y)=(-1,0)$. Inflection points $(x_3\,,\,y_3)$ and $(x_4\,,\,y_4)$ (where $3x^2+18x+19=(x-x_3)(x-x_4)$) are determined as

x		x_4		x_3		1	
2	+	+	+	+	+	+	+
$x - x_4$	−	0	+	+	+	+	+
$x - x_3$	−	−	−	0	+	+	+
$(x-1)^6$	+	+	+	+	+	0	+
$f''(x)$	+	0	−	0	+	n.d.	+
$f(x)$	∪	y_4	∩	y_3	∪	n.d.	∪

There are two inflection points, at $(x, y) = (x_4, y_4)$ and $(x, y) = (x_3, y_3)$. In addition, $f(x)$ changes its convexity after passing its vertical asymptote.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} \quad & x \neq 1 \\
\underline{\text{v. asymptote:}} \quad & x = 1 \\
\underline{\text{h. asymptote:}} \quad & y = 0 \\
\underline{\text{y-axis crossing point } f(0)\text{:}} \quad & y = 1 \\
\underline{\text{zero:}} \quad & (x, y) = (-1, 0) \\
\underline{\text{extremes:}} \quad & (x, y) = (-3, {}^1\!/_{64}) \ \ \text{(max)} \\
& (x, y) = (-1, 0) \ \ \text{(min)} \\
\underline{\text{inflections:}} \quad & (x, y) = \left(\frac{-9 - 2\sqrt{6}}{3}, \frac{3}{10000}(29 - 6\sqrt{6}) \right) \\
& (x, y) = \left(\frac{-9 + 2\sqrt{6}}{3}, \frac{3}{10000}(29 + 6\sqrt{6}) \right)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.34 are summarized as the complete graph of $f(x)$ in Fig. 3.26 (left) and zoom-in around its extreme and inflection points in Fig. 3.26 (right).

3.35. Given rational function

$$f(x) = \frac{x+1}{x+1} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x + 1 \quad \text{and} \quad Q(x) = x + 1$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x + 1 \neq 0 \quad \therefore \quad x_1 \neq -1$$

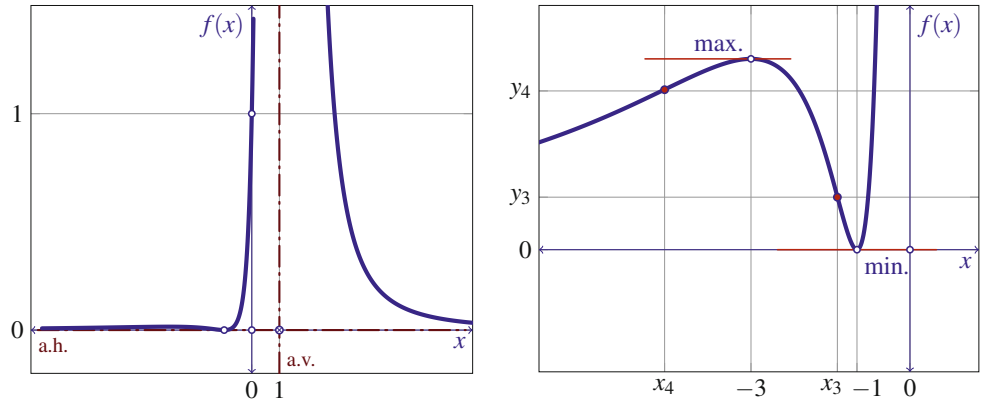


Fig. 3.26 Example P.3.34

There is one *first-order pole* of $f(x)$ at $(x = -1)$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq -1\}$.

It is important to note that $(x = -1)$ is also *first-order zero* of this function and that this function is actually *not defined* (i.e., it takes one of the forms: $0/0$, ∞/∞, 0^0, ...) at this specific point, as

$$f(-1) = \frac{(-1)+1}{(-1)+1} = \frac{0}{0}$$

It is a common mistake to assume that $f(x) = 1$ as

$$f(x) = \frac{\cancel{x+1}}{\cancel{x+1}} = 1$$

However, this result is valid everywhere except in one single point $(x = -1)$, which turns out to be a very important property of a function.

This and the following examples illustrate this situation, commonly known as *pole-zero cancellation*, which may create a very interesting effect of suppressing the expected vertical asymptote. Depending on how many pole-zero pairs are cancelled and how many poles or zeros remain, the function's behavior is drastically modified.

2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)+1}{(-x)+1} = \frac{-x+1}{-x+1} = \frac{(-1)(x-1)}{(-1)(x-1)} = \frac{x-1}{x-1} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \quad -f(-x) = -\left(\frac{x-1}{x-1}\right) = -\frac{x-1}{x-1} \neq f(x) \Rightarrow \quad \text{not odd}$$

This result may appear bizarre, because one would assume that a constant function $f(x) = 1$ is even. But it must not be forgotten that in $x < 0$ side, there is *one single point* $f(-1) =$ n.d. missing; however, $f(1) = 1$ is well defined. By consequence, this function is *not symmetric* relative to the vertical axis.

In conclusion, this function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$f(x) = 0: \quad \therefore \quad P(x) = 0 \quad \therefore \quad x + 1 = 0 \quad \therefore \quad x_1 = -1 \quad \text{(first-order zero)}$$

As there is pole-zero cancellation, the positivity/negativity intervals are:

x		-1	
$x+1$	$-$	0	$+$
$x+1$	$-$	0	$+$
$f(x)$	$+$	n.d.	$+$

In conclusion, $f(x) = (x+1)/(x+1)$ is always positive, except in $(x = -1)$ where it is not defined because of $0/0$ division.

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at $(0, 1)$ as,

$$f(0) = \frac{(0)+1}{(0)+1} = 1$$

In conclusion, this function crosses the vertical axis at $y = 1$.

5. *Limits:* there is $(x = -1)$ point where the function is not defined, thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{x+1}1_-}{\cancel{x+1}1_-} = 1_+ \quad \text{(function is always positive)}$$

$$\lim_{\substack{x\to-1\\ \to}} f(x) = \lim_{\substack{x\to-1\\ \to}} \frac{x+1}{x+1} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to-1\\ \to}} \frac{(x+1)'}{(x+1)'} = \lim_{\substack{x\to-1\\ \to}} \frac{\overset{-1}{\to}}{\overset{-1}{\to}} = 1\downarrow$$

$$\lim_{\substack{x\to-1\\ \leftarrow}} f(x) = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{x+1}{x+1} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to-1\\ \leftarrow}} \frac{(x+1)'}{(x+1)'} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{\overset{1}{\leftarrow}}{\overset{1}{\leftarrow}} = 1\downarrow$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x+1}1_+}{\cancel{x+1}1_+} = \underset{\leftarrow}{1} \quad \text{(function is always positive)}$$

In conclusion, $f(x) = (x+1)/(x-1)$ is not defined at $(x = -1)$. However, left- and right-side limits $\lim_{x\to1} f(x) = 1$ are equal; therefore, the limit exist. Horizontal asymptote is $y = 1$ because $\lim_{x\to\pm\infty} f(x) = 1$.

6. *Oblique asymptote:* while $(x \to \infty)$, this function is well defined, and being far away from $(x = -1)$ point, division $(x+1)/(x+1) = 1$ is valid.

$$f(x) = \frac{\cancel{x+1}}{\cancel{(x+1)}} = 1 \quad \therefore \quad a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

In conclusion, oblique asymptote $y_{\mathtt{aa}} = ax + b$ does not exist because $a = 0$.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{\cancel{x+1}1}{\cancel{x+1}1}\right)' = (1)' = 0 \quad \therefore \quad f(x) = \text{const.}$$

$$\underline{f''(x) = 0:} \qquad f'(x) = 0 \Rightarrow f''(x) = 0 = \text{const.}$$

In conclusion, due to pole-zero cancelation, this function indeed reduces to $f(x) = 1$ for all x except $(x = -1)$ where it is not defined but the limit exists. Being constant function, its horizontal asymptote is superimposed at $y = 1$.

8. *Summary of the important points:*

defined:	$x \neq -1$
y-axis crossing point $f(0)$:	$y = 1$
zeros:	(none)
v. asymptote:	(none)
h. asymptote:	$y = 1$ (the function itself)

9. *Graphical representation:* results of the analysis for function in P.3.35 are summarized by $f(x)$ graph; see Fig. 3.27. The discontinuity point $(x = -1)$ is indicated by left- and right-side-limiting arrows and $\otimes$ symbol.

3.36. Given rational function

$$f(x) = \frac{x+1}{(x+1)^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x + 1 \quad \text{and} \quad Q(x) = (x+1)^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e. $Q(x) \neq 0$, i.e.

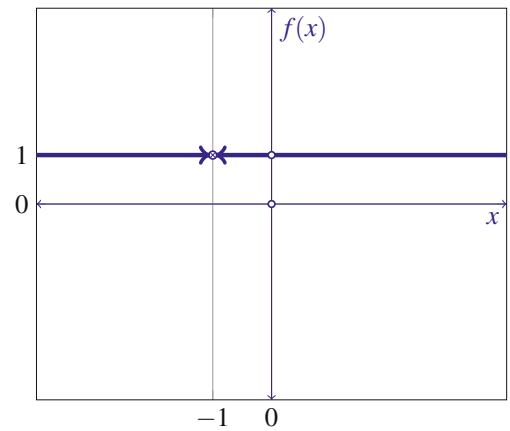


Fig. 3.27 Example P.3.35

$$Q(x) \neq 0 \;\; \therefore \;\; (x+1)^2 \neq 0 \;\; \therefore \;\; x_{1,2} \neq -1$$

There is *second-order pole* of $f(x)$ at $(x = -1)$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq -1\}$.

It is important to note that $(x = -1)$ is also *first-order zero* of this function and that this function is actually *not defined* (i.e., it takes one of the forms: $0/0$, ∞/∞, 0^0, ...) at this specific point, as

$$f(-1) = \frac{(-1)+1}{\big((-1)+1\big)^2} = \frac{0}{0}$$

The factorization

$$f(x) = \frac{x+1}{(x+1)^2} = \frac{1}{x+1}$$

is valid everywhere except in one single point $(x = -1)$. In addition, it is important to note that the same point is also *first-order zero* of this function, hence causing pole-zero pair cancelation by division.

2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)+1}{\big((-x)+1\big)^2} = \frac{-x+1}{(-x+1)^2} = \frac{(-1)(x-1)}{(-1)^2(x-1)^2} = -\frac{x-1}{(x-1)^2}$$

$$\neq f(x) \Rightarrow \quad \text{not even}$$

$$\text{odd:} \; -f(-x) = -\left(-\frac{x-1}{(x-1)^2}\right) = \frac{x-1}{(x-1)^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$f(x) = 0: \quad \therefore \;\; P(x) = 0 \;\; \therefore \;\; x+1 = 0 \;\; \therefore \;\; x_1 = -1 \quad \text{(first-order zero)}$$

As there is pole-zero cancellation, the positivity/negativity intervals are:

x		-1	
1	$+$	$+$	$+$
$x+1$	$-$	0	$+$
$f(x)$	$-$	n.d.	$+$

In conclusion, $f(x) = (x+1)/(x+1)^2$ indeed changes its sign after passing its discontinuity point $(x = -1)$ where it is not defined because of $0/0$ division.

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at $(0, 1)$ as,

$$f(0) = \frac{(0)+1}{\big((0)+1\big)^2} = 1$$

This function crosses the vertical axis at $y = 1$.

5. *Limits:* there is discontinuity point ($x = -1$) where the function is not defined, thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{x+1}^{1}}{(x+1)^{\cancel{2}}} = \lim_{x\to-\infty} \frac{1}{(x+1)} = \lim_{x\to-\infty} \frac{1}{-\infty} = 0\uparrow$$

$$\lim_{\substack{x\to-1\\ \to}} f(x) = \lim_{\substack{x\to-1\\ \to}} \frac{x+1}{(x+1)^2} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to-1\\ \to}} \frac{(x+1)'}{\left((x+1)^2\right)'} = \lim_{\substack{x\to-1\\ \to}} \frac{1}{2(x+1)} = \lim_{\substack{x\to-1\\ \to}} \frac{1}{2(\underset{\to}{0})} = -\infty$$

$$\lim_{\substack{x\to-1\\ \leftarrow}} f(x) = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{x+1}{(x+1)^2} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to-1\\ \leftarrow}} \frac{(x+1)'}{\left((x+1)^2\right)'} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{1}{2(x+1)} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{1}{2(\underset{\leftarrow}{0})} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x+1}^{1}}{(x+1)^{\cancel{2}}} = \lim_{x\to+\infty} \frac{1}{(x+1)} = \lim_{x\to+\infty} \frac{1}{+\infty} = 0\downarrow$$

In conclusion, $f(x) = (x+1)/(x-1)^2$ is not defined at $(x = -1)$. While approaching its discontinuity point, this function tends to $\pm\infty$, that is to say, $(x = -1)$ is its vertical asymptote. Horizontal asymptote is $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:* while $(x \to \infty)$, this function is well defined, and being far away from $(x = -1)$ point, division $(x+1)/(x+1) = 1$ is valid.

$$f(x) = \frac{\cancel{x+1}}{(x+1)^{\cancel{2}}} = \frac{1}{(x+1)} \quad \therefore \quad a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x(x+1)} = \frac{1}{\infty} = 0$$

In conclusion, oblique asymptote $y_{\text{aa}} = ax + b$ does not exist because $a = 0$.

7. *Critical points:* $\{\text{if } (x \neq -1)\}$

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{1}{x+1}\right)' = \frac{0-(x+1)}{(x+1)^2} = \frac{-\cancel{(x+1)}^{1}}{(x+1)^{\cancel{2}}} = \frac{-1}{(x+1)^2} < 0$$

$$\therefore \quad f'(x) < 0 \ \text{(i.e. const.)} \quad \therefore \quad f(x) \quad \text{always descends} \quad \searrow$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{-1}{(x+1)^2}\right)' = \frac{0+2(x+1)}{(x+1)^4} = \frac{\cancel{2(x+1)}}{(x+1)^{\cancel{4}3}} = \frac{2}{(x+1)^3}$$

Sign of $f'(x)$ is fixed because its numerator is always negative and denominator positive (note: square functions), while $f''(x)$ is

x		-1	
2	+	+	+
$(x+1)^3$	−	0	+
$f''(x)$	−	n.d.	+
f(x)	∩		∪

Thus, this and the following examples illustrate that even though order of $f(x)$ is effectively reduced by the cancelation of one pole-zero pair, it is still necessary to examine the remaining binomial terms in $f'(x)$ and $f''(x)$ under condition $(x \neq -1)$.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{defined:}} \quad & x \neq -1 \\ \underline{\text{y-axis crossing point } f(0)\text{:}} \quad & y = 1 \\ \underline{\text{zeros:}} \quad & \text{(none)} \\ \underline{\text{v. asymptote:}} \quad & x = -1 \\ \underline{\text{h. asymptote:}} \quad & y = 0 \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.36 are summarized by $f(x)$ graph; see Fig. 3.28. The discontinuity point $(x = -1)$ is indeed location of a vertical asymptote, due to the remaining binomial term $(x + 1)$ as

$$f(x) = \frac{\cancel{x+1}}{(x+1)^{\cancel{2}}} \quad \{\text{if } (x \neq -1)\} = \frac{1}{x-1}$$

3.37. Given rational function

$$f(x) = \frac{x+1}{(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x+1 \quad \text{and} \quad Q(x) = (x+1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x+1)^3 \neq 0 \quad \therefore \quad x_{1,2,3} \neq -1$$

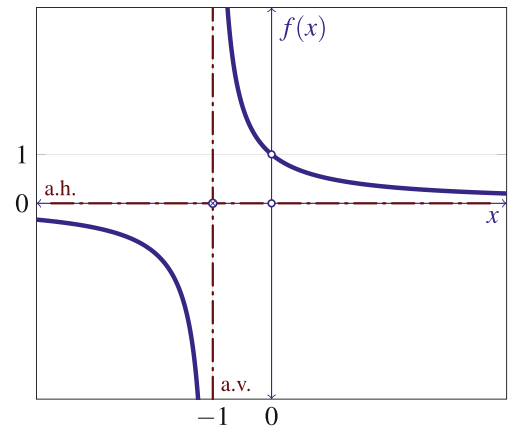


Fig. 3.28 Example P.3.36

There is *third-order pole* of $f(x)$ at $(x = -1)$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq -1\}$.

It is important to note that $(x = -1)$ is also *first-order zero* of this function and that this function is actually *not defined* (i.e., it takes one of the forms: $0/0$, ∞/∞, 0^0, ...) at this specific point, as

$$f(-1) = \frac{(-1)+1}{\big((-1)+1\big)^3} = \frac{0}{0}$$

The factorization

$$f(x) = \frac{\cancel{x+1}}{(x+1)^{\cancel{3}2}} = \frac{1}{(x+1)^2}$$

is valid everywhere except in one single point $(x = -1)$. In addition, it is important to note that the same point is also *first-order zero* of this function, hence causing pole-zero pair cancelation of the binomial term in the numerator.

2. *Function's parity:*

$$\text{even:} \quad f(-x) = \frac{(-x)+1}{\big((-x)+1\big)^3} = \frac{-x+1}{(-x+1)^3} = \frac{(-1)(x-1)}{(-1)^3(x-1)^3} = \frac{x-1}{(x-1)^3} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\text{odd:} \quad -f(-x) = -\left(\frac{x-1}{(x-1)^3}\right) = -\frac{x-1}{(x-1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

In conclusion, this function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$f(x) = 0: \quad \therefore \quad P(x) = 0 \quad \therefore \quad x+1 = 0 \quad \therefore \quad x_1 = -1 \quad \text{(first-order zero)}$$

As there is pole-zero cancellation, the positivity/negativity intervals are:

x		-1	
1	$+$	$+$	$+$
$(x+1)^2$	$+$	0	$+$
$f(x)$	$+$	n.d.	$+$

In conclusion, $f(x) = (x+1)/(x+1)^3$ is positive, it does not change its sign at $(x = -1)$.

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{(0)+1}{\big((0)+1\big)^3} = 1$$

In conclusion, this function crosses the vertical axis at $y = 1$.

5. *Limits:* there is discontinuity point $(x = -1)$ where the function is not defined, thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{x+1}^{1}}{(x+1)^{\not 2}} = \lim_{x\to-\infty} \frac{1}{(x+1)^2} = \lim_{x\to-\infty} \frac{1}{(-\infty)^2} = 0\downarrow$$

$$\lim_{x\to\underset{\to}{-1}} f(x) = \lim_{x\to\underset{\to}{-1}} \frac{x+1}{(x+1)^3} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\underset{\to}{-1}} \frac{(x+1)'}{\left((x+1)^3\right)'} = \lim_{x\to\underset{\to}{-1}} \frac{1}{3(x+1)^2} = \lim_{x\to\underset{\to}{-1}} \frac{1}{3(\underset{\to}{0})^2} = +\infty$$

$$\lim_{x\to\underset{\leftarrow}{-1}} f(x) = \lim_{x\to\underset{\leftarrow}{-1}} \frac{x+1}{(x+1)^3} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\underset{\leftarrow}{-1}} \frac{(x+1)'}{\left((x+1)^3\right)'} = \lim_{x\to\underset{\leftarrow}{-1}} \frac{1}{3(x+1)^2} = \lim_{x\to\underset{\leftarrow}{-1}} \frac{1}{3(\underset{\leftarrow}{0})^2} = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x+1}^{1}}{(x+1)^{\not 2}} = \lim_{x\to+\infty} \frac{1}{(x+1)^2} = \lim_{x\to+\infty} \frac{1}{(+\infty)^2} = 0\downarrow$$

In conclusion, $f(x) = (x+1)/(x-1)^3$ is not defined at $(x = -1)$. While approaching its discontinuity point, this function tends to $+\infty$, that is to say, $(x = -1)$ is its vertical asymptote. Horizontal asymptote is $y = 0$ because $\lim_{x\to\pm\infty} f(x) = 0$.

6. *Oblique asymptote:* while $(x \to \infty)$, this function is well defined, and being far away from $(x = -1)$ point, division $(x+1)/(x+1) = 1$ is valid.

$$f(x) = \frac{\cancel{x+1}}{(x+1)^{\not 2}} = \frac{1}{(x+1)^2} \quad \therefore \quad a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x(x+1)^2} = \frac{1}{\infty} = 0$$

In conclusion, oblique asymptote $y_{\text{aa}} = ax + b$ does not exist because $a = 0$.

7. *Critical points:* $\{\text{if } (x \neq -1)\}$

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{1}{(x+1)^2}\right)' = \frac{0 - 2(x+1)}{(x+1)^4} = \frac{-2\cancel{(x+1)}}{(x+1)^{\not 4}} = \frac{-2}{(x+1)^3}$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{-2}{(x+1)^3}\right)' = \frac{0 + 6(x+1)^2}{(x+1)^6} = \frac{6\cancel{(x+1)^2}}{(x+1)^{\not 6}} = \frac{6}{(x+1)^4} > 0$$

$$\therefore \quad \underline{f''(x) > 0 \quad \therefore \quad f(x) \quad \text{is always concave} \quad \cup}$$

Sign of $f'(x)$ is (the second derivative is always positive)

x		-1	
-2	$-$	$-$	$-$
$(x+1)^3$	$-$	0	$+$
$f'(x)$	$+$	n.d.	$-$
f(x)	$\nearrow$	n.d.	$\searrow$

8. *Summary of the important points:*

defined:	$x \neq -1$
y-axis crossing point $f(0)$:	$y = 1$
zeros:	(none)
v. asymptote:	$x = -1$
h. asymptote:	$y = 0$

9. *Graphical representation:* see Fig. 3.29. The discontinuity point ($x = -1$) is indeed location of a vertical asymptote, as

$$f(x) = \frac{\cancel{x+1}}{(x+1)^{\cancel{2}}} \quad \{\text{if}(x \neq -1)\} = \frac{1}{(x+1)^2}$$

3.38. Given rational function

$$f(x) = \frac{(x+1)^3}{(x+1)} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = (x+1)^3 \quad \text{and} \quad Q(x) = x+1$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x+1 \neq 0 \quad \therefore \quad x_1 \neq -1$$

There is *first-order pole* of $f(x)$ at $x = -1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq -1\}$.

However, it is important to note that $x = -1$ is also *third-order zero* of this function causing pole-zero pair cancelation. Nevertheless, if $x = -1$, this function is actually not defined as

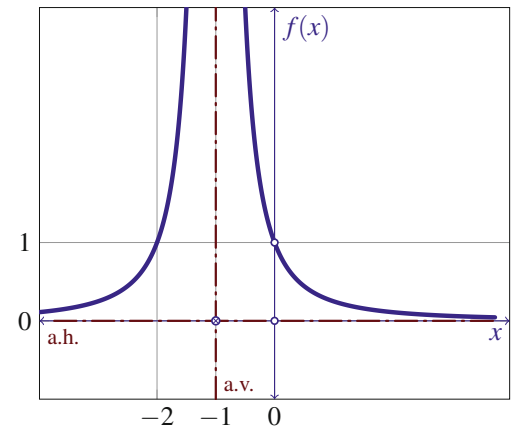


Fig. 3.29 Example P.3.37

$$f(-1) = \frac{\big((-1)+1\big)^3}{(-1)+1} = \frac{0}{0}$$

Effectively, this function is quadratic, not cubic, as

$$f(x) = \frac{(x+1)^{\cancel{3}2}}{\cancel{x+1}} = (x+1)^2 \quad \{\text{if } x \neq -1\}$$

that is to say, this is a quadratic function with very important distinction of one point missing at $x = -1$, where this function is not defined due to $0/0$ division.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\big((-x)+1\big)^3}{(-x)+1} = \frac{(-x+1)^3}{-x+1} = \frac{(-1)^3(x-1)^3}{(-1)(x-1)} = \frac{(x-1)^3}{x-1} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\frac{(x-1)^3}{x-1}\right) = -\frac{(x-1)^3}{x-1} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad (x+1)^3 = 0 \quad \therefore \quad x_{1,2,3} = -1 \quad \text{(third order zero)}$$

In general, the positivity/negativity intervals of this function are bound by its poles and zeros. However, as there is pole-zero cancellation,

x		-1	
$(x+1)^3$	$-$	0	$+$
$x+1$	$-$	0	$+$
$f(x)$	$+$	n.d.	$+$

In conclusion, $f(x) = (x+1)^3/(x+1)$ does not change its sign after passing its discontinuity point $x = -1$ where it is not defined because of $0/0$ division.

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at $(0, 1)$ as,

$$f(0) = \frac{\big((0)+1\big)^3}{(0)+1} = 1$$

In conclusion, this function crosses the vertical axis at $y = 1$.

5. *Limits:* there is discontinuity point $x = -1$ where the function is not defined, thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x+1)^{\cancel{3}2}}{\cancel{x+1}} = \lim_{x\to-\infty} (x+1)^2 = \lim_{x\to-\infty} (-\infty)^2 = +\infty$$

$$\lim_{\substack{x\to-1\\ \to}} f(x) = \lim_{\substack{x\to-1\\ \to}} \frac{(x+1)^3}{x+1} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to -1\\ \to}} \frac{\left((x+1)^3\right)'}{(x+1)'} = \lim_{\substack{x\to -1\\ \to}} \frac{3(x+1)^2}{1} = \lim_{\substack{x\to -1\\ \to}} \frac{3(0\rfloor)^2}{1} = 0\rfloor$$

$$\lim_{\substack{x\to -1\\ \leftarrow}} f(x) = \lim_{\substack{x\to -1\\ \leftarrow}} \frac{(x+1)^3}{x+1} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to -1\\ \leftarrow}} \frac{\left((x+1)^3\right)'}{(x+1)'} = \lim_{\substack{x\to -1\\ \leftarrow}} \frac{3(x+1)^2}{1} = \lim_{\substack{x\to -1\\ \leftarrow}} \frac{3(0\rfloor)^2}{1} = 0\rfloor$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{(x+1)^{\cancel{3}2}}{\cancel{x+1}} = \lim_{x\to +\infty} (x+1)^2 = \lim_{x\to +\infty} (+\infty)^2 = 0_+$$

In conclusion, $f(x) = (x-1)^3/(x+1)$ is not defined at $x = -1$. While approaching its discontinuity point ($x \to -1$), this function limits to $0\rfloor$, because both left-side and right-side limits are equal. (Note: the square of both negative and positive numbers is positive.)

6. *Oblique asymptote:* while ($x \to \infty$), this function is well defined, and being far away from ($x = -1$) point, division $(x+1)/(x+1) = 1$ is valid.

$$f(x) = \frac{(x+1)^{\cancel{3}2}}{\cancel{x+1}} = (x+1)^2 \quad \therefore$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(x+1)^2}{x} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\left((x+1)^2\right)'}{(x+1)'} = \lim_{x\to\infty} 2(x+1) = \infty$$

In conclusion, oblique asymptote $y_{\text{aa}} = ax + b$ does not exist because $a = \infty$.

7. *Critical points:* $\{\text{if } (x \neq -1)\}$

$$\underline{f'(x) = 0:} \quad f'(x) = \left((x+1)^2\right)' = 2(x+1) \quad \therefore \quad f'(x) = 0 \quad \therefore \quad 2(x+1) = 0$$

$$\therefore \quad x_1 = -1, \quad y_1 = f(-1) = \text{n.d.}$$

$$\underline{f''(x) = 0:} \quad f''(x) = (2(x+1))' = 2 > 0 \quad \therefore \quad \underline{f(x) \quad \text{is always concave} \quad \cup}$$

Thus, sign of $f'(x)$ is:

x		-1	
2	+	+	+
$x+1$	−	0	+
$f'(x)$	−	0	+
f(x)	↘	n.d.	↗

that is to say, the only possible minimum of this function is located at the point where the function is not defined. The second derivative is always positive.

8. *Summary of the important points:*

defined:	$x \neq -1$
zeros:	(none)
y-axis crossing point $f(0)$:	$y = 1$
v. asymptote:	(none)
h. asymptote:	(none)

9. *Graphical representation:* results of the analysis for function in P.3.38 are summarized by $f(x)$ graph (see Fig. 3.30). The discontinuity point $x = -1$ does not contain vertical asymptote, because the binomial term $(x + 1)$ in the denominator is cancelled. Equally, the extreme point is also missing. The absence of $f(-1)$ point is indicated by the left/right side limiting arrows and $\otimes$ symbol, as

$$f(x) = \frac{(x+1)^{\cancel{2}}}{\cancel{x+1}} \quad \{\text{if } (x \neq -1)\} = (x+1)^2$$

3.39. Given rational function

$$f(x) = \frac{(x+1)^3}{(x+1)^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = (x+1)^3 \quad \text{and} \quad Q(x) = (x+1)^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x+1)^2 \neq 0 \quad \therefore \quad x_{1,2} \neq -1$$

There is *second-order pole* of $f(x)$ at $x = -1$; thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R} | x \neq -1\}$.

However, it is important to note that $x = -1$ is also *third-order zero* of this function causing pole-zero pair cancelation. Nevertheless, if $x = -1$, this function is actually not defined as

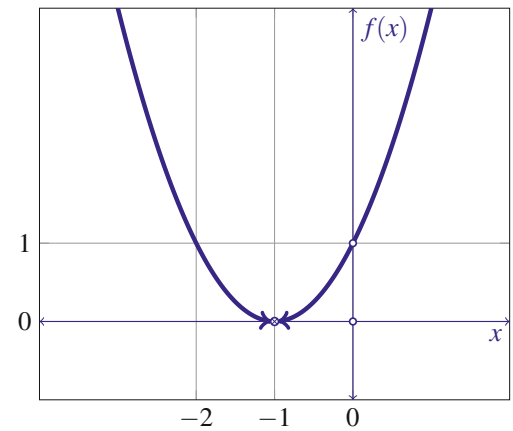


Fig. 3.30 Example P.3.38

$$f(-1) = \frac{\big((-1)+1\big)^3}{\big((-1)+1\big)^2} = \frac{0}{0}$$

Effectively, this function is linear, not cubic, as

$$f(x) = \frac{(x+1)^{\cancel{3}}}{\cancel{\big((-1)+1\big)^2}} = x+1 \quad \{\text{if } x \neq -1\}$$

that is to say, this is a linear function with very important distinction of one point missing at $x = -1$, where this function is not defined due to its $0/0$ form.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\big((-x)+1\big)^3}{\big((-1)+1\big)^2} = \frac{(-x+1)^3}{(-x+1)^2} = \frac{(-1)^3(x-1)^3}{(-1)^2(x-1)^2} = -\frac{(x-1)^3}{(x-1)^2} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{(x-1)^3}{(x-1)^2}\right) = \frac{(x-1)^3}{(x-1)^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad (x+1)^3 = 0 \quad \therefore \quad x_{1,2,3} = -1 \quad \text{(third order zero)}$$

In general, the positivity/negativity intervals of this function are bound by its poles and zeros. However, as there is pole-zero cancellation, we find:

x		-1	
$(x+1)^3$	$-$	0	$+$
$(x+1)^2$	$+$	0	$+$
$f(x)$	$-$	n.d.	$+$

In conclusion, $f(x) = (x+1)^3/(x+1)^2$ changes its sign after passing its discontinuity point $x = -1$ where it is not defined because of $0/0$ division.

4. *y-axis crossing point* $f(0)$: this function crosses the y-axis at (0, 1) as,

$$f(0) = \frac{\big((0)+1\big)^3}{\big((0)+1\big)^2} = 1$$

In conclusion, this function crosses the vertical axis at $y = 1$.

5. *Limits:* there is discontinuity point $x = -1$, where the function is not defined, thus

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{(x+1)^{\cancel{3}}}{\cancel{(x+1)^2}} = \lim_{x\to-\infty} x+1 = \lim_{x\to-\infty} -\infty = -\infty$$

$$\lim_{x\to -1_-} f(x) = \lim_{x\to -1_-} \frac{(x+1)^3}{(x+1)^2} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to -1_-} \frac{\left((x+1)^3\right)'}{\left((x+1)^2\right)'} = \lim_{x\to -1_-} \frac{3(x+1)^2}{2(x+1)} = \lim_{x\to -1_-} \frac{3(0_-)^2}{2(0_-)} = 0_+$$

$$\lim_{x\to -1_+} f(x) = \lim_{x\to -1_+} \frac{(x+1)^3}{(x+1)^2} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to -1_+} \frac{\left((x+1)^3\right)'}{\left((x+1)^2\right)'} = \lim_{x\to -1_+} \frac{3(x+1)^2}{2(x+1)} = \lim_{x\to -1_+} \frac{3(0_+)^2}{2(0_+)} = 0_+$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{(x+1)^{\cancel{3}}}{\cancel{(x+1)^2}} = \lim_{x\to +\infty} (x+1) = \lim_{x\to +\infty} +\infty = 0_+$$

In conclusion, $f(x) = (x-1)^3/(x+1)^2$ is not defined at $x = -1$. While approaching its discontinuity point ($x \to -1$), this function limits to 0_+, where the left-side and right-side limits are equal.

6. *Oblique asymptote:* while ($x \to \infty$) this function is well defined and, being far away from ($x = -1$) point, division $(x+1)/(x+1) = 1$ is valid.

$$f(x) = \frac{(x+1)^{\cancel{3}}}{\cancel{(x+1)^2}} = x+1 \quad \therefore$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x+1}{x} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(x+1)'}{(x+1)'} = \lim_{x\to\infty} 1 = 1$$

and,

$$b = \lim_{x\to\infty} \Big(f(x) - 1\,x\Big) = \lim_{x\to\infty} ((\cancel{x}+1) - \cancel{x}) = \lim_{x\to\infty} 1 = 1$$

There is oblique asymptote $y_{\texttt{aa}} = x+1$, which is the same form of linear function as $f(x)$ under condition ($x \neq -1$) (this is trivial case).

7. *Critical points:* $\{\text{if } (x \neq -1)\}$

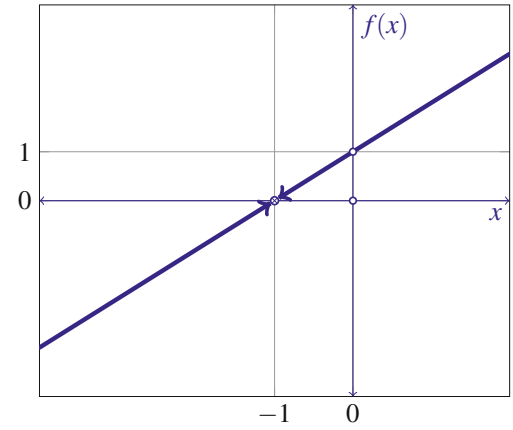
$$f'(x) = 0: \quad f'(x) = (x+1)' = 1 > 0 \quad \therefore \quad f(x) \quad \text{(always ascending} \quad \nearrow)$$

$$f''(x) = 0: \quad f''(x) = (1)' = 0 \quad \therefore \quad f(x) \quad \text{(neither concave nor convex)}$$

8. *Summary of the important points:*

defined:	$x \neq -1$
y-axis crossing point $f(0)$:	$y = 1$
zeros:	(none)

Fig. 3.31 Example P.3.39



v. asymptote: (none)

h. asymptote: $y = 1$ (the function itself)

9. *Graphical representation:* results of the analysis for function in P.3.39 are summarized by $f(x)$ graph; see Fig. 3.31. The discontinuity point $x = -1$ does not contain vertical asymptote, because the binomial term $(x+1)$ in the denominator is cancelled. The absence of $f(-1)$ point is indicated by the left-/right-side-limiting arrows and $\otimes$ symbol, as

$$f(x) = \frac{(x+1)^{\cancel{3}}}{\cancel{(x+1)^2}} \quad \{\text{if } (x \neq -1)\} = x + 1$$

3.40. Given rational function

$$f(x) = \frac{(x+1)^3}{(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = (x+1)^3 \quad \text{and} \quad Q(x) = (x+1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x+1)^3 \neq 0 \quad \therefore \quad x_{1,2,3} \neq -1$$

There is *third-order pole* of $f(x)$ at $(x = -1)$, thus $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R} | x \neq -1\}$.

However, it is important to note that $(x = -1)$ is also *third-order zero* of this function resulting in pole-zero cancelation of all three pairs. Nevertheless, if $(x = -1)$, this function is actually not defined as

$$f(-1) = \frac{\big((-1)+1\big)^3}{\big((-1)+1\big)^3} = \frac{0}{0}$$

Effectively, this function is constant, not cubic, as

$$f(x) = \frac{\cancel{(x+1)^3}}{\cancel{(x+1)^3}} = 1$$

that is to say, this is a constant function $f(x) = 1$ with very important distinction of one point missing at $(x = -1)$, where this function is not defined due to its $0/0$ form.

2. *Summary of the important points:*

defined:	$x \neq -1$
zeros:	(none)
y-axis crossing point $f(0)$:	$y = 1$
v. asymptote:	(none)
h. asymptote:	$y = 1$ (the function itself)

3. *Graphical representation:* results of the analysis for function in P.3.40 are summarized by $f(x)$ graph; see Fig. 3.32. The discontinuity point $(x = -1)$ does not contain vertical asymptote, because the binomial term $(x + 1)$ in the denominator is cancelled. The absence of $f(-1)$ point is indicated by the left/right side limiting arrows and $\otimes$ symbol.

3.41. Given rational function

$$f(x) = \frac{x}{x(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x \quad \text{and} \quad Q(x) = x(x+1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x(x+1)^3 \neq 0 \quad \therefore \quad x_1 \neq 0,\ x_{2,3,4} \neq -1$$

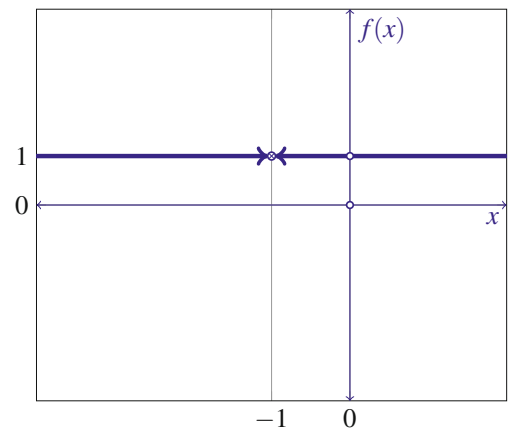


Fig. 3.32 Example P.3.40

There is *first-order pole* of $f(x)$ at $x = 0$, as well as *third-order pole* at $(x = -1)$. Thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq 0, x \neq -1\}$.

It is important to note that $x = 0$ is also *first-order zero* of this function causing pole-zero pair cancelation at $(x = 0)$. Nevertheless, whether $(x = 0)$ or $(x = -1)$, $f(x)$ is not defined as

$$f(0) = \frac{(0)}{(0)\left((0)+1\right)^3} = \frac{0}{0}$$

$$f(-1) = \frac{(-1)}{(-1)\left((-1)+1\right)^3} = \frac{-1}{0} = \infty$$

It is important to note slightly different nature of these two discontinuity points: due to pole-zero cancelation at $(x = 0)$, there is no vertical asymptote, while at $(x = -1)$, function tends to infinity due to the existence of vertical asymptote.

Effectively, (for $x \neq 0$)

$$f(x) = \frac{\cancel{x}}{\cancel{x}(x+1)^3} = \frac{1}{(x+1)^3}$$

That is to say, this is an inverse cubic function delayed by "1" with very important distinction: one point is missing at $x = 0$ where this function is not defined due to $0/0$ division.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{(-x)}{(-x)\left((-x)+1\right)^3} = \frac{-x}{-x\,(-1)^3(x-1)^3} = -\frac{x}{x\,(x-1)^3} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{x}{x\,(x-1)^3}\right) = \frac{x}{x\,(x-1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x = 0 \quad \text{(first-order zero)}$$

After taking into account all poles and zero:

x		-1		0	
x	$-$	$-$	$-$	0	$+$
$x(x+1)^3$	$+$	0	$-$	0	$+$
$f(x)$	$-$	n.d.	$+$	n.d.	$+$

In conclusion, $f(x) = x/x(x+1)^3$ is affected by the pole-zero cancelled pair at $(x = 0)$ as well as by its non-cancelled pole at $(x = -1)$.

4. *y-axis crossing point* $f(0)$: this function is not defined at $f(0)$.
5. *Limits:* with two discontinuity points, there are six limits in total to be resolved.

$\underline{x \to \infty:}$ $(x \neq -1)$ and, $(x \neq 0)$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{x}}{\cancel{x}(x+1)^3} = \lim_{x\to-\infty} \frac{1}{(x+1)^3} = \lim_{x\to-\infty} \frac{1}{(-\infty)^3} = 0\downarrow$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x}}{\cancel{x}(x+1)^3} = \lim_{x\to+\infty} \frac{1}{(x+1)^3} = \lim_{x\to+\infty} \frac{1}{(+\infty)^3} = 0\uparrow$$

$\underline{x \to -1:}$ $(x \neq 0)$

$$\lim_{\substack{x\to-1\\ \to}} f(x) = \lim_{\substack{x\to-1\\ \to}} \frac{\cancel{(-1)}\,1}{\cancel{(-1)}(x+1)^3} = \lim_{\substack{x\to-1\\ \to}} \frac{1}{(0)^3} = -\infty$$

$$\lim_{\substack{x\to-1\\ \leftarrow}} f(x) = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{\cancel{(-1)}\,1}{\cancel{(-1)}(x+1)^3} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{1}{(0)^3} = +\infty$$

$\underline{x \to 0:}$ $(x \neq -1)$

$$\lim_{\substack{x\to0\\ \to}} f(x) = \lim_{\substack{x\to0\\ \to}} \frac{x}{x\,\cancel{(x+1)^3}^{\,1}} = \lim_{\substack{x\to0\\ \to}} \frac{x}{x} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to0\\ \to}} \frac{(x)'}{(x)'} = \lim_{\substack{x\to0\\ \to}} \frac{\frac{1}{\to}}{\frac{1}{\to}} = 1\downarrow$$

$$\lim_{\substack{x\to0\\ \leftarrow}} f(x) = \lim_{\substack{x\to0\\ \leftarrow}} \frac{x}{x\,\cancel{(x+1)^3}^{\,1}} = \lim_{\substack{x\to0\\ \leftarrow}} \frac{x}{x} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to0\\ \leftarrow}} \frac{(x)'}{(x)'} = \lim_{\substack{x\to0\\ \leftarrow}} \frac{\frac{1}{\leftarrow}}{\frac{1}{\leftarrow}} = 1\uparrow$$

In conclusion, $f(x) = x/x(x-1)^3$ is not defined neither at $(x = -1)$ nor $(x = 0)$. However, due to pole-zero cancellation, as $(x \to 0)$ both sides limits are equal, by consequence $\lim_{x\to0} f(x) = 1$. In addition, there is a horizontal asymptote at $y = 0$.

6. *Oblique asymptote:* while $(x \to \infty)$, this function is well defined, and being far away from $(x = -1)$ and $(x = 0)$ discontinuities, division $x/x = 1$ is valid; at the same time, $1/(x+1)^3$ term is defined; thus

$$f(x) = \frac{\cancel{x}\,1}{\cancel{x}(x+1)^3} = \frac{1}{(x+1)^3} \quad \therefore$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,(x+1)^3} = \lim_{x\to\infty} \frac{1}{\infty} = 0$$

In conclusion, oblique asymptote $y_{\texttt{aa}} = ax + b$ does not exist because $a = 0$.

7. *Critical points:* $\{\text{if } (x \neq -1),\ (x \neq 0)\}$

$\underline{f'(x) = 0:}$

$$f'(x) = \left(\frac{1}{(x+1)^3}\right)' = \frac{-3\cancel{(x+1)^2}}{(x+1)^{\cancel{6}4}} = -\frac{3}{(x+1)^4}$$

$$\therefore \quad f'(x) < 0 \quad \therefore \quad f(x) \quad \text{always descents} \quad \searrow$$

$$f''(x) = 0: \quad f''(x) = \left(-\frac{3}{(x+1)^4}\right)' = \frac{12\cancel{(x+1)^3}}{(x+1)^{\cancel{8}5}} = \frac{12}{(x+1)^5}$$

Sign of $f''(x)$ is

x		-1		0	
12	$+$	$+$	$+$	$+$	$+$
$(x+1)^5$	$-$	0	$+$	$+$	$+$
$f''(x)$	$-$	n.d.	$+$	$+$	$+$
$f(x)$	$\cap$	n.d.	$\cup$	n.d.	$\cup$

8. *Summary of the important points:*

defined: $x \neq -1$ and, $x \neq 0$

y-axis crossing point $f(0)$: (none)

zeros: (none)

v. asymptote: $x = -1$

h. asymptote: $y = 0$

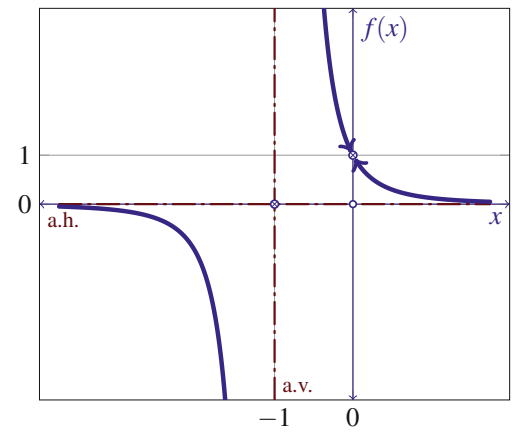
9. *Graphical representation:* see Fig. 3.33. The discontinuity point ($x = -1$) is indeed location of a vertical asymptote, as

$$f(x) == \frac{\cancel{x}}{\cancel{x}(x+1)^3} \{\text{if } x \neq 0\} = \frac{1}{(x+1)^3}$$

3.42. Given rational function

$$f(x) = \frac{x(x+1)}{x(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x(x+1) \quad \text{and} \quad Q(x) = x(x+1)^3$$

Fig. 3.33 Example P.3.41



1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \;\; \therefore \;\; x(x+1)^3 \neq 0 \;\; \therefore \;\; x_1 \neq 0,\; x_{2,3,4} \neq -1$$

There is *first-order pole* of $f(x)$ at $x = 0$, as well as *third-order pole* at $(x = -1)$. Thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R} | x \neq 0, x \neq -1\}$.

It is important to note that both $(x = -1)$ and $(x = 0)$ are also *first-order zeros* of this function causing pole-zero pair cancelations at both points. Nevertheless, whether $(x = 0)$ or $(x = -1)$, then $f(x)$ is not defined as

$$f(0) = \frac{(0)\left((0)+1\right)}{(0)\left((0)+1\right)^3} = \frac{0}{0} \;\; \text{n.d.}$$

$$f(-1) = \frac{(-1)\left((-1)+1\right)}{(-1)\left((-1)+1\right)^3} = \frac{0}{0} \;\; \text{n.d.}$$

It is important to note slightly different natures of these two discontinuity points: due to pole-zero cancelation at $(x = 0)$, there is no vertical asymptote, while at $(x = -1)$, function tends to infinity due to the remaining poles, thus the existence of vertical asymptote.

Effectively, (for $(x \neq 0), \;\; (x \neq -1)$)

$$f(x) = \frac{\cancel{x(x+1)}}{\cancel{x}(x+1)^{\cancel{3}2}} = \frac{1}{(x+1)^2}$$

That is to say, this is an inverse quadratic function with very important distinction: one point is missing at $f(0)$ where this function is not defined due to $0/0$ division.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{(-x)\left((-x)+1\right)}{(-x)\left((-x)+1\right)^3} = \frac{x\,(x-1)}{x\,(x-1)^3} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \;\; -f(-x) = -\left(\frac{x\,(x-1)}{x\,(x-1)^3}\right) = -\frac{x\,(x-1)}{x\,(x-1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; P(x) = 0 \;\; \therefore \;\; x(x+1) = 0$$

$$\therefore \;\; x_1 = 0, \;\; x_2 = -1 \quad \text{(two first-order zeros)}$$

After taking into account all poles and zero:

x		-1		0	
x	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$x(x+1)^3$	$+$	0	$-$	0	$+$
$f(x)$	$+$	n.d.	$+$	n.d.	$+$

In conclusion, $f(x) = x(x+1)/x(x+1)^3$ is always positive (when defined).

4. *y-axis crossing point* $f(0)$: this function is not defined at $f(0)$.
5. *Limits:* with two discontinuity points, there are six limits in total to be resolved.

$$\underline{x \to \infty:} \quad (x \neq -1) \quad \text{and,} \quad (x \neq 0)$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{x(x+1)}}{\cancel{x}(x+1)^{\cancel{2}}} = \lim_{x\to-\infty} \frac{1}{(x+1)^2} = \lim_{x\to-\infty} \frac{1}{(-\infty)^2} = 0\downarrow$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x(x+1)}}{\cancel{x}(x+1)^{\cancel{2}}} = \lim_{x\to+\infty} \frac{1}{(x+1)^2} = \lim_{x\to+\infty} \frac{1}{(+\infty)^2} = 0\downarrow$$

$$\underline{x \to -1:} \quad (x \neq 0)$$

$$\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{\cancel{(-1)(x+1)}}{\cancel{(-1)}(x+1)^{\cancel{2}}} = \lim_{x\to-1} \frac{1}{(0)^2} = +\infty$$

$$\lim_{x\to-1} f(x) = \lim_{x\to-1} \frac{\cancel{(-1)(x+1)}}{\cancel{(-1)}(x+1)^{\cancel{2}}} = \lim_{x\to-1} \frac{1}{(0)^2} = +\infty$$

$$\underline{x \to 0:} \quad (x \neq -1)$$

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{\cancel{x(x+1)}}{x\cancel{(x+1)^2}^{1}} = \lim_{x\to0} \frac{x}{x} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to0} \frac{(x)'}{(x)'} = \lim_{x\to0} \frac{1}{1} = 1\downarrow$$

$$\lim_{x\to0} f(x) = \lim_{x\to0} \frac{\cancel{x(x+1)}}{x\cancel{(x+1)^2}^{1}} = \lim_{x\to0} \frac{x}{x} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to0} \frac{(x)'}{(x)'} = \lim_{x\to0} \frac{1}{1} = 1\uparrow$$

In conclusion, $f(x) = x(x+1)/x(x-1)^3$ is not defined neither at $(x = -1)$ nor $(x = 0)$. However, due to pole-zero cancellation, as $(x \to 0)$, both sides limits are equal, and by consequence, $\lim_{x\to0} f(x) = 1$. In addition, there is a horizontal asymptote at $y = 0$.

6. *Oblique asymptote:* while $(x \to \infty)$, this function is well defined, and being far away from $(x = -1)$ and $(x = 0)$ discontinuities, division $x/x = 1$ is valid, and at the same time, $1/(x+1)^3$ term is defined; thus

$$f(x) = \frac{\cancel{x(x+1)}}{\cancel{x}(x+1)^{\not 2}} = \frac{1}{(x+1)^2} \quad \therefore$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,(x+1)^2} = \lim_{x\to\infty} \frac{1}{\infty} = 0$$

As $a = 0$ oblique asymptote $y_{\mathtt{aa}} = ax + b$ does not exist.

7. *Critical points:* $\{\text{if } (x \neq -1),\ (x \neq 0)\}$

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{1}{(x+1)^2}\right)' = \frac{-2\cancel{(x+1)}}{(x+1)^{\not 3}} = -\frac{2}{(x+1)^3}$$

$$\therefore \quad f'(x) \neq 0$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-\frac{2}{(x+1)^3}\right)' = \frac{6\cancel{(x+1)^2}}{(x+1)^{\not 4}} = \frac{6}{(x+1)^4} > 0$$

$\therefore$ $f(x)$ is always concave $\cup$

Sign of $f'(x)$ is

x		-1		0	
-2	$-$	$-$	$-$	$-$	$-$
$(x+1)^3$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	n.d.	$-$	$-$	$-$
$f(x)$	$\nearrow$	n.d.	$\searrow$	n.d.	$\searrow$

that is to say, the sign of $f'(x)$ changes after passing $x = -1$ vertical asymptote, but is not affected by passing $x = 0$ discontinuity where the pole-zero cancellation occurred.

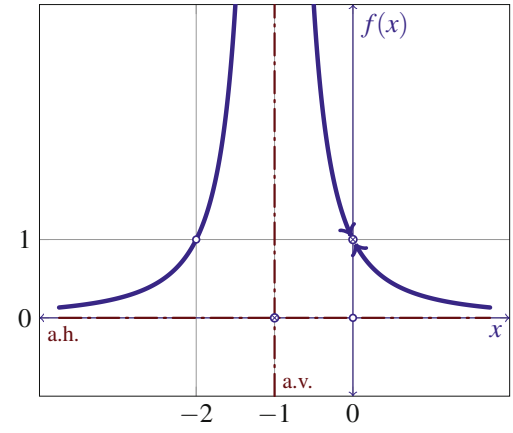
8. *Summary of the important points:*

defined:	$x \neq -1$ and, $x \neq 0$
y-axis crossing point $f(0)$:	(none)
zeros:	(none)
v. asymptote:	$x = -1$
h. asymptote:	$y = 0$

9. *Graphical representation:* results of the analysis for function in P.3.42 are summarized by $f(x)$ graph; see Fig. 3.34. The discontinuity point $(x = -1)$ belongs to vertical asymptote, while $f(0)$ tends to one. The absence of $f(0)$ point is indicated by the left/right side limiting arrows and $\otimes$ symbol, as

$$f(x) = \frac{\cancel{x(x+1)}}{\cancel{x}(x+1)^{\not 2}} \ \{\text{if } x \neq 0\} = \frac{1}{(x+1)^2}$$

Fig. 3.34 Example P.3.42



3.43. Given rational function

$$f(x) = \frac{x(x+1)^2}{x(x+1)^3} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x(x+1)^2 \quad \text{and} \quad Q(x) = x(x+1)^3$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad x(x+1)^3 \neq 0 \quad \therefore \quad x_1 \neq 0,\ x_{2,3,4} \neq -1$$

There is *first-order pole* of $f(x)$ at $x = 0$, as well as *third-order pole* at $(x = -1)$. Thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R} | x \neq 0, x \neq -1\}$.

It is important to note that $(x = -1)$ is also *second-order zero* as well as $(x = 0)$ being *first-order zero* of this function, thus causing respective pole-zero pair cancelations at both points. Nevertheless, whether $(x = 0)$ or $(x = -1)$, then $f(x)$ is not defined as

$$f(0) = \frac{(0)\left((0)+1\right)^2}{(0)\left((0)+1\right)^3} = \frac{0}{0} \quad \text{n.d.}$$

$$f(-1) = \frac{(-1)\left((-1)+1\right)^2}{(-1)\left((-1)+1\right)^3} = \frac{0}{0} \quad \text{n.d.}$$

It is important to note slightly different natures of these two discontinuity points: due to pole-zero cancelation at $(x = 0)$, there is no vertical asymptote (that pole-zero pair is cancelled). At the same time, even though one pole-zero pair is cancelled, at $(x = -1)$, function tends to infinity due to the remaining poles, thus the existence of vertical asymptote.

Effectively, (for $(x \neq 0)$, $(x \neq -1)$)

$$f(x) = \frac{\cancel{x}(x+1)^{\cancel{2}}}{\cancel{x}(x+1)^{\cancel{3}}} = \frac{1}{(x+1)}$$

That is to say, this is an inverse function delayed by "1" with very important distinction: there is one point missing at $f(0)$ where this function is not defined due to $0/0$ division.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{(-x)\left((-x)+1\right)^2}{(-x)\left((-x)+1\right)^3} = -\frac{x\,(x-1)^2}{x\,(x-1)^3} \neq f(x)$$

$$\Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{x\,(x-1)^2}{x\,(x-1)^3}\right) = \frac{x\,(x-1)^2}{x\,(x-1)^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x(x+1)^2 = 0$$

$$\therefore \quad x_1 = 0, \quad x_{2,3} = -1 \quad \text{(second-order zero)}$$

After taking into account all poles and zero:

x		-1		0	
x	$-$	$-$	$-$	0	$+$
$(x+1)^2$	$+$	0	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$
$(x+1)^3$	$-$	0	$+$	$+$	$+$
$f(x)$	$-$	n.d.	$+$	n.d.	$+$

In conclusion, $f(x) = x(x+1)^2/x(x+1)^3$ is negative before and positive after its vertical asymptote ($x = -1$) (when defined). Otherwise, there are no zero points.

4. *y-axis crossing point* $f(0)$: this function is not defined at $f(0)$.
5. *Limits:* with two discontinuity points, there are six limits in total to be resolved.

$$\underline{x \to \infty:} \quad (x \neq -1) \quad \text{and,} \quad (x \neq 0)$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\cancel{x(x+1)^2}}{\cancel{x(x+1)^3}} = \lim_{x\to-\infty} \frac{1}{(x+1)} = \lim_{x\to-\infty} \frac{1}{(-\infty)} = 0\uparrow$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x(x+1)^2}}{\cancel{x(x+1)^3}} = \lim_{x\to+\infty} \frac{1}{(x+1)} = \lim_{x\to+\infty} \frac{1}{(+\infty)} = 0\downarrow$$

$$\underline{x \to -1:} \quad (x \neq 0)$$

$$\lim_{\substack{x\to-1\\ \to}} f(x) = \lim_{\substack{x\to-1\\ \to}} \frac{\cancel{(-1)(x+1)^2}}{\cancel{(-1)(x+1)^3}} = \lim_{\substack{x\to-1\\ \to}} \frac{1}{(0)} = -\infty$$

$$\lim_{\substack{x\to-1\\ \leftarrow}} f(x) = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{\cancel{(-1)(x+1)^2}}{\cancel{(-1)(x+1)^3}} = \lim_{\substack{x\to-1\\ \leftarrow}} \frac{1}{(0)} = +\infty$$

$\underline{x \to 0:}\quad (x \neq -1)$

$$\lim_{\substack{x\to 0\\ \to}} f(x) = \lim_{\substack{x\to 0\\ \to}} \frac{x(x+1)^2}{x(x+1)^3} = \lim_{\substack{x\to 0\\ \to}} \frac{x}{x} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to 0\\ \to}} \frac{(x)'}{(x)'} = \lim_{\substack{x\to 0\\ \to}} \frac{1}{1} = 1$$

$$\lim_{\substack{x\to 0\\ \leftarrow}} f(x) = \lim_{\substack{x\to 0\\ \leftarrow}} \frac{x(x+1)^2}{x(x+1)^3} = \lim_{\substack{x\to 0\\ \leftarrow}} \frac{x}{x} = \left(\frac{0}{0}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{\substack{x\to 0\\ \leftarrow}} \frac{(x)'}{(x)'} = \lim_{\substack{x\to 0\\ \leftarrow}} \frac{1}{1} = 1$$

In conclusion, $f(x) = x(x+1)^2/x(x-1)^3$ is not defined neither at $(x = -1)$ nor $(x = 0)$. However, due to pole-zero cancellation, as $(x \to 0)$, both sides limits are equal, and by consequence $\lim_{x\to 0} f(x) = 1$. In addition, there is a horizontal asymptote at $y = 0$.

6. *Oblique asymptote:* while $(x \to \infty)$, this function is well defined, and being far away from $(x = -1)$ and $(x = 0)$ discontinuities, division $x/x = 1$ is valid, and at the same time, $1/(x+1)^3$ term is defined; thus

$$f(x) = \frac{x(x+1)^2}{x(x+1)^3} = \frac{1}{x+1} \quad \therefore$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x\,(x+1)} = \lim_{x\to\infty} \frac{1}{\infty} = 0$$

As $a = 0$, oblique asymptote $y_{\mathtt{aa}} = ax + b$ does not exist.

7. *Critical points:* $\{\text{if } (x \neq -1),\ (x \neq 0)\}$

$$\underline{f'(x) = 0:}\quad f'(x) = \left(\frac{1}{x+1}\right)' = \frac{-1}{(x+1)^2} \neq 0$$

$$\therefore\ f'(x) < 0 \quad \text{always descents} \quad \searrow$$

and,

$$\underline{f''(x) = 0:}\quad f''(x) = \left(-\frac{1}{(x+1)^2}\right)' = \frac{2(x+1)}{(x+1)^4} = \frac{2}{(x+1)^3} \neq 0$$

Sign of $f''(x)$ is

x		-1		0	
2	$+$	$+$	$+$	$+$	$+$
$(x+1)^3$	$-$	0	$+$	$+$	$+$
$f''(x)$	$-$	n.d.	$+$	$+$	$+$
$f(x)$	$\cap$	n.d.	$\cup$	n.d.	$\cup$

that is to say, this function changes its form from convex ∩ to concave ∪ after passing pole at $(x = -1)$ but not after passing cancelled pole-zero pair at $(x = 0)$.

8. *Summary of the important points:*

$$\begin{aligned} \underline{\text{defined:}} \quad & x \neq -1 \quad \text{and,} \quad x \neq 0 \\ \underline{\text{zeros:}} \quad & \text{(none)} \\ \underline{\text{y-axis crossing point } f(0)\text{:}} \quad & \text{(none)} \\ \underline{\text{v. asymptote:}} \quad & x = -1 \\ \underline{\text{h. asymptote:}} \quad & y = 0 \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.43 are summarized by $f(x)$ graph (see Fig. 3.35). The discontinuity point $(x = -1)$ belongs to vertical asymptote, while $\lim_{x\to 0} f(x) = 1$. The absence of $f(0)$ point is indicated by the left/right side limiting arrows and ⊗ symbol, as

$$f(x) = \frac{\cancel{x}(x+1)^{\cancel{2}}}{\cancel{x}(x+1)^{\cancel{1}}} \left\{ \text{if } x \neq 0, \quad x \neq -1 \right\} = \frac{1}{x+1}$$

3.44. Given rational function

$$f(x) = \frac{x^3}{(x-1)^2} = \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x^3 \quad \text{and} \quad Q(x) = (x-1)^2$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$, i.e.,

$$Q(x) \neq 0 \quad \therefore \quad (x-1)^2 \neq 0 \quad \therefore \quad x_{1,2} \neq 1$$

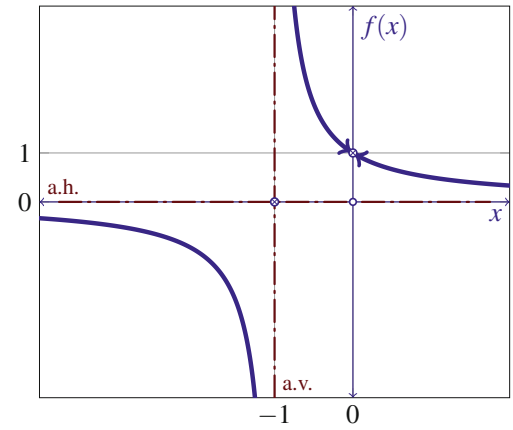


Fig. 3.35 Example P.3.43

There is *second-order pole* of $f(x)$ at $x = 1$. Thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq 1\}$, because

$$f(1) = \frac{(1)^3}{\big((1)-1\big)^2} = \frac{1}{0} = \infty$$

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{(-x)^3}{\big((-x)-1\big)^2} = \frac{-x^3}{\big(x+1\big)^2} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \; -f(-x) = -\left(-\frac{x^3}{\big(x+1\big)^2}\right) = \frac{x^3}{\big(x+1\big)^2} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x^3 = 0 \quad \therefore \quad x_{1,2,3} = 0 \quad \text{(third order zero)}$$

After taking into account all poles and zero:

x		0		1	
x^3	$-$	0	$+$	$+$	$+$
$(x-1)^2$	$+$	$+$	$+$	0	$+$
$f(x)$	$-$	0	$+$	n.d.	$+$

In conclusion, $f(x) = x^3/(x+1)^2$ is negative for $(x < 0)$ and positive thereafter (when defined).

4. *y-axis crossing point* $f(0)$: $f(0) = 0$, which is at the same time this function's zero.
5. *Limits:* with one discontinuity point, there are four limits in total to be resolved.

$$\underline{x \to \infty:} \quad (x \neq 1)$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{\big(x^3\big)'}{\big((x-1)^2\big)'} = \lim_{x\to-\infty} \frac{3x^2}{2(x-1)}$$

$$= \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\} \overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{6x}{2} = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3}{(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{3x^2}{2(x-1)} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{6x}{2} = +\infty$$

$\underline{x \to 1:}$

$$\lim_{x \to 1_{\to}} f(x) = \lim_{x \to 1_{\to}} \frac{x^3}{(x-1)^2} = \frac{1}{(0)^2_{\to}} = +\infty$$

$$\lim_{x \to 1_{\leftarrow}} f(x) = \lim_{x \to 1_{\leftarrow}} \frac{x^3}{(x-1)^2} = \frac{1}{(0)^2_{\leftarrow}} = +\infty$$

In conclusion, $f(x) = x^3/(x-1)^2$ tends to $+\infty$ on both sides of its vertical asymptote ($x = 1$) as well as when ($x \to +\infty$); otherwise, on the negative side, this function tends to $-\infty$.

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^{\cancel{3}\,2}}{\cancel{x}\,(x-1)^2} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{\cancel{2x}\,1}{\cancel{2x}\,1} = 1$$

$$b = \lim_{x \to \infty} \Big(f(x) - ax\Big) = \lim_{x \to \infty} \left(\frac{x^3}{(x-1)^2} - x\right) = \lim_{x \to \infty} \left(\frac{x^3 - x(x-1)^2}{(x-1)^2}\right)$$

$$= \lim_{x \to \infty} \left(\frac{\cancel{x^3} - \cancel{x^3} + 2x^2 - x}{(x-1)^2}\right) = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{4x-1}{2(x-1)} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{\cancel{4}\,2}{\cancel{2}} = 2$$

There is oblique asymptote $y_{\text{aa}} = x + 2$.

7. *Critical points:* $\{\text{if } (x \neq 1)\}$

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{x^3}{(x-1)^2}\right)' = \frac{3x^2(x-1)^{\cancel{2}} - 2x^3\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} = \frac{x^3 - 3x^2}{(x-1)^3} = 0$$

$$\therefore \quad x^2(x-3) = 0 \quad \therefore \quad x_{1,2} = 0, \quad x_3 = 3$$

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{x^3 - 3x^2}{(x-1)^3}\right)' = \frac{(3x^2 - 6x)(x-1)^{\cancel{3}} - 3(x^3 - 3x^2)\cancel{(x-1)^2}}{(x-1)^{\cancel{6}\,4}}$$

$$= \frac{\cancel{3x^3} - \cancel{9x^2} + 6x - \cancel{3x^3} + \cancel{9x^2}}{(x-1)^4} = \frac{6x}{(x-1)^4} = 0$$

$$\therefore \quad 6x = 0 \quad \therefore \quad x = 0$$

where,

$$f(0) = \frac{0^3}{(0-1)^2} = 0$$

$$f(3) = \frac{3^3}{(3-1)^2} = \frac{27}{4}$$

Sign of $f'(x)$ is

x		0		1		3	
x^2	+	0	+	+	+	+	+
$x-3$	−	−	−	−	−	0	+
$(x-1)^3$	−	−	−	0	+	+	+
$f'(x)$	+	0	+	n.d.	−	0	+
$f(x)$	↗	0	↗	n.d.	↘	$27/4$	↗

There is minimum point at $(x, y) = (3, 27/4)$. Sign of $f''(x)$ is

x		0		1	
$6x$	−	0	+	1	+
$(x-1)^4$	+	+	+	0	+
$f''(x)$	−	0	+	n.d.	+
$f(x)$	∩	0	∪	n.d.	∪

that is to say, this function changes its form from convex ∩ to concave ∪ after crossing zero at $(x = 0)$ but not after passing even order pole at $(x = 1)$.

8. *Summary of the important points:*

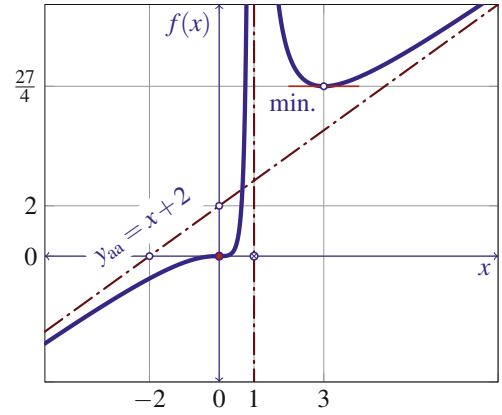
$$\begin{aligned}
\underline{\text{defined:}} &\quad x \neq 1 \\
\underline{\text{y-axis crossing point } f(0)\text{:}} &\quad (x, y) = (0, 0) \\
\underline{\text{v. asymptote:}} &\quad x = 1 \ \text{ (second order)} \\
\underline{\text{a.a. asymptote:}} &\quad y = x + 2 \\
\underline{\text{zeros:}} &\quad (x, y) = (0, 0) \\
\underline{\text{extreme:}} &\quad (x, y) = (3, 27/4) \ \text{ (min)} \\
\underline{\text{inflections:}} &\quad (x, y) = (0, 0)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.44 are summarized by $f(x)$ graph (see Fig. 3.36). The discontinuity point $(x = 1)$ belongs to vertical asymptote, and inflection point coincides with zero.

3.45. Given rational function

$$f(x) = 2 - \frac{2(1-x)}{x^2+1} = \frac{2(x^2+1) - 2(1-x)}{x^2+1} = \frac{2x^2 + \cancel{2} - \cancel{2} + 2x}{x^2+1} = \frac{2x(x+1)}{x^2+1}$$
$$= \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 2x(x+1) \quad \text{and} \quad Q(x) = x^2 + 1$$

Fig. 3.36 Example P.3.44



1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e., $Q(x) \neq 0$,

$$Q(x) = x^2 + 1 \neq 0 \quad \forall x \in \{\mathbb{R}\}$$

   There are no poles of $f(x)$, thus this function is defined for all real x.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{2(-x)\left((-x)+1\right)}{\left((-x)^2+1\right)} = \frac{2x(x-1)}{x^2+1} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\frac{2x(x-1)}{x^2+1}\right) = -\frac{2x(x-1)}{x^2+1} \neq f(x) \Rightarrow \quad \text{not odd}$$

   This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad 2x(x+1) = 0 \quad \therefore \quad x_1 = 0, \quad x_2 = -1 \quad \text{(first-order zeros)}$$

x		-1		0	
$2x$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$
x^2+1	$+$	$+$	$+$	$+$	$+$
$f(x)$	$+$	0	$-$	0	$+$

   In conclusion, $f(x) = 2x(x+1)/(x^2+1)$ is continuous; it has two first-order zeros, at $(x, y) = (-1, 0)$ and $(x, y) = (0, 0)$.
4. *y-axis crossing point* $f(0)$: $f(0) = 0$, which is at the same time this function's zero.
5. *Limits:* there are no discontinuity points; there are two limits in total to be resolved,

$$\underline{x \to \infty:}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x(x+1)}{x^2+1} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$
$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{\left(2x(x+1)\right)'}{\left(x^2+1\right)'} = \lim_{x\to-\infty} \frac{4x+2}{2x}$$
$$= \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\} \overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{4\cancel{2}}{\cancel{2}} = 2$$
$$lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{2x(x+1)}{x^2+1} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$
$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{\left(2x(x+1)\right)'}{\left(x^2+1\right)'} = \lim_{x\to+\infty} \frac{4x+2}{2x}$$
$$= \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\} \overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{4\cancel{2}}{\cancel{2}} = 2$$

As $x \to \pm\infty$, $f(x) = 2x(x+1)/(x^2+1) \to 2$, thus $y = 2$ is horizontal asymptote.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2\cancel{x}(x+1)}{\cancel{x}(x^2+1)} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1}{2x} = 0$$

As $a = 0$ there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{2x^2+2x}{x^2+1}\right)' = \frac{(4x+2)(x^2+1) - 2x(2x^2+2x)}{(x^2+1)^2}$$
$$= \frac{\cancel{4x^3} + 2x^2 + 4x + 2 - \cancel{4x^3} - 4x^2}{(x^2+1)^2} = -2\,\frac{x^2-2x-1}{(x^2+1)^2} = 0$$
$$\therefore \quad x^2 - 2x - 1 = 0 \quad \therefore \quad \underbrace{x^2 - 2x + 1}\ -1 - 1 = 0$$
$$(x-1)^2 - 2 = 0 \quad \therefore \quad (x-1)^2 = 2 \quad \therefore \quad x_{1,2} = 1 \pm \sqrt{2}$$

Coordinates of the two extremes are:

$$f(x_1) = 2 - \frac{2(1-x_1)}{x_1^2+1} = 2 - \frac{2(1-(1-\sqrt{2})}{\left(1-\sqrt{2}\right)^2+1} = 2 - \frac{2\sqrt{2}}{(1-2\sqrt{2}+2)+1}$$
$$= 2 - \frac{\cancel{2}\sqrt{2}}{\cancel{2}(2-\sqrt{2})}\ \frac{2+\sqrt{2}}{2+\sqrt{2}} = 2 - \frac{\cancel{2}\sqrt{2}+\cancel{2}1}{\cancel{2}} = 1 - \sqrt{2}$$
$$f(x_2) = 2 - \frac{2(1-x_2)}{x_2^2+1} = 2 - \frac{2(1-(1+\sqrt{2})}{\left(1+\sqrt{2}\right)^2+1} = \cdots = 1 + \sqrt{2}$$

Signs of $f'(x)$ are

x		x_1		x_2	
-2	$-$	$-$	$-$	$-$	$-$
$x - x_1$	$-$	0	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	0	$+$
$(x^2+1)^2$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	$+$	$-$
$f(x)$	$\searrow$	y_1	$\nearrow$	y_2	$\searrow$

There are two extreme points, minimum at $(x, y) = (1 - \sqrt{2}, 1 - \sqrt{2})$ and maximum at $(x, y) = (1 + \sqrt{2}, 1 + \sqrt{2})$.

$$f''(x) = 0: \quad f''(x) = \left(-2\,\frac{x^2 - 2x - 1}{(x^2+1)^2}\right)'$$

$$= -2\,\frac{(2x-2)(x^2+1)^2 - 4x(x^2-2x-1)(x^2+1)}{(x^2+1)^4}$$

$$= 4\,\frac{(x+1)(x^2-4x+1)}{(x^2+1)^3} = 0$$

$$\therefore \quad (x+1)(x^2-4x+1) = 0$$

$$\therefore \quad x_3 = -1, \quad x_{4,5} = 2 \pm \sqrt{3}$$

Coordinates of the three inflection points are:

$$f(x_3) = 2 - \frac{2(1-x_3)}{x_3^2+1} = 2 - \frac{2(1-(-1))}{(-1)^2+1} = 0 = y_3$$

$$f(x_4) = 2 - \frac{2(1-x_4)}{x_4^2+1} = 2 - \frac{2(1-(2-\sqrt{3}))}{\left(2-\sqrt{3}\right)^2+1} = 2 + \frac{2(1-\sqrt{3})}{2(2-\sqrt{3})}\,\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= 2 + \frac{1}{2}(1-\sqrt{3})(2+\sqrt{3}) = 2 - \frac{1+\sqrt{3}}{2} = \frac{3-\sqrt{3}}{2} = y_4$$

$$f(x_5) = 2 - \frac{2(1-x_5)}{x_5^2+1} = 2 - \frac{2(1-(2+\sqrt{3}))}{\left(2+\sqrt{3}\right)^2+1} = \cdots = 2 - \frac{1-\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2} = y_5$$

Signs of $f''(x)$ are

x		x_3		x_4		x_5	
$x - x_3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - x_4$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x - x_5$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x^2+1)^3$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$f''(x)$	$-$	0	$+$	0	$+$	0	$+$
$f(x)$	$\cap$	y_3	$\cup$	y_4	$\cap$	y_5	$\cup$

There are three inflection points, $(x, y) = (x_3, y_3)$, $(x, y) = (x_4, y_4)$, and $(x, y) = (x_5, y_5)$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{y-axis crossing point } f(0)}: &\quad (x, y) = (0, 0)\\
\underline{\text{h. asymptotes}}: &\quad y = 2\\
\underline{\text{zeros}}: &\quad (x, y) = (-1, 0)\\
&\quad (x, y) = (0, 0)\\
\underline{\text{extremes}}: &\quad (x, y) = (1 - \sqrt{2}, 1 - \sqrt{2}) \ \text{(min)}\\
&\quad (x, y) = (1 + \sqrt{2}, 1 + \sqrt{2}) \ \text{(max)}\\
\underline{\text{inflections}}: &\quad (x, y) = (-1, 0)\\
&\quad (x, y) = \left(2 - \sqrt{3}, \frac{3 - \sqrt{3}}{2}\right)\\
&\quad (x, y) = \left(2 + \sqrt{3}, \frac{3 + \sqrt{3}}{2}\right)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.45 are summarized by $f(x)$ graph (see Fig. 3.37): complete function plot (left) and zoom-in around its inflection points (right).

3.46. Given rational function

$$\begin{aligned}
f(x) &= \frac{x}{(x+1)(x-4)}\\
&= \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = x \quad \text{and} \quad Q(x) = (x+1)(x-4)
\end{aligned}$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, i.e.,

$$Q(x) = (x+1)(x-4) \neq 0 \quad \therefore \quad (x+1)(x-4) \neq 0 \quad \therefore \quad x_1 \neq -1, \quad x_2 \neq 4$$

There are two *first-order poles* of $f(x)$ at $x = -1$ and at $(x = 4)$. Thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R} | x \neq -1, x \neq 4\}$,

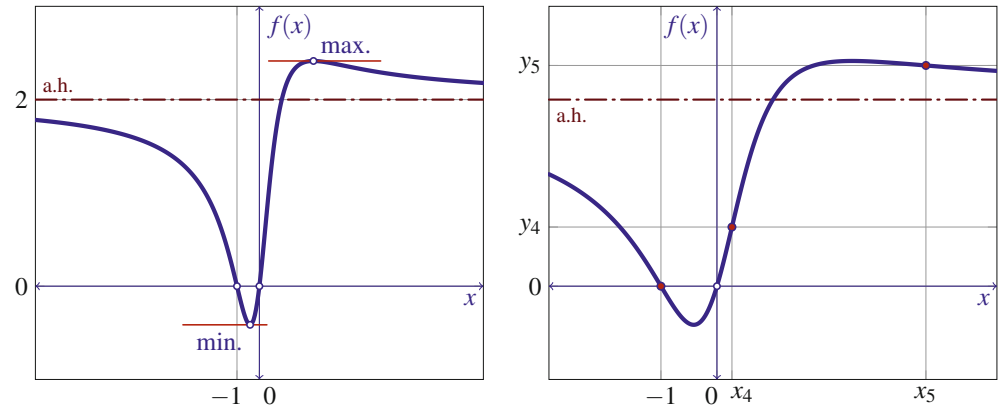


Fig. 3.37 Example P.3.45

as

$$f(-1) = \frac{(-1)}{((-1)+1)((-1)-4)} = \frac{-1}{0} = \infty$$
$$f(4) = \frac{(4)}{((4)+1)((4)-4)} = \frac{4}{0} = \infty$$

In summary, there are two vertical asymptotes at $x = -1$ and $x = 4$.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{(-x)}{((-x)+1)((-x)-4)} = -\frac{x}{(x-1)(x+4)} \neq f(x) \Rightarrow \quad \text{not even}$$
$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{x}{(x-1)(x+4)}\right) = \frac{x}{(x-1)(x+4)} \neq f(x) \Rightarrow \quad \text{not odd}$$

This function is not odd because $f(x) \neq -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad x = 0 \quad \therefore \quad x_1 = 0, \qquad \text{(first-order zero)}$$

After taking into account all poles and zero:

x		-1		0		4	
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	n.d.	$+$	0	$-$	n.d.	$+$

In conclusion, $f(x) = x/(x+1)(x-4)$ has two vertical asymptotes, one first-order zero at $(x, y) = (0, 0)$, and it changes sign every time after passing either pole or zero.

4. *y-axis crossing point* $f(0)$: $f(0) = 0$, which is at the same time this function's zero.
5. *Limits:* there are two discontinuity points and thus six limits in total to be resolved.

$\underline{x \to \infty:}$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{(x+1)(x-4)} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$
$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{1}{2x-3} = \lim_{x\to-\infty} \frac{1}{-\infty} = 0\uparrow$$
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{(x+1)(x-4)} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$
$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{1}{2x-3} = \lim_{x\to+\infty} \frac{1}{+\infty} = 0\downarrow$$

$\underline{x \to -1:}$

$$\lim_{x\to-1\atop\to} f(x) = \lim_{x\to-1\atop\to} \frac{x}{(x+1)(x-4)} = \lim_{x\to-1\atop\to} \frac{-1}{(x+1)(-5)}$$
$$= \lim_{x\to-1\atop\to} \frac{1}{0\atop\to} = -\infty$$
$$\lim_{x\to-1\atop\leftarrow} f(x) = \lim_{x\to-1\atop\leftarrow} \frac{x}{(x+1)(x-4)} = \lim_{x\to-1\atop\leftarrow} \frac{-1}{(x+1)(-5)}$$
$$= \lim_{x\to-1\atop\leftarrow} \frac{1}{0\atop\leftarrow} = +\infty$$

$\underline{x \to 4:}$

$$\lim_{x\to 4\atop\to} f(x) = \lim_{x\to 4\atop\to} \frac{x}{(x+1)(x-4)} = \lim_{x\to 4\atop\to} \frac{4}{(5)(x-4)}$$
$$= \lim_{x\to 4\atop\to} \frac{4}{0\atop\to} = -\infty$$
$$\lim_{x\to 4\atop\leftarrow} f(x) = \lim_{x\to 4\atop\leftarrow} \frac{x}{(x+1)(x-4)} = \lim_{x\to 4\atop\leftarrow} \frac{4}{(5)(x-4)}$$
$$= \lim_{x\to 4\atop\leftarrow} \frac{4}{0\atop\leftarrow} = +\infty$$

In conclusion, as $x \to \pm\infty$, $f(x) = x/(x+1)(x-4) \to 0$; in other words, $y = 0$ is horizontal asymptote.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}(x+1)(x-4)} = \lim_{x\to\infty} \frac{1}{\infty} = 0$$

In conclusion, as $a = 0$ consequently there is no oblique asymptote.

7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = \left(\frac{x}{(x+1)(x-4)}\right)' = \frac{(x+1)(x-4) - x(2x-3)}{(x+1)^2(x-4)^2}$$

$$= -\frac{x^2+4}{(x+1)^2(x-4)^2} < 0 \quad \therefore \quad f(x) \quad \text{always descends} \quad \searrow$$

because, in addition to one negative sign, each of the three factors in $f'(x)$ are positive (note the squares). And,

$$f''(x) = 0: \quad f''(x) = \left(-\frac{x^2+4}{(x+1)^2(x-4)^2}\right)'$$

$$= -\frac{2x(x+1)^2(x-4)^2 - (x^2+4)(4x^3-18x^2+2x+24)}{(x+1)^4(x-4)^4}$$

$$= -\frac{2x(x+1)^{\cancel{2}}(x-4)^{\cancel{2}} - 2(x^2+4)\cancel{(x+1)(x-4)}(2x-3)}{(x+1)^{\cancel{4}3}(x-4)^{\cancel{4}3}}$$

$$= \frac{2(x^3+12x-12)}{(x+1)^3(x-4)^3}$$

Cubic function $P_3(x) = x^3 + 12x - 12$ must be resolved by graphical/numerical methods. It is a third-order polynomial, and therefore there must be at least one real zero.

Reminder: Complex zeros always come as complex conjugate pairs, which in this case leaves the third zero without its associated companion.

Graphical method: first, a simple graphics method is sufficient to evaluate approximately x, after rearranging the root equation

$$x^3 + 12x - 12 = 0 \quad \therefore \quad x^3 = -12x + 12 = 12(1-x)$$

which is to say that the intersect between a simple cubic function on the left and linear function on the right is the only common point between the two, that is to say at x_0 (see Fig. 3.38 (left)). Evidently, in the first iteration $x_0 \approx 1$, (note: $1^3 = 1$).

Numerical method: after taking the initial guess $x_0 = 1$, the iterative Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$$

where,

$$f(x_0) = x_0^3 + 12x_0 - 12$$

$$f'(x_0) = 3x_0^2 + 12$$

and

n	x_0	$f(x_0)$	$f'(x_0)$
0	1	1.000	15.000
1	0.933	0.013	14.800
2	0.932	0.000	14.797
2	0.932		

with this initial guess $x_0 = 1$, down to the third decimal place $x = 0.932$. To verify,

$$P_3(0.932) = 0.932^3 + 12 \times 0.932 - 12 \approx 1.666 \times 10^{-4}$$

where,

$$f(0.932) \approx \frac{0.932}{(0.932+1)(0.932-4)} \approx -0.157$$

Illustratively, the numerical solution $(x_0, y_0) = (0.932, -0.157)$ is in this case very close to the approximate graphical solution $(x_0, y_0) \approx (1, f(1) = -1/6)$, and sign of $f''(x)$ is therefore

x		-1		x_0		4	
2	+	+	+	+	+	+	+
$P_3(x)$	−	−	−	0	+	+	+
$(x+1)^3$	−	0	+	+	+	+	+
$(x-4)^3$	−	−	−	−	−	0	+
$f''(x)$	−	n.d	+	0	−	n.d.	+
$f(x)$	∩	n.d	∪	y_0	∩	n.d	∪

that is to say, this function changes its form from convex ∩ to concave ∪ after passing each of its vertical asymptotes, as well as after crossing $f''(x)$ point.

8. *Summary of the important points:*

defined:	$x \in \{\mathbb{R} \mid x \neq -1, x \neq 4\}$
y-axis crossing point $f(0)$:	$(x, y) = (0, 0)$
h. asymptotes:	$y = 0$
v. asymptotes:	$x = -1$
	$x = 4$
zero:	$(x, y) = (0, 0)$
extremes:	(none, $f(x)$ always descends ↘)
inflection:	$(x, y) = (0.932, -0.157)$

9. *Graphical representation:* results of the analysis for function in P.3.46 are summarized by $f(x)$ graph (see Fig. 3.38).

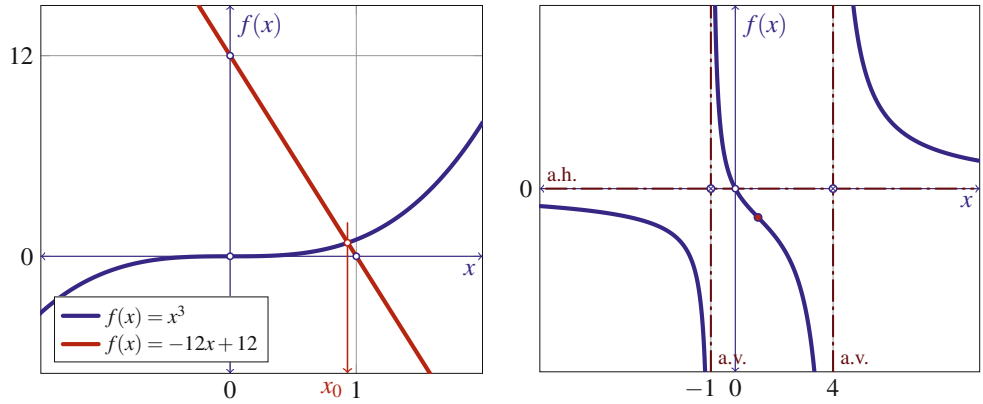


Fig. 3.38 Example P.3.46

3.47. Given rational function

$$f(x) = \frac{4x}{4-x^2} = \left\{a^2 - b^2 = (a-b)(a+b)\right\} = \frac{4x}{(2-x)(2+x)}$$
$$= \frac{P(x)}{Q(x)} \quad \therefore \quad P(x) = 4x \quad \text{and} \quad Q(x) = (2-x)(2+x)$$

1. *Domain of definition* $\mathbb{D}$: all x where $f(x) \neq \infty$, that is to say,

$$Q(x) = (2-x)(2+x) \neq 0 \quad \therefore \quad x_1 \neq -2, \quad x_2 \neq 2$$

There are two *first-order poles* of $f(x)$ at $x = \pm 2$. Thus, $f(x)$ is defined for all real numbers x except where the function becomes infinite and/or *not defined*, i.e., $\{\mathbb{R}|x \neq \pm 2\}$, as

$$f(\pm 2) = \frac{4(\pm 2)}{(4-(\pm 2)^2)} = \frac{4(\pm 2)}{0} = \infty$$

In summary, $x \in \{\mathbb{R}|x \neq \pm 2\}$, there are two vertical asymptotes, $x = -2$ and $x = 2$.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{4(-x)}{(4-(-x)^2)} = -\frac{4x}{4-x^2} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{4x}{4-x^2}\right) = \frac{4x}{4-x^2} = f(x) \Rightarrow \quad \text{odd}$$

In conclusion, this rational function is not even because $f(x) \neq f(-x)$, and it is odd (i.e., symmetric relative to the origin) because $f(x) = -f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad P(x) = 0 \quad \therefore \quad 4x = 0 \quad \therefore \quad x_1 = 0, \quad \text{(first-order zero)}$$

After taking into account all poles and zero:

x		-2		0		2	
4	$+$	$+$	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$2+x$	$-$	0	$+$	$+$	$+$	$+$	$+$
$2-x$	$+$	$+$	$+$	$+$	$+$	0	$-$
$f(x)$	$+$	n.d.	$-$	0	$+$	n.d.	$-$

In conclusion, $f(x) = 4x/(4-x^2)$ has two vertical asymptotes, $x = -2$ and $x = 2$, one first-order zero $(x, y) = (0, 0)$, and it changes sign every time after passing either pole or zero.

4. *y-axis crossing point* $f(0)$: $f(0) = 0$, which is at the same time this function's zero.
5. *Limits:* there are two discontinuity points, and there are six limits in total to be resolved,

$\underline{x \to \infty:}$ $(x \neq -2)$ and, $(x \neq +2)$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x}{4-x^2} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{4}{-2x} = \lim_{x\to-\infty} \frac{1}{(-2)(-\infty)} = 0\downarrow$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{4x}{4-x^2} = \left(\frac{\infty}{\infty}\right) \{\text{l'Hôpital's rule}\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{4}{-2x} = \lim_{x\to+\infty} \frac{1}{(-2)(+\infty)} = 0\uparrow$$

$\underline{x \to -2:}$ $(x \neq 2)$

$$\lim_{x\to-2\overset{\to}{}} f(x) = \lim_{x\to-2\overset{\to}{}} \frac{4x}{(2-x)(2+x)} = \lim_{x\to-2\overset{\to}{}} \frac{-8}{(4)(2+x)} = \lim_{x\to-2\overset{\to}{}} \frac{-8}{0\overset{\to}{}} = +\infty$$

$$\lim_{x\to-2\overset{\leftarrow}{}} f(x) = \lim_{x\to-2\overset{\leftarrow}{}} \frac{4x}{(2-x)(2+x)} = \lim_{x\to-2\overset{\leftarrow}{}} \frac{-8}{(4)(2+x)} = \lim_{x\to-2\overset{\leftarrow}{}} \frac{-8}{0\overset{\leftarrow}{}} = -\infty$$

$\underline{x \to 2:}$ $(x \neq -2)$

$$\lim_{x\to 2\overset{\to}{}} f(x) = \lim_{x\to 2\overset{\to}{}} \frac{8}{(2-x)(2+x)} = \lim_{x\to 2\overset{\to}{}} \frac{8}{(2-x)(4)} = \lim_{x\to 2\overset{\to}{}} \frac{8}{0\overset{\leftarrow}{}} = +\infty$$

$$\lim_{x\to 2\overset{\leftarrow}{}} f(x) = \lim_{x\to 2\overset{\leftarrow}{}} \frac{4x}{(2-x)(2+x)} = \lim_{x\to 2\overset{\leftarrow}{}} \frac{8}{(2-x)(4)} = \lim_{x\to 2\overset{\leftarrow}{}} \frac{8}{0\overset{\to}{}} = -\infty$$

As $x \to \pm\infty$, $f(x) = 4x/(4-x^2) \to 0$, i.e., $y = 0$ is horizontal asymptote.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{4\cancel{x}}{\cancel{x}(4-x^2)} = \lim_{x\to\infty} \frac{1}{\infty} = 0$$

As $a = 0$ there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{4x}{4-x^2}\right)' = 4\,\frac{(4-x^2) - x(-2x)}{(4-x^2)^2}$$

$$= 4\,\frac{x^2+4}{(4-x^2)^2} > 0 \quad \therefore \quad f(x) \quad \text{always ascends} \quad \nearrow$$

and,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(4\,\frac{x^2+4}{(4-x^2)^2}\right)'$$

$$= 4\,\frac{2x\,(4-x^2)^{\cancel{2}^{1}} - 2(x^2+4)\cancel{(4-x^2)}(-2x)}{(4-x^2)^{\cancel{4}^{3}}}$$

$$= 4\,\frac{8x - 2x^3 + 4x^3 + 16x}{(4-x^2)^3} = 8\,\frac{x(x^2+12)}{(4-x^2)^3} = 0$$

$$\therefore \quad x(x^2+12) = 0 \quad \therefore \quad x = 0 \quad (\text{because, } x^2+12 > 0)$$

Sign of $f''(x)$ is therefore

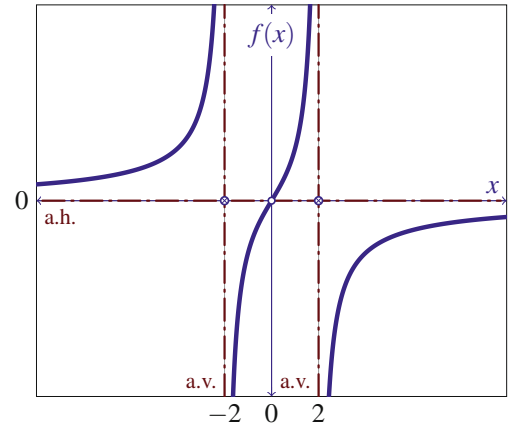
x		-2		0		2	
8	$+$	$+$	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
x^2+12	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$(2-x)^3$	$+$	$+$	$+$	$+$	$+$	0	$-$
$(2+x)^3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f''(x)$	$+$	n.d.	$-$	0	$+$	n.d.	$-$
$f(x)$	$\cup$	n.d.	$\cap$	0	$\cup$	n.d.	$\cap$

There is one inflection point at $(x, y) = (0, 0)$, and this function changes its form from concave $\cup$ to convex $\cap$ after passing each of its vertical asymptotes; note that $f''(\pm 2)$ is not defined..

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} \quad & x \in \{\mathbb{R} | x \neq \pm 2\} \\
\underline{\text{y-axis crossing point } f(0)\text{:}} \quad & (x, y) = (0, 0) \\
\underline{\text{h. asymptote:}} \quad & y = 0 \\
\underline{\text{v. asymptotes:}} \quad & x = -2 \\
& x = 2 \\
\underline{\text{zero:}} \quad & (x, y) = (0, 0)
\end{aligned}$$

Fig. 3.39 Example P.3.47



extremes: (none, $f(x)$ always ascends ↗)

inflection: $(x, y) = (0, 0)$

9. *Graphical representation:* results of the analysis for function in P.3.47 are summarized by $f(x)$ graph (see Fig. 3.39).

3.3 Radical Functions

3.48. Given square root radical function

$$f(x) = \sqrt{x}$$

it must to be noted that, in the strict mathematical sense, a square root is *not a function* because for any given argument it produces *two* different results: one positive and one negative. For example, $\sqrt{1}$ equals both $+1$ and -1. However, by definition, a function must have *one to one mapping*, i.e., one input must produce only one output. Nevertheless, in engineering sciences, radicals are referred to as "functions." In practical analysis, even order radical functions are first analyzed only for the $y = f(x) > 0$, and then the resulting graph is simply mirrored relative to the horizontal axis to complete the function's form.

1. *Domain of definition* $\mathbb{D}$: in the real domain $\mathbb{R}$, even order radical functions are defined only for arguments $x \geq 0$.

   Otherwise, negative arguments of even order radical functions result in *complex number* solutions. This is due to the fact that $\sqrt{-1} = i$, aka *imaginary unit*, that is to say, these solutions belong to the complex domain $\mathbb{C}$ numbers.
2. *Function's parity:*

even: $f(-x) = \sqrt{-x} \Rightarrow f(-x)$ is not defined for $x > 0$, (not even)

odd: consequently $-f(-x)$ is also not defined, (not odd)

3. *Function's sign:* $f(x) = 0 \Rightarrow x = 0$. There is first-order zero at $(x, y) = (0, 0)$, while even order radical functions are always positive at $y \geq 0$ side.
4. *y-axis crossing point* $f(0)$: $f(0) = 0$, which is at the same time this function's zero.
5. *Limits:* there is one limit in total to be resolved (for $y > 0$)

$$x \to +\infty: \quad (x \geq 0) \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt{x} = +\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\sqrt{x}}{x} = \lim_{x \to \infty} \frac{x^{\frac{1}{2}}}{x} = \lim_{x \to \infty} x^{-\frac{1}{2}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

   As $a = 0$ consequently there is no oblique asymptote.
7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

$$> 0 \quad (\text{for} \quad y > 0) \quad \therefore \quad f(x) \quad \text{always ascends} \quad \nearrow$$

$$< 0 \quad (\text{for} \quad y < 0) \quad \therefore \quad f(x) \quad \text{always ascends} \quad \searrow$$

   and,

$$f''(x) = 0: \quad f''(x) = \left(\frac{1}{2\sqrt{x}}\right)' = -\frac{1}{4x\sqrt{x}}$$

$$< 0 \quad (\text{for} \quad y > 0, \quad \text{because} \quad x > 0) \quad \therefore \quad f(x) : \cap$$

$$> 0 \quad (\text{for} \quad y < 0, \quad \text{because} \quad x > 0) \quad \therefore \quad f(x) : \cup$$

8. *Summary of the important points:* (for $y > 0$)

defined:	$x \geq 0$
y-axis crossing point $f(0)$:	$(x, y) = (0, 0)$
zero:	$(x, y) = (0, 0)$
extremes:	(none)
inflection:	(none)

9. *Graphical representation:* results of the analysis for function in P.3.48 are summarized by $f(x)$ graph (see Fig. 3.40). Note that $y < 0$ side is simply mirrored image of $f(x)$, which is already deduced for $y > 0$. Non-defined region $x < 0$ is indicated by shading.

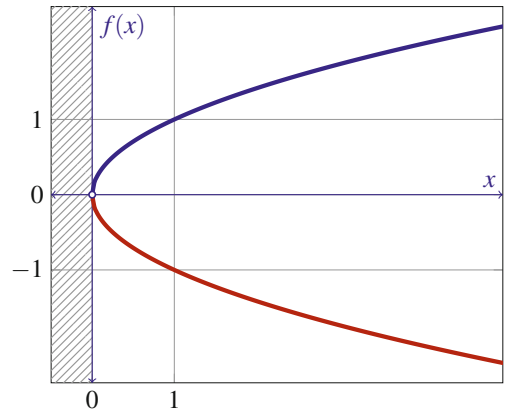


Fig. 3.40 Example P.3.48

3.49. Given square root radical function

$$f(x) = \sqrt{x+1}$$

note that argument $x + 1$ of this function is a simple delayed version of the argument x in P.3.48. Therefore, the function's form does not change; instead, it is simply delayed by "1," as illustrated by $f(x)$ graph (see Fig. 3.41).

1. *Domain of definition* $\mathbb{D}$: in the real domain $\mathbb{R}$, even order radical functions are defined only for positive arguments, i.e.,

$$x + 1 \geq 0 \quad \therefore \quad x \geq -1$$

2. *y-axis crossing point* $f(0)$:

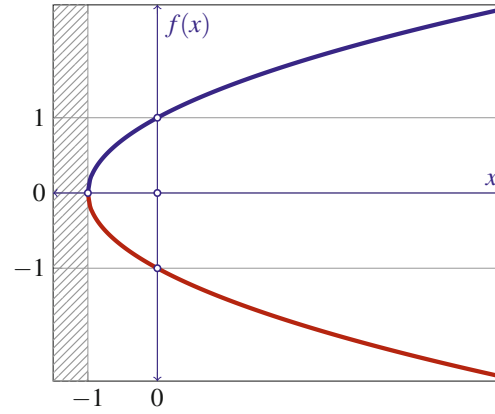
$$f(0) = \sqrt{(0)+1} = \pm 1$$

3. *ummary of the important points:* (for $y > 0$)

defined:	$x \geq -1$
y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
zero:	$(x, y) = (-1, 0)$
extremes:	(none)
inflection:	(none)

4. *Graphical representation:* results of the analysis for function in P.3.49 are summarized by $f(x)$ graph (see Fig. 3.41).

Fig. 3.41 Example P.3.49



3.50. Given square root radical function, the following transformation

$$f(x) = \sqrt{x^2} = \left(x^2\right)^{1/2} = |x|$$

shows that this is actually $|x|$ function because x^2 generates only $x \geq 0$ numbers; therefore, argument of this square root function is always positive even for $x < 0$.

Reminder:

$$|x| = \begin{cases} x; & (x \geq 0) \\ -x; & (x < 0) \end{cases} \qquad \text{sign}\,(x) = \begin{cases} 1; & (x > 0) \\ 0; & (x = 0) \\ -1; & (x < 0) \end{cases} \quad \text{(by convention)}$$

1. *Domain of definition* $\mathbb{D}$: this function is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}}\ f(-x) = \sqrt{(-x)^2} = \sqrt{x^2} = f(x) \quad \therefore \quad \text{(even)}$$

$$\underline{\text{odd:}}\ -f(-x) = -\sqrt{x^2} \neq f(x) \quad \therefore \quad \text{(not odd)}$$

3. *Function's sign:* $f(x) = 0 \Rightarrow |x| = 0$, therefore $f(x) = \sqrt{x^2} \geq 0$.
4. *y-axis crossing point* $f(0)$*:* $f(0) = \sqrt{0} = 0$, which is at the same time this function's zero.
5. *Limits:* there are two limits in total to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} |x| = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} |x| = +\infty$$

6. *Oblique asymptote:* note that $\text{sign}\,(x) = |x|/x = x/|x| = \pm 1$, that is to say

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{|x|}{x} = \lim_{x\to+\infty} \operatorname{sign}(x) = +1 \quad (x \neq 0)$$
$$= \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{|x|}{x} = \lim_{x\to-\infty} \operatorname{sign}(x) = -1 \quad (x \neq 0)$$

As a is well not defined, consequently there is no oblique asymptote.

7. *Critical points:* $\left\{(|x|)' = \dfrac{x}{|x|} = \dfrac{|x|}{x} = \operatorname{sign}(x)\right\}$

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\sqrt{x^2}\right)' = (|x|)' = \frac{x}{|x|} = \operatorname{sign}(x) = 0 \quad \therefore \quad x = 0$$

Note that $f'(0) = 0/0$ is not defined; however, $f(0) = 0$ is well defined because $f(x)$ is a continuous function. Therefore, for $x < 0$, it is true that $f'(x) < 0$ and function descends $\searrow$, while for $x > 0$, it is true that $f'(x) > 0$ and function ascends $\nearrow$. In conclusion, there is minimum at $(x, y) = (0, 0)$ even though $f'(0) = 0/0$ is not defined. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = (\pm 1)' = 0$$

There are no extreme points, and this function is neither concave nor convex.

8. *Summary of the important points:* (for $y > 0$)

zero:	$(x, y) = (0, 0)$	
y-axis crossing point $f(0)$:	$(x, y) = (0, 0)$	
extremes:	$(x, y) = (0, 0)$	(min)
inflection:	(none)	

9. *Graphical representation:* results of the analysis for function in P.3.50 are summarized by $f(x)$ graph (see Fig. 3.42).

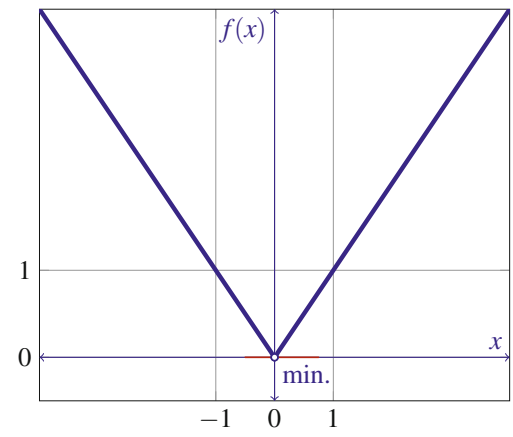
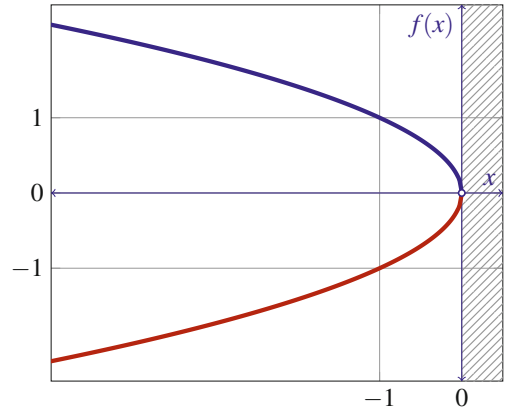


Fig. 3.42 Example P.3.50

Fig. 3.43 Example P.3.51



3.51. Given radical function in P.3.51,

$$f(x) = \sqrt{-x}$$

note that argument ' $-x$ ' of this function is a simple inverted argument version of the argument 'x' in P.3.48. Therefore, the function's form does not change; instead, it is simply mirrored relative to the vertical axis, that is, valid for $x \leq 0$, as illustrated by $f(x)$ graph (see Fig. 3.43).

1. *Domain of definition* $\mathbb{D}$: in the real domain $\mathbb{R}$, even order radical functions are defined only for positive arguments, i.e.,

$$-x \geq 0 \quad \therefore \quad x \leq 0$$

   This function is defined $x \in \mathbb{R} | x \leq 0$.
2. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{(0)} = 0$, which is at the same time this function's zero.
3. *Summary of the important points:* (for $y > 0$)

defined:	$x \leq 0$
y-axis crossing point $f(0)$:	$(x, y) = (0, 0)$
zero:	$(x, y) = (0, 0)$
extremes:	(none)
inflection:	(none)

4. *Graphical representation:* results of the analysis for function in P.3.51 are summarized by $f(x)$ graph (see Fig. 3.43).

3.52. Given radical function in P.3.52,

$$f(x) = \sqrt{-x-1} = \sqrt{-(x+1)}$$

note that argument '$-(x+1)$' of this function is a simple inverted argument version of the argument '$x+1$' in P.3.49. Therefore, the function's form does not change; instead, it is simply mirrored relative to the vertical axis, that is, valid for $x \leq -1$, as illustrated by $f(x)$ graph (see Fig. 3.44).

1. *Domain of definition* $\mathbb{D}$: in the real domain $\mathbb{R}$, even order radical functions are defined only for positive arguments, i.e.,

$$-x-1 \geq 0 \quad \therefore \quad x \leq -1$$

   In conclusion, $x \in \mathbb{R} | x \leq -1$.
2. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{(0)} \Rightarrow f(x)$ not defined
3. *Summary of the important points:* (for $y > 0$)

$$\begin{aligned} \underline{\text{defined:}} \quad & x \leq -1 \\ \underline{\text{zero:}} \quad & (x, y) = (-1, 0) \\ \underline{\text{extremes:}} \quad & \text{(none)} \\ \underline{\text{inflection:}} \quad & \text{(none)} \end{aligned}$$

4. *Graphical representation:* results of the analysis for function in P.3.52 are summarized by $f(x)$ graph (see Fig. 3.44).

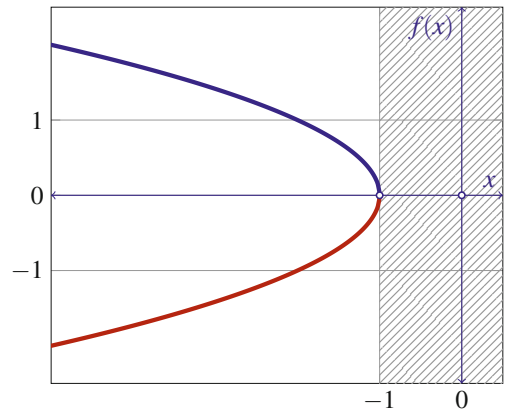
3.53. Given radical function and its mirrored version

$$f(x) = \begin{cases} 1 - \sqrt{-x} & \text{for } +\sqrt{*} \\ 1 + \sqrt{-x} & \text{for } -\sqrt{*} \end{cases}$$

1. *Domain of definition* $\mathbb{D}$: in the real domain $\mathbb{R}$, even order radical functions are defined only for positive arguments, i.e.,

$$-x \geq 0 \quad \therefore \quad x \leq 0$$

Fig. 3.44 Example P.3.52



This function is defined for $x \in \mathbb{R} | x \leq 0$.

2. *Function's parity:*

$$\underline{\text{even:}}\ f(-x) = 1 - \sqrt{-(-x)} = 1 - \sqrt{x} \neq f(x) \quad \therefore \quad \text{not even}$$
$$\underline{\text{odd:}}\ -f(-x) = -(1 - \sqrt{x}) = -1 + \sqrt{x} \neq f(x) \quad \therefore \quad \text{not odd}$$

This function is not odd (i.e., symmetric relative to the origin) because $f(x) = -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad 1 - \sqrt{-x} = 0 \quad \therefore \quad \sqrt{-x} = 1 \quad \therefore \quad -x = 1^2$$
$$\therefore \quad x = -1 \quad \text{(first-order zero)}$$

and, therefore $f(x) < 0$ for $(x < -1)$, and $f(x) > 0$ for $(-1 < x \leq 0)$. Note that there are *two* points that correspond to $f(-1)$:

$$f(-1) = 1 - (\pm\sqrt{-(-1)}) = 1 - (\pm\sqrt{1}) \Rightarrow y_1 = 1 - 1 = 0 \quad (y < 1)$$
$$\Rightarrow y_2 = 1 + 1 = 2 \quad (y > 1)$$

Consequently, this radical function is mirrored along a horizontal axis of symmetry, which in this case is elevated to $y = 1$.

4. *y-axis crossing point* $f(0)$*:* $f(0) = 1 - \sqrt{0} = 1$.
5. *Limits:* on the positive side, i.e., $x > 0$, this function is not defined, thus

$$\underline{x \to -\infty:} \quad (x \leq 0) \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 1 - \sqrt{-x} = +\infty \quad (\text{for } y > 1)$$
$$= \lim_{x \to -\infty} 1 - \sqrt{-x} = -\infty \quad (\text{for } y < 1)$$
$$\underline{x \to \underset{\to}{0}:} \quad (x \leq 0) \quad \lim_{x \to 0} f(x) = \lim_{x \to \underset{\to}{0}} 1 - \sqrt{-x} = 0\downarrow \quad (\text{for } y > 1)$$
$$= \lim_{x \to \underset{\to}{0}} 1 - \sqrt{-x} = 0\uparrow \quad (\text{for } y < 1)$$

6. *Oblique asymptote:* (only for the $x < 0$ side)

$$a = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{1 - \sqrt{-x}}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x \to -\infty} \frac{1}{2\sqrt{-x}} = 0$$

As $a = 0$ there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(1 - \sqrt{-x}\right)' = \frac{1}{2\sqrt{-x}} > 0 \quad (\text{for} \quad y < 1)$$
$$\therefore \quad f(x) \quad \text{ascends} \quad \nearrow$$

$$= \left(1 + \sqrt{-x}\right)' = -\frac{1}{2\sqrt{-x}} < 0 \quad (\text{for} \quad y > 1)$$

$$\therefore \quad f(x) \quad \text{ascends} \quad \searrow$$

and,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{1}{2\sqrt{-x}}\right)' = \left(\frac{1}{4}\right)\frac{1}{\pm\sqrt{-x^3}}$$

$$< 0 \quad (\text{for} \quad y > 1, \quad \text{because} \quad x < 0) \quad \therefore \quad f(x) : \cap$$

$$> 0 \quad (\text{for} \quad y < 1, \quad \text{because} \quad x < 0) \quad \therefore \quad f(x) : \cup$$

8. *Summary of the important points:* (for $y > 0$)

defined:	$x \leq 1$
y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
zero:	$(x, y) = (-1, 0)$
extremes:	(none)
inflection:	(none)

9. *Graphical representation:* see Fig. 3.45.

3.54. Given radical function

$$f(x) = \sqrt{x^2 - x^3} = \sqrt{x^2(1 - x)} = |x|\sqrt{1 - x}$$

Being a square root, this function is symmetric relative to the horizontal axis $y = 0$, that is to say:

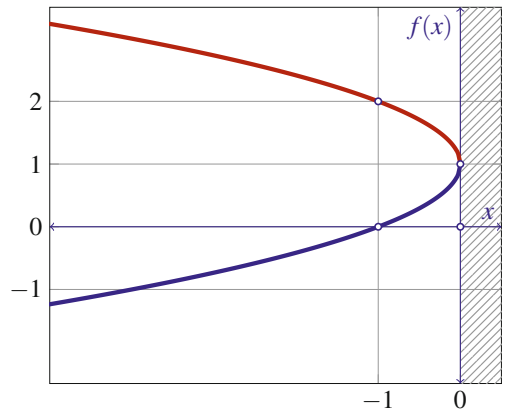


Fig. 3.45 Example P.3.53

$$f(x) = +\sqrt{x^2 - x^3} \quad (y > 0)$$
$$= -\sqrt{x^2 - x^3} \quad (y < 0)$$

1. *Domain of definition* $\mathbb{D}$: even order radical functions are defined only for positive arguments. Note that $x^2 \geq 0$ for all x, i.e.,

$$(x^2 - x^3) \geq 0 \;\; \therefore \;\; x^2(1-x) \geq 0 \Rightarrow 1 - x \geq 0 \;\; \therefore \;\; 1 \geq x$$

   In conclusion, $x \in \mathbb{R} | x \leq 1$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt{(-x)^2 - (-x)^3} = \sqrt{x^2 + x^3} \neq f(x) \Rightarrow \quad \text{not even}$$
$$\underline{\text{odd:}} \; -f(-x) = -\left(\sqrt{x^2 + x^3}\right) = -\sqrt{x^2 + x^3} \neq f(x) \Rightarrow \quad \text{not odd}$$

   This radical function is not odd (i.e., symmetric relative to the origin) because $f(x) = -f(-x)$, and it is not even because $f(x) \neq f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; \sqrt{x^2(1-x)} \;\; \therefore \;\; x_{1,2} = 0, \quad \text{(second order zero)}$$
$$\therefore \;\; x_3 = 1, \quad \text{(first-order zero)}$$

   so that (for $y > 0$):

x		0		1	
$\lvert x \rvert$	+	0	+	+	+
$+\sqrt{1-x}$	+	+	+	0	n.d.
$f(x)$	+	0	+	0	n.d.

   There is first-order zero at $(x, y) = (1, 0)$ and second-order zero at $(x, y) = (0, 0)$; however, $f(x)$ does not change its sign (it is always positive for $y > 0$, and it is always negative for $y < 0$).
4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{(0)^2 - (0^3)} = 0$, also this function's zero.
5. *Limits:* there is one limit in total to be resolved,

$$\underline{x \to -\infty:} \quad (x \leq 1) \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} |x|\sqrt{1-x} = +\infty \quad (\text{for } y > 0)$$
$$= \lim_{x \to -\infty} -|x|\sqrt{1-x} = -\infty \quad (\text{for } y < 0)$$

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{|x|\sqrt{1-x}}{x} = \lim_{x \to \infty} (\text{sign}\,(x) \cdot \infty) = \pm\infty$$

   As $a = \infty$ there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad \left\{ \text{sign}\,(x) = \frac{x}{|x|} = \frac{|x|}{x} \right\}$$

$$f'(x) = \left(\sqrt{x^2 - x^3}\right)' = \frac{2x - 3x^2}{2\sqrt{x^2 - x^3}} = \frac{x}{|x|}\,\frac{2 - 3x}{2\sqrt{1 - x}}$$

$$= \text{sign}\,(x)\,\frac{2 - 3x}{2\sqrt{1 - x}} = 0 \quad \therefore \quad 2 - 3x = 0$$

$$\therefore \quad x_1 = \frac{2}{3} \quad \text{and,} \quad f(2/3) = \left|\frac{2}{3}\right| \sqrt{1 - \frac{2}{3}} = \frac{2}{3}\sqrt{\frac{1}{3}} = y_1$$

Note: $f'(0) = 0/0$ is not defined, and $f'(1) \to \infty$; thus (for $y > 0$)

x		0		x_1		1
$\text{sign}\,(x)$	$-$	0	$+$	$+$	$+$	$+$
$2 - 3x$	$+$	$+$	$+$	0	$-$	$-$
$\sqrt{1 - x}$	$+$	$+$	$+$	$+$	$+$	0
$f'(x)$	$-$	n.d.	$+$	0	$-$	n.d.
$f(x)$	$\searrow$	0	$\nearrow$	y_1	$\searrow$	0

There is minimum at $(x, y) = (0, 0)$ and maximum at $(x, y) = (1, 2/3\sqrt{1/3})$. Also,

$$\underline{f''(x) = 0:} \quad (x \neq 0) \quad \left\{ \text{sign}\,(x) = \pm 1 \quad \text{(i.e. a constant} \right\}$$

$$f''(x) = \left(\text{sign}\,(x)\,\frac{2 - 3x}{2\sqrt{1 - x}}\right)' = \text{sign}\,(x)\,\frac{1}{2}\left(\frac{2 - 3x}{\sqrt{1 - x}}\right)'$$

$$= \text{sign}\,(x)\,\frac{1}{2}\,\frac{-3\sqrt{1 - x} - \dfrac{(2 - 3x)(-1)}{2\sqrt{1 - x}}}{\left(\sqrt{1 - x}\right)^2}$$

$$= \text{sign}\,(x)\,\frac{1}{2}\,\frac{\dfrac{-6(1 - x) - (2 - 3x)(-1)}{2\sqrt{1 - x}}}{(1 - x)}$$

$$= \text{sign}\,(x)\,\frac{1}{4}\,\frac{3x - 4}{(1 - x)\sqrt{1 - x}} = 0$$

Therefore, $f''(1)$ is not defined because of division by zero, while

$$f''(x) = 0 \quad \therefore \quad 3x - 4 = 0 \quad \therefore \quad x = \frac{4}{3} \Rightarrow$$

$$\text{but } f(4/3) \quad \text{not defined, because} \quad x \in \mathbb{R} | x \leq 1$$

Sign of $f''(x)$, for $(y > 0)$, is then

x		0		1
$1/4$	$+$	$+$	$+$	$+$
$\text{sign}\,(x)$	$-$	0	$+$	$+$
$3x-4$	$-$	$-$	$-$	$-$
$1-x$	$+$	$+$	$+$	0
$\sqrt{1-x}$	$+$	$+$	$+$	0
$f''(x)$	$+$	0	$-$	0
$f(x)$	$\cup$	0	$\cap$	0

There is one inflection point at $(x, y) = (0, 0)$ that coincides with both function's zero and minimum ($y > 0$).

8. *Summary of the important points:* (for $y > 0$)

$$\begin{aligned}
\underline{\text{defined:}} \quad & x \leq 1 \\
\underline{\text{y-axis crossing point } f(0)\text{:}} \quad & (x, y) = (0, 0) \\
\underline{\text{zeros:}} \quad & (x, y) = (0, 0) \\
& (x, y) = (0, 0) \\
& (x, y) = (1, 0) \\
\underline{\text{extremes:}} \quad & (x, y) = (0, 0) \quad \text{(min)} \\
& (x, y) = \left(\frac{2}{3}, \frac{2}{3}\sqrt{\frac{1}{3}}\right) \quad \text{(max)} \\
\underline{\text{inflection:}} \quad & (x, y) = (0, 0)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.54 are summarized by $f(x)$ graph, Fig. 3.46. Note that mirrored image points are symmetric relative to the horizontal axis. In this case, note that minimum $(x, y) = (0, 0)$ on $y > 0$ side falls at the same point as the maximum on $y < 0$ side.

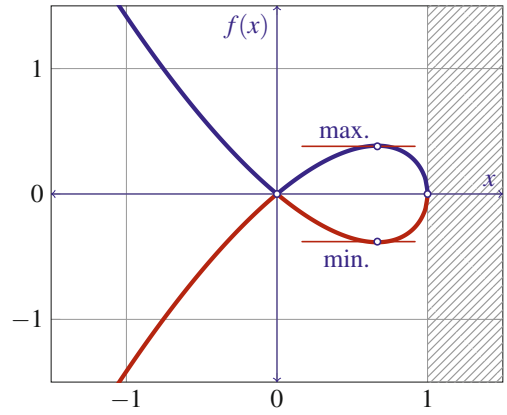
3.55. Given radical function

$$f(x) = \sqrt[3]{x^2 - x^3} = \sqrt[3]{x^2(1-x)}$$

1. *Domain of definition* $\mathbb{D}$: in the real domain $\mathbb{R}$, odd order radical functions are defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\begin{aligned}
\underline{\text{even:}} \quad & f(-x) = \sqrt[3]{(-x)^2 - (-x)^3} = \sqrt[3]{x^2 + x^3} \neq f(x) \Rightarrow \quad \text{not even} \\
\underline{\text{odd:}} \quad & -f(-x) = -\left(\sqrt[3]{x^2 + x^3}\right) = -\sqrt[3]{x^2 + x^3} \neq f(x) \Rightarrow \quad \text{not odd}
\end{aligned}$$

Fig. 3.46 Example P.3.54



This radical function is not odd (i.e., symmetric relative to the origin) because $f(x) = -f(-x)$, and it is not even because $f(x) \neq f(-x)$.

3. *Function's sign:*

$$f(x) = 0: \quad \therefore \quad \sqrt[3]{x^2(1-x)} \quad \therefore \quad x_{1,2} = 0, \quad \text{(second order zero)}$$
$$\therefore \quad x_2 = 1, \quad \text{(first-order zero)}$$

so that:

x		0		1	
x^2	+	0	+	+	+
$1-x$	+	+	+	0	−
$\sqrt[3]{x^2(1-x)}$	+	0	+	0	−
$f(x)$	+	0	+	0	−

There is one first-order zero at $(x, y) = (1, 0)$ and one second-order zero at $(x, y) = (0, 0)$. Function changes its sign after crossing its first-order zero, but not after touching its second (even)-order zero.

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt[3]{(0)^2 - (0^3)} = 0$, also this function's zero.
5. *Limits:* there are two limits in total to be resolved.

$$x \to -\infty: \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt[3]{x^2(1-x)} = -(-\infty) = +\infty$$
$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt[3]{x^2(1-x)} = -(+\infty) = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt[3]{x^2 - x^3)}}{x} = \lim_{x\to\infty} \frac{\sqrt[3]{x^3\,(1/x - 1)}}{x} = \lim_{x\to\infty} \frac{x\sqrt[3]{\left(\cancel{1/x}^{\,0} - 1\right)}}{\cancel{x}}$$
$$= -1$$

$$b = \lim_{x\to\infty}(f(x) - ax) = \lim_{x\to\infty}\left(\sqrt[3]{x^2 - x^3} + x\right) = (\infty + \infty) \text{ n.d.}$$

$$= \lim_{x\to\infty} x\left(\sqrt[3]{1/x - 1} + 1\right) = (\infty \cdot 0) \text{ n.d.}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{1/x - 1} + 1}{1/x} = (0/0) \text{ n.d.}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\sqrt[3]{1/x - 1}}{-3x\left(1/x^2\right)(x-1)} = \lim_{x\to\infty} \frac{\sqrt[3]{1/x - 1}}{-3\left(1 - 1/x\right)} = \frac{1}{3}$$

There is oblique asymptote $y_{\text{aa}} = -x + 1/3$.

7. *Critical points:*

$f'(x) = 0$:

$$f'(x) = \left((x^2 - x^3)^{1/3}\right)' = \frac{1}{3}(x^2 - x^3)^{-2/3}(2x - 3x^2)$$

$$= \frac{1}{3}\frac{x(2-3x)}{\sqrt[3]{\left(x^2(1-x)\right)^2}} = 0 \quad \therefore \quad \begin{cases} x_1 = 0 \\ 2 - 3x = 0 \quad \therefore \quad x_2 = \dfrac{2}{3} \end{cases}$$

$$\therefore \quad f(0) = 0 = y_1 \quad \text{and,}$$

$$f(2/3) = \sqrt[3]{(2/3)^2 - (2/3)^3} = \sqrt[3]{\frac{4}{27}} = \frac{\sqrt[3]{4}}{3} = y_2$$

There are two poles of $f'(x)$ at $x = 0$ and $x = 1$; therefore sign of $f'(x)$ is,

x		0		x_2		1	
$1/3$	+	+	+	+	+	+	+
x	−	0	+	+	+	+	+
$2 - 3x$	+	+	+	0	−	−	−
$\sqrt[3]{\left(x^2(1-x)\right)^2}$	+	0	+	+	+	0	+
$f'(x)$	−	n.d.	+	0	−	n.d.	−
$f(x)$	↘	y_1	↗	y_2	↘	0	↘

There is minimum point at $(x, y) = (0, 0)$ and maximum point at $(x, y) = (2/3, \sqrt[3]{4}/3)$. Note that $f'(0)$ and $f'(1)$ are not defined. Also,

$f''(x) = 0$:

$$f''(x) = \left(\frac{1}{3}\frac{x(2-3x)}{\sqrt[3]{\left(x^2(1-x)\right)^2}}\right)'$$

$$
\begin{aligned}
&= \frac{1}{3}\,\frac{(2-6x)\big(x^2(1-x)\big)^{2/3} - {}^2\!/_3\,(2x-3x^2)\big(x^2(1-x)\big)^{-1/3}(2x-3x^2)}{\big(x^2(1-x)\big)^{4/3}} \\
&= \frac{2}{9}\,\frac{\cancel{\big(x^2(1-x)\big)^{2/3}}\left(3(1-3x) - (2x-3x^2)^2\big(x^2(1-x)\big)^{-1}\right)}{\big(x^2(1-x)\big)^{\cancel{4}2/3}} \\
&= \frac{2}{9}\,\frac{3(1-3x)x^2(1-x) - (2x-3x^2)^2}{\big(x^2(1-x)\big)^{2/3}x^2(1-x)} \\
&= \frac{2}{9}\,\frac{1}{\sqrt[3]{\big(x^2(1-x)\big)^2}\;(x-1)} \\
&\neq 0
\end{aligned}
$$

There are two poles of $f''(x)$ at $x = 0$ (second order) and at $x = 1$ (first order). Sign of $f''(x)$, for $(y > 0)$, is then

x		0		1	
$2/9$	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
$\sqrt[3]{\big(x^2(1-x)\big)^2}$	$+$	0	$+$	0	$+$
$f''(x)$	$-$	n.d.	$-$	n.d.	$+$
$f(x)$	$\cap$	0	$\cap$	0	$\cup$

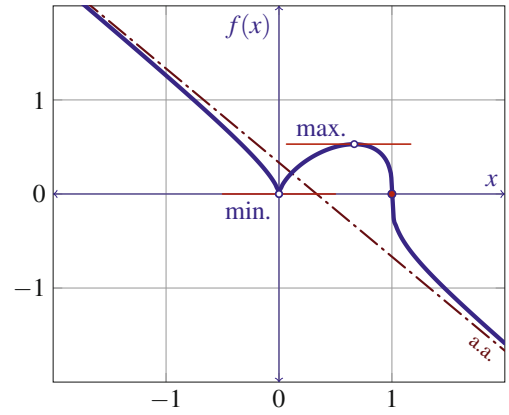
There is one inflection point at $(x, y) = (1, 0)$, which is also its first-order zero point. Note that $f''(0)$ and $f''(1)$ are not defined.

8. *Summary of the important points:*

$$
\begin{aligned}
\text{y-axis crossing point } f(0)\text{:} \quad & (x, y) = (0, 0) \\
\text{a.a. asymptote:} \quad & y = -x + \frac{1}{3} \\
\text{zeros:} \quad & (x, y) = (0, 0) \\
& (x, y) = (0, 0) \\
& (x, y) = (1, 0) \\
\text{extremes:} \quad & (x, y) = (0, 0) \quad \text{(min)} \\
& (x, y) = \left(\frac{2}{3}, \frac{\sqrt[3]{4}}{3}\right) \quad \text{(max)} \\
\text{inflection:} \quad & (x, y) = (1, 0)
\end{aligned}
$$

9. *Graphical representation:* results of the analysis for function in P.3.55 are summarized by $f(x)$ graph (see Fig. 3.47).

Fig. 3.47 Example P.3.55



3.56. Given radical function in P.3.56,

$$f(x) = \sqrt[4]{1+|x|}$$

1. *Domain of definition* $\mathbb{D}$: absolute function $|*|$ is always positive, by consequence argument $(1+|x|)$ of fourth root function always positive, therefore $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt[4]{1+|-x|} = \sqrt[4]{1+|x|} = f(x) \Rightarrow \quad \text{even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\sqrt[4]{1+|x|}\right) = -\sqrt[4]{1+|x|} \neq f(x) \Rightarrow \quad \text{not odd}$$

   In conclusion, this radical function is not odd (i.e., symmetric relative to the origin) because $f(x) = -f(-x)$, and it is even because $f(x) = f(-x)$.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt[4]{1+|x|} \neq 0 \quad \therefore \quad f(x) > 0 \quad \forall x \quad \text{(no zeros)}$$

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt[4]{1+|0|} = 1$.
5. *Limits:* there are two limits in total to be resolved,

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt[4]{1+|x|} = +\infty$$

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt[4]{1+|x|} = +\infty$$

6. *Oblique asymptote:* it is useful to recall technique for finding derivative of $|x|$, i.e.,

$$(|x|)' = \left(\sqrt{x^2}\right)' = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|} \equiv \frac{|x|}{x} = \text{sign}\,(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \text{(by convention)}$$

so that,

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt[4]{1+|x|}}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\left((1+|x|)^{1/4}\right)'}{(x)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{4}\,(1+|x|)^{-3/4}\,(|x|)'}{1} = \frac{1}{4} \lim_{x\to\infty} \frac{\cancel{\text{sign}\,(x)}^{\pm 1}}{\sqrt[4]{(1+|x|)^3}}$$

$$= 0$$

As $a = 0$ thus there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad \left\{(|x|)' = \text{sign}\,(x) = \frac{x}{|x|} = \frac{|x|}{x}\right\}$$

$$f'(x) = \left((1+|x|)^{1/4}\right)' = \frac{1}{4}\,\frac{\text{sign}\,(x)}{\sqrt[4]{(1+|x|)^3}} = 0 \quad \therefore \quad \text{sign}\,(x) = 0$$

$$\therefore \quad x = 0 \quad \text{and,} \quad f(0) = 1$$

by convention, sign $(0) = 0$, so that sign of $f'(x)$ is,

x		0	
$1/4$	+	+	+
sign (x)	−	0	+
$\sqrt[4]{(1+\|x\|)^3}$	+	+	+
$f'(x)$	−	n.d	+
$f(x)$	↘	1	↗

There is minimum point at $(x, y) = (0, 1)$ because $f(x)$ is a continuous function. Note that $f'(0)$ is not defined because its left- and right-side limits to zero are different. Also,

$$\underline{f''(x) = 0:} \quad \left\{\text{sign}\,(x) = \pm 1 \quad \therefore \quad (\text{sign}\,(x))' = 0\right\}$$

$$f''(x) = \left(\frac{1}{4}\,\frac{\text{sign}\,(x)}{\sqrt[4]{(1+|x|)^3}}\right)'$$

$$= \frac{1}{4}\,\frac{\cancel{(\text{sign}\,(x))'}^{0}\sqrt[4]{(1+|x|)^3} - \text{sign}\,(x)\,\left(\sqrt[4]{(1+|x|)^3}\right)'}{\sqrt[4]{(1+|x|)^6}}$$

$$= \frac{1}{4}\,\frac{-\text{sign}\,(x)\;(3/4)\,(1+|x|)^{-1/4}(|x|)'}{\sqrt[4]{(1+|x|)^6}}$$

$$= -\frac{3}{16}\,\frac{\text{sign}\,(x)\;\text{sign}\,(x)}{\sqrt[4]{(1+|x|)}\,\sqrt[4]{(1+|x|)^6}}$$

$$= \begin{cases} -\dfrac{3}{16}\,\dfrac{1}{\sqrt[4]{(1+|x|)^7}} & ;\,(x \neq 0) \\ 0 & ;\,(x = 0) \end{cases}$$

because,

$$\operatorname{sign}(x)\ \operatorname{sign}(x) = \begin{cases} 0; & (x = 0) \\ 1; & (x \neq 0) \end{cases}$$

by convention, $\operatorname{sign}(0) = 0$. However, note that $f''(0)$ is not defined because its left- and right-side limits to zero are different.

x		0	
$-3/16$	$-$	$-$	$-$
$\operatorname{sign}(x)\ \operatorname{sign}(x)$	$+$	0	$+$
$\sqrt[3]{(1+\lvert x\rvert)^7}$	$+$	$+$	$+$
$f''(x)$	$-$	n.d.	$-$
$f(x)$	$\cap$	1	$\cap$

There are no inflection points, and $f(x)$ is always convex $\cap$.

8. *Summary of the important points:*

y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
zeros:	(none, $f(x)$ is always positive)
extremes:	$(x, y) = (0, 1)$ (min)
inflections:	(none)

9. *Graphical representation:* results of the analysis for function in P.3.56 are summarized by $f(x)$ graph (see Fig. 3.48).

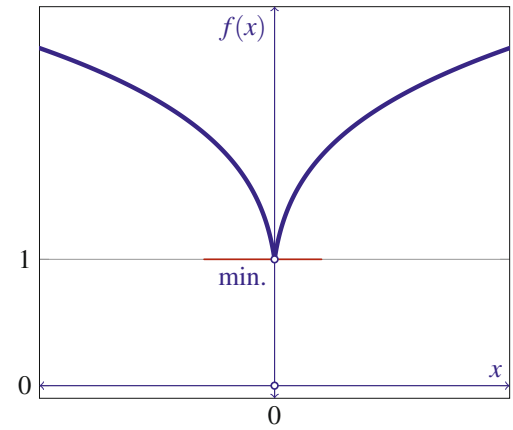


Fig. 3.48 Example P.3.56

3.57. Given radical function

$$f(x) = \sqrt{x + |x|}$$

1. *Domain of definition* $\mathbb{D}$: absolute function $|*|$ is always positive, and argument of square root must be non-negative; thus

$$x + |x| \geq 0 \quad \text{and,} \quad \begin{cases} (x \geq 0) \Rightarrow |x| = x\,; & f(x) = \sqrt{x + x} = \sqrt{2x} \quad \therefore \quad x \in \mathbb{R} | x \geq 0 \\ (x < 0) \Rightarrow |x| = -x\,; & f(x) = \sqrt{x - x} = 0 \quad \therefore \quad x \in \mathbb{R} | x < 0 \end{cases}$$

   in conclusion, $x \in \mathbb{R}$. It is therefore noted that

$$f(x) = \sqrt{x + |x|} = \begin{cases} \sqrt{2x} = \sqrt{2}\sqrt{x}, & (x \geq 0) \\ 0, & (x < 0) \end{cases}$$

   That is to say, on the positive side ($x \geq 0$), this function is a simple square root function, already analyzed in P.3.48, but this time scaled by the $\sqrt{2}$ factor. At the negative side ($x < 0$), this function always equals zero.
2. *Summary of the important points:* (for $+\sqrt{*}$)

y-axis crossing point $f(0)$:	$(x, y) = (0, 0)$
zero:	$(x, y) = (0, 0)$
extremes:	(none, $f(x)$ always ascends $\nearrow$)
inflections:	(none, $f(x)$ always is always convex $\cap$)

3. *Graphical representation:* see Fig. 3.49, where negative part of square root function is mirrored along the horizontal axis,

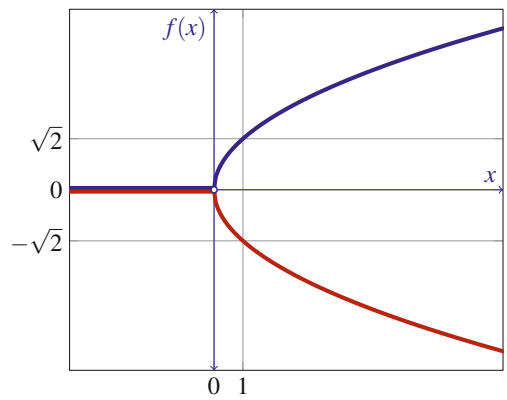


Fig. 3.49 Example P.3.57

3.58. Given radical function in P.3.58,

$$f(x) = \sqrt{x - \sqrt{x}}$$

1. *Domain of definition* $\mathbb{D}$: arguments of both square roots must be non-negative; thus

$$x - \sqrt{x} \geq 0 \;\; \therefore \;\; x \geq \sqrt{x} \;\; \therefore \;\; x^2 \geq x \;\; \therefore \;\; x^2 - x \geq 0 \;\; \therefore \;\; x(x-1) \geq 0$$

$$\begin{cases} x & \geq 0 \;\; \therefore \;\; x = 0 \quad \text{is included} \\ & \text{and,} \\ x - 1 & \geq 0 \;\; \therefore \;\; x \geq 1 \;\; \therefore \;\; x \geq 1 \end{cases}$$

$$\therefore \;\; x \in \mathbb{R} | x = 0,\; x \geq 1$$

   In summary, $x \in \mathbb{R} | x = 0, x \geq 1$, note that $x = 0$ is an isolated point.
2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = \sqrt{-x - \sqrt{-x}} \;\; \therefore \qquad \text{not defined}$$

   In conclusion, this radical function is neither odd nor even because $f(-x)$ is not defined.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; \sqrt{x - \sqrt{x}} = 0 \;\; \therefore \;\; \begin{cases} & x_1 = 0, \\ x - \sqrt{x} \geq 0 \;\; \therefore & x_2 = 1 \end{cases}$$

   There is one isolated first-order zero at $(x, y) = (0, 0)$, and one first-order zero at $(x, y) = (1, 0)$.
4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{0 - \sqrt{0}} = 0$ (a single isolated point).
5. *Limits:* in $x \geq 1$, there is only one limit in total to be resolved,

$$\begin{aligned} \underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) &= \lim_{x \to +\infty} \sqrt{x - \sqrt{x}} = (\infty - \infty) \;\; \text{n.d.} \\ &= \sqrt{\lim_{x \to +\infty} (x - \sqrt{x})} = \sqrt{\lim_{x \to +\infty} x \left(1 - \frac{\sqrt{x}}{x}\right)} \\ &= \sqrt{\lim_{x \to +\infty} x \left(1 - \sqrt{1/x}^{\,0}\right)} \\ &= +\infty \;\; \text{for } (y > 0) \end{aligned}$$

6. *Oblique asymptote:* $(x \geq 1)$

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\sqrt{x - \sqrt{x}}}{x} = \sqrt{\lim_{x \to \infty} \frac{x - \sqrt{x}}{x^2}}$$

$$= \sqrt{\lim_{x\to\infty}\left(\cancel{(1/x)}^{0} - \sqrt{\cancel{1/x^3}}^{0}\right)}$$
$$= 0$$

As $a = 0$ thus there is no oblique asymptote.

7. *Critical points:*

$\underline{f'(x) = 0:}$

$$f'(x) = \left((x-\sqrt{x})^{1/2}\right)' = \frac{1}{2}(x-\sqrt{x})^{-1/2}\left(1-\frac{1}{2\sqrt{x}}\right)$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{x-\sqrt{x}}} - \frac{1}{2\sqrt{x}\sqrt{x-\sqrt{x}}}\right)$$
$$= \frac{1}{4}\frac{2\sqrt{x}-1}{\sqrt{x}\sqrt{x-\sqrt{x}}}$$
$$= 0$$
$$\therefore\ 2\sqrt{x}-1=0\ \ \therefore\ \ \sqrt{x}=\frac{1}{2}\ \ \therefore\ \ x=\frac{1}{4}\notin\mathbb{R}|x\geq 1$$

thus, $f'(x) > 0$ because if $(x \geq 1)$ then $2\sqrt{x} > 1$ and, by consequence $f(x)$ ascends $\nearrow$ for $y > 0$ (and descends $\searrow$ for $y < 0$). Also,

$\underline{f''(x) = 0:}\ (x \geq 1)$

$$f''(x) = \left(\frac{1}{4}\frac{2\sqrt{x}-1}{\sqrt{x}\sqrt{x-\sqrt{x}}}\right)' = \frac{1}{4}\left(\frac{2\sqrt{x}-1}{\sqrt{x(x-\sqrt{x})}}\right)' = \frac{1}{4}\left(\frac{2\sqrt{x}-1}{\sqrt{x^2-x^{3/2}}}\right)'$$
$$= \frac{1}{4}\frac{\frac{1}{\sqrt{x}}\sqrt{x(x-\sqrt{x})} - (2\sqrt{x}-1)\frac{2x-\frac{3\sqrt{x}}{2}}{2\sqrt{x(x-\sqrt{x})}}}{x(x-\sqrt{x})}$$
$$= \frac{1}{4}\frac{\sqrt{(x-\sqrt{x})} - \frac{(2\sqrt{x}-1)(4x-3\sqrt{x})}{4\sqrt{x(x-\sqrt{x})}}}{x(x-\sqrt{x})}$$
$$= \frac{1}{16}\frac{4\sqrt{(x-\sqrt{x})}\sqrt{x(x-\sqrt{x})} - (2\sqrt{x}-1)(4x-3\sqrt{x})}{x(x-\sqrt{x})\sqrt{x(x-\sqrt{x})}}$$
$$= \frac{1}{16}\frac{4\sqrt{x}(x-\sqrt{x}) - (2\sqrt{x}-1)(4x-3\sqrt{x})}{x(x-\sqrt{x})\sqrt{x(x-\sqrt{x})}}$$
$$= \frac{1}{16}\frac{6x-(4x+3)\sqrt{x}}{x(x-\sqrt{x})\sqrt{x(x-\sqrt{x})}} = 0\ \ \therefore\ \ 6x-(4x+3)\sqrt{x}=0$$

Note that, given $x \geq 1$, all terms in the denominator are positive (for $+\sqrt{*}$):

$$16 \geq 10,\ x \geq 0,\ (x - \sqrt{x}) \geq 0,\ \sqrt{x(x - \sqrt{x})} \geq 0$$

therefore, the denominator is also greater than zero. In order to decide sign of $f''(x)$, the question is then: given that $x \geq 1$ which term is greater: "$6x$" or "$(4x + 3)\sqrt{x}$?" One of the possible ways is to deduce the answer is

$$\underline{x \geq 1} \quad \therefore \ (4x + 3) > 6x, \quad \text{in addition} \quad \sqrt{x} \geq 1 \quad \therefore \ \sqrt{x}(4x + 3) > 6x$$
$$\therefore \ 6x - (4x + 3)\sqrt{x} < 0$$

for all $x \geq 1$, and by consequence $f''(0) < 0$, i.e., this function is always convex $\cap$ (for $+\sqrt{*}$).

8. *Summary of the important points:* (for $y > 0$)

defined:	$x = 0,\ x \geq 1$
zeros:	$(x, y) = (1, 0)$
	$(x, y) = (0, 0)$
extremes:	(none, $f(x)$ always ascends $\nearrow$)
inflections:	(none, $f(x)$ is always convex $\cap$)

9. *Graphical representation:* results of the analysis for function in P.3.58 are summarized by $f(x)$ graph (see Fig. 3.50).

3.59. Given radical function

$$f(x) = \sqrt[3]{x^3 - 1}$$

1. *Domain of definition* $\mathbb{D}$: odd order radicals are defined for both positive and negative arguments, as well as polynomials; thus $x \in \mathbb{R}$.

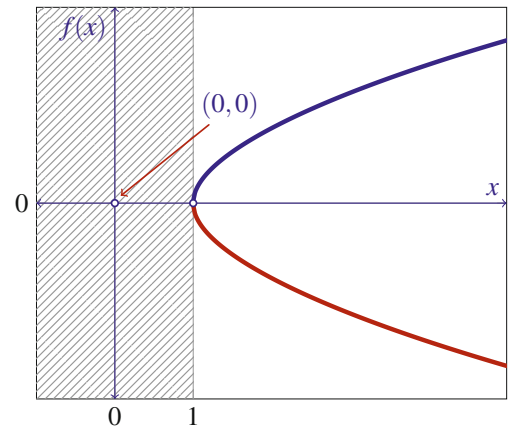


Fig. 3.50 Example P.3.58

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt[3]{(-x)^3 - 1} = \sqrt[3]{-x^3 - 1} \neq f(x) \Rightarrow \quad \text{not even}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\sqrt[3]{-x^3 - 1}\right) = \sqrt[3]{x^3 + 1} \neq f(x) \Rightarrow \quad \text{not odd}$$

This radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt[3]{x^3 - 1} = 0 \quad \therefore \quad x^3 - 1 = 0 \quad \therefore \quad x = 1 \quad \text{(first-order real zero)}$$

note that, $\sqrt[3]{x^3} = x$. With only one odd zero and no vertical asymptotes, $f(x)$ is negative before its zero and positive after. The remaining two x^3 zeros are complex.

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt[3]{0^3 - 1} = -1$.

5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt[3]{x^3 - 1} = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \sqrt[3]{x^3 - 1} = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\sqrt[3]{x^3 - 1}}{x} = \sqrt[3]{\lim_{x \to \infty} \frac{x^3 - 1}{x^3}} = \sqrt[3]{\lim_{x \to \infty} \left(1 - \cancel{1/x^3}^{\,0}\right)}$$

$$= 1$$

$$b = \lim_{x \to \infty} (f(x) - ax) = \lim_{x \to \infty} \left(\sqrt[3]{x^3 - 1} - x\right) = \lim_{x \to \infty} \left(x \sqrt[3]{1 - \cancel{1/x^3}^{\,0}} - x\right)$$

$$= \lim_{x \to \infty} (x - x) = 0$$

There is $y_{\mathtt{aa}} = x$ oblique asymptote.

7. *Critical points:*

$\underline{f'(x) = 0:}$

$$f'(x) = \left((x^3 - 1)^{(1/3)}\right)' = \frac{1}{\cancel{3}} (x^3 - 1)^{(-2/3)} \, \cancel{3}x^2 = \frac{x^2}{\sqrt[3]{(x^3 - 1)^2}} = 0$$

$$\therefore \quad x^2 = 0 \quad \therefore \quad x_{1,2} = 0 \quad \text{(second-order zero)}$$

$$\text{and, } f(0) = -1$$

In addition, $x = 1$ is pole of $f'(x)$, so that,

x		0		1	
x^2	+	0	+	+	+
$\sqrt[3]{(x^3-1)^2}$	+	+	+	0	+
$f'(x)$	+	0	+	n.d.	+
$f(x)$	↗	−1	↗	0	↗

Note that by crossing even order zero $f'(x)$ does not change its sign. Similarly, crossing even order pole at $x = 1$ (note the quadratic term) does not change sign of $f'(x)$. Consequently, due to even order poles and zeros, there are no extreme points. Also,

$f''(x) = 0$:

$$f''(x) = \left(\frac{x^2}{\sqrt[3]{(x^3-1)^2}}\right)' = \frac{2x\sqrt[3]{(x^3-1)^2} - \frac{x^2\,(2/3)\,3x^2}{(x^3-1)^{1/3}}}{\sqrt[3]{(x^3-1)^4}}$$

$$= \frac{\frac{2x\sqrt[3]{(x^3-1)^3} - 2x^4}{(x^3-1)^{1/3}}}{\sqrt[3]{(x^3-1)^4}} = 2x\frac{\sqrt[3]{(x^3-1)^3} - x^3}{\sqrt[3]{(x^3-1)^5}}$$

$$= 2x\frac{\sqrt[3]{(x^3-1)^3} - x^3}{\sqrt[3]{(x^3-1)^5}} = -\frac{2x}{\sqrt[3]{(x^3-1)^5}}$$

$$= 0 \quad \therefore \quad x = 0 \text{ (first (odd) order zero)}$$

Also, note the fifth-order term in the denominator; thus there is odd order pole of $f''(x)$ at $x = 1$ (i.e. $f''(1) = \infty$), so that

x		0		1	
-2	−	−	−	−	−
x	−	0	+	+	+
$\sqrt[3]{(x^3-1)^5}$	−	−	−	0	+
$f''(x)$	−	0	+	n.d.	−
$f(x)$	∩	−1	∪	0	∩

Even though $f''(1)$ is not defined (fifth-order pole), there are two inflection points at $(x, y) = (0, -1)$ and $(x, y) = (1, 0)$ because $f(x)$ is continuous function, i.e., $f(1)$ is defined.

8. *Summary of the important points:*

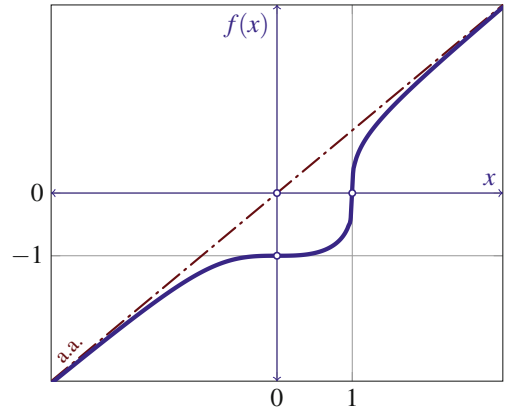
y-axis crossing point $f(0)$: $(x, y) = (0, -1)$

a.a. asymptote: $y = x$

zero: $(x, y) = (1, 0)$

extremes: (none, $f(x)$ always ascends ↗)

Fig. 3.51 Example P.3.59



$$\underline{\text{inflections:}} \quad (x, y) = (0, -1)$$
$$(x, y) = (1, 0)$$

9. *Graphical representation:* see Fig. 3.51. Note the function shape at $x = 1$.

3.60. Given radical function and its alternative forms

$$f(x) = \sqrt{x + \sqrt{\frac{1}{x}}}$$

note that, depending on $\pm\sqrt{*}$ signs, there are *four* possible functions:

$$f_1(x) = +\sqrt{x + \sqrt{\frac{1}{x}}}, \quad f_2(x) = -\sqrt{x + \sqrt{\frac{1}{x}}}$$

$$f_3(x) = +\sqrt{x - \sqrt{\frac{1}{x}}}, \quad f_4(x) = -\sqrt{x - \sqrt{\frac{1}{x}}}$$

where $f_{3,4}$ ($y < 0$) are mirrored images of $f_{1,2}$ ($y > 0$), respectively.

Branch #1: "default" $f_1(x)$, i.e., both square roots positive. Note that its mirrored image $f_2(x)$ is with the negative sign of the exterior square root.

$$f(x) = \sqrt{x + \sqrt{\frac{1}{x}}}$$

1. *Domain of definition* $\mathbb{D}$: inverse function $1/x$ is defined for $x \neq 0$ and square root for $x \geq 0$, thus in total $x \in \mathbb{R}|x > 0$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt{(-x) + \sqrt{\frac{1}{(-x)}}} = \sqrt{-x + \sqrt{\frac{1}{-x}}} \quad \therefore \quad (\text{ not defined })$$

because if $x > 0$, then square root is not defined for negative numbers. In conclusion, this radical function is neither odd nor even.

3. *Function's sign:* (for $+\sqrt{*}$)

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt{x + \sqrt{\frac{1}{x}}} = 0 \Rightarrow x + \sqrt{\frac{1}{x}} = 0 \quad \text{but} \quad + \sqrt{\frac{1}{x}} \neq -x$$

because left side of the equation is positive and the right negative. Therefore, there are no zeros.

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{0 + \sqrt{\frac{1}{0}}}$ (not defied).
5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt{x^{+\infty} + \sqrt{1/x^{0}}} = +\infty$$

$$\underline{x \to \underset{\leftarrow}{0}:} \quad \lim_{x \to \underset{\leftarrow}{0}} f(x) = \lim_{x \to \underset{\leftarrow}{0}} \sqrt{x^{0} + \sqrt{1/x^{+\infty}}} = +\infty$$

6. *Oblique asymptote:* $(x > 0)$

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\sqrt{x + \sqrt{\frac{1}{x}}}}{x} = \lim_{x \to \infty} \sqrt{\frac{x}{x^2} + \frac{1}{x^2}\sqrt{\frac{1}{x}}} = \lim_{x \to \infty} \sqrt{1/x^{0} + \sqrt{1/x^{5}}^{0}} = 0$$

In conclusion, $a = 0$, thus there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:}$$

$$f'(x) = \left(\sqrt{x + \sqrt{\frac{1}{x}}}\right)' = \frac{1 + \frac{-1/x^2}{2\sqrt{1/x}}}{2\sqrt{x + \sqrt{1/x}}} = \frac{2\sqrt{1/x} - 1/x^2}{4\sqrt{1/x}\sqrt{x + \sqrt{1/x}}}$$

$$= \frac{1}{4}\frac{2\sqrt{1/x} - 1/x^2}{\sqrt{1/x}\sqrt{x + \sqrt{1/x}}} = \frac{1}{4}\frac{2\sqrt{x^3} - 1}{\sqrt{x^3}\sqrt{x + \sqrt{\frac{1}{x}}}} = \frac{1}{4}\frac{2\sqrt{x^3} - 1}{\sqrt{x^4 + \sqrt{x^5}}} = 0$$

$$\therefore \quad 2\sqrt{x^3} - 1 = 0 \quad \therefore \quad x_1 = \frac{1}{\sqrt[3]{4}} = \frac{\sqrt[3]{2}}{2}$$

$$\text{and, } f\left(\frac{1}{\sqrt[3]{4}}\right) = \sqrt{\frac{1}{\sqrt[3]{4}} + \sqrt{\sqrt[3]{4}}} = \sqrt{\frac{1}{\sqrt[3]{4}} + \sqrt[3]{2}} = y_1 \approx 1.375$$

so that,

x	0		x_1	
$1/4$	$1/4$	$1/4$	$1/4$	$1/4$
$2\sqrt{x^3}-1$	$-$	$-$	0	$+$
$\sqrt{x^4+\sqrt{x^5}}$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	$-$	0	$+$
$f(x)$	n.d.	$\searrow$	y_1	$\nearrow$

There is one minimum point at $(x, y) = (x_1, y_1)$. Also,

$f''(x) = 0$:

$$
\begin{aligned}
f''(x) &= \left(\frac{1}{4}\,\frac{2\sqrt{x^3}-1}{\sqrt{x^4+\sqrt{x^5}}}\right)' \\
&= \frac{1}{4}\,\frac{\cancel{2}\,\dfrac{3x^2}{\cancel{2}\sqrt{x^3}}\sqrt{x^4+\sqrt{x^5}} - (2\sqrt{x^3}-1)\,\dfrac{4x^3+\dfrac{5x^4}{2\sqrt{x^5}}}{2\sqrt{x^4+\sqrt{x^5}}}}{x^4+\sqrt{x^5}} \\
&= \frac{1}{4}\,\frac{\dfrac{3x^2\sqrt{x^4+\sqrt{x^5}}}{\sqrt{x^3}} - \dfrac{(2\sqrt{x^3}-1)\,(8x^3\sqrt{x^5}+5x^4)}{4\sqrt{x^5}\sqrt{x^4+\sqrt{x^5}}}}{x^4+\sqrt{x^5}} \\
&= \frac{1}{16}\,\frac{12x^2\,(x^4+\sqrt{x^5})\sqrt{x^5} - \sqrt{x^3}\,(2\sqrt{x^3}-1)\,(8x^3\sqrt{x^5}+5x^4)}{(x^4+\sqrt{x^5})\sqrt{x^3}\sqrt{x^5}\sqrt{x^4+\sqrt{x^5}}} \\
&= \frac{1}{16}\,\frac{12x^2\,(x^4+\sqrt{x^5})\cancel{\sqrt{x^3}}\sqrt{x^2} - \cancel{\sqrt{x^3}}\,(2\sqrt{x^3}-1)\,(8x^3x^2\sqrt{x}+5x^4)}{(x^4\sqrt{x^5}+x^5)\cancel{\sqrt{x^3}}\sqrt{x^4+\sqrt{x^5}}} \\
&= \frac{1}{16}\,\frac{12x^2\,(x^4+\sqrt{x^5})\cancel{\sqrt{x^2}} - (2\sqrt{x^3}-1)\,x^4\,(8x\sqrt{x}+5)}{(x^4x^2\sqrt{x}+x^5)\sqrt{x^4+\sqrt{x^5}}} \\
&= \frac{1}{16}\,\frac{12\cancel{x^2}x^2\,(x^2+\sqrt{x})\cancel{\sqrt{x^2}} - (2\sqrt{x^3}-1)\,x^{\cancel{4}2}\,(8\sqrt{x^3}+5)}{x^{\cancel{3}3}\,(\sqrt{x^3}+1)\sqrt{x^4+\sqrt{x^5}}} \\
&= \frac{1}{16}\,\frac{12\cancel{x^2}\,(x^2+\sqrt{x})\,x - (2\sqrt{x^3}-1)\cancel{x^2}\,(8\sqrt{x^3}+5)}{x^{\cancel{3}}\,(\sqrt{x^3}+1)\sqrt{x^4+\sqrt{x^5}}} \\
&= \frac{1}{16}\,\frac{12x^3+12\sqrt{x^3}-16x^3-10\sqrt{x^3}+8\sqrt{x^3}+5}{x\,(\sqrt{x^3}+1)\sqrt{x^4+\sqrt{x^5}}} \\
&= \frac{1}{16}\,\frac{10\sqrt{x^3}-4x^3+5}{x\,(\sqrt{x^3}+1)\sqrt{x^4+\sqrt{x^5}}} \\
&= 0 \quad \therefore \quad P_3(x) = 10\sqrt{x^3}-4x^3+5 = 0
\end{aligned}
$$

This third-order polynomial with radical terms cannot be factorized by using the algebraic identities, so the solution is found by graphical/numerical methods.

Graphical method: first,

$$10\sqrt{x^3} - 4x^3 + 5 = 0 \quad \therefore \quad 10\sqrt{x^3} = 4x^3 - 5$$

which is to say that the intersect between a radical on the left and cubic function on the right is the only common point between the two functions. Cubic function is "growing faster"; thus there must be one intersect point the $x > 0$ side (see Fig. 3.52 (left)). As the first estimate, evidently, these two functions are close to each other in the proximity of $x_0 \approx 2$.

Numerical method: after taking the initial guess $x_0 = 2$, the iterative Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where,

$$f(x_n) = 10\sqrt{x_n^3} - 4x_n^3 + 5$$

$$f'(x_n) = \left(10\sqrt{x_n^3} - 4x_n^3 + 5\right)' = 10\frac{3x_n^2}{2\sqrt{x_n^3}} - 12x_n^2 = 15\sqrt{x_n} - 12x_0^2$$

results in

n	x	$f(x)$	$f'(x)$
0	2.000	1.284	−26.787
1	2.048	−0.495	−28.863
2	2.046	-6.465×10^{-5}	−28.788
3	2.046	-1.106×10^{-10}	−28.903

with this initial guess $x_0 = 2$, to the third decimal place $x = 2.046$. To illustrate the result,

$$P_3(2.046) = 10\sqrt{2.046^3} - 4 \times 2.046^3 + 5 \approx -6.0468 \times 10^{-10}$$

and,

$$f(2.046) \approx \sqrt{2.046 + \sqrt{\frac{1}{2.046}}} \approx 1.657$$

In summary, there is one zero of $f''(x)$ found at $(x_1, y_1) = (2.046, 1.657)$. Note that even the first graphical estimation $x_0 \approx 2$ is not far off, because $y_0 = f(x_0) \approx \sqrt{1 + \sqrt{1/2}} \approx 1.645$. In order to simplify the following tabular analysis, note that $f''(x)$ denominator is always positive for $+\sqrt{*}$, i.e., $D(f''(x)) = 16x\,(\sqrt{x^3} + 1)\sqrt{x^4 + \sqrt{x^5}} > 0$ given that $(x > 0)$, and by adopting the numerator notation as $N(f''(x)) = 10\sqrt{x^3} - 4x^3 + 5$, it follows

x	0		x_1	
$N(f''(x))$	+	+	0	−
$D(f''(x))$	+	+	+	+
$f''(x)$	0	+	0	−
$f(x)$	n.d.	$\cup$	y_1	$\cap$

There is one inflection point at $(x, y) = (2.026, 1, 657)$.

8. *Summary of the important points, brunch #1:* (for $y > 0$ only)

$$\begin{aligned} \underline{\text{defined:}} \quad & x > 0 \\ \underline{\text{zeros:}} \quad & (\text{none}, f(x) \geq 0) \\ \underline{\text{v. asymptotes:}} \quad & x = 0 \\ \underline{\text{extremes:}} \quad & (x, y) = \left(\frac{1}{\sqrt[3]{4}}, \sqrt{\frac{1}{\sqrt[3]{4}} + \sqrt[3]{2}} \right) \quad (\text{min}) \\ \underline{\text{inflection:}} \quad & (x, y) = (2.046, 1.657) \end{aligned}$$

Branch #2: mirrored image $f_3(x)$: i.e., when the interior square root is negative.

$$f(x) = \sqrt{x - \sqrt{\frac{1}{x}}}$$

1. *Domain of definition* $\mathbb{D}$: inverse function $1/x$ is defined for $x \neq 0$ and square root for $x \geq 0$,

$$x - \sqrt{\frac{1}{x}} \geq 0 \quad \therefore \quad x \geq \sqrt{\frac{1}{x}} \quad \therefore \quad x^2 \geq \frac{1}{x} \quad \therefore \quad x^3 \geq 1 \quad \therefore \quad x \geq 1$$

thus in total $x \in \mathbb{R} | x \geq 1$.

2. *Function's parity:* $(x \geq 1)$

$$\underline{\text{even:}} \quad f(-x) = \sqrt{(-x) - \sqrt{\frac{1}{(-x)}}} = \sqrt{-x - \sqrt{\frac{1}{-x}}} \quad \therefore \quad (\text{ not defined })$$

because $x > 0$ therefore square root is not defined for negative numbers. In conclusion, this radical function is neither odd nor even.

3. *Function's sign:* $(x \geq 1)$

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt{x - \sqrt{\frac{1}{x}}} = 0 \quad \therefore \quad x - \sqrt{\frac{1}{x}} = 0 \quad \therefore \quad \sqrt{\frac{1}{x}} = x \quad \therefore \quad x = 1$$

There is one zero at $(x, y) = (1, 0)$, while $f(x) > 0$ for all $x > 1$.

4. *y-axis crossing point* $f(0)$: $f(0)\sqrt{0-\sqrt{\frac{1}{0}}}$ (not defied).
5. *Limits:* there is only one limit in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x^{+\infty} - \sqrt{1/x^{0}}} = +\infty$$

6. *Oblique asymptote:* $(x > 0)$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{x-\sqrt{\frac{1}{x}}}}{x} = \lim_{x\to\infty} \sqrt{\frac{x}{x^2}\frac{1}{x} - \frac{1}{x^2}\sqrt{\frac{1}{x}}} = \lim_{x\to\infty} \sqrt{1/x^{0} - \sqrt{1/x^{5}}^{0}}$$
$$= 0$$

In conclusion, $a = 0$, thus there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:}$$

$$f'(x) = \left(\sqrt{x-\sqrt{\frac{1}{x}}}\right)' = \frac{1-\frac{-1/x^2}{2\sqrt{1/x}}}{2\sqrt{x-\sqrt{1/x}}} = \frac{1}{4}\frac{2\sqrt{1/x}+1/x^2}{\sqrt{1/x}\sqrt{x-\sqrt{1/x}}} \quad \frac{x^2}{x^2}$$
$$= \frac{1}{4}\frac{2\sqrt{x^3}+1}{\sqrt{x^3}\sqrt{x-\sqrt{\frac{1}{x}}}} = \frac{1}{4}\frac{2\sqrt{x^3}+1}{\sqrt{x^4-\sqrt{x^5}}} = 0$$
$$\therefore \quad 2\sqrt{x^3}+1 \neq 0 \quad (x \geq 0)$$

Due to $f'(x) > 0$, there are no extreme points, i.e., $f(x)$ always ascends ↗. Also,

$$\underline{f''(x) = 0:}$$

$$f''(x) = \left(\frac{1}{4}\frac{2\sqrt{x^3}+1}{\sqrt{x^4-\sqrt{x^5}}}\right)'$$
$$= \{\text{similarly as } f''(x) \text{ of branch \#1}\}$$
$$= \frac{1}{16}\frac{-10\sqrt{x^3}-4x^3+5}{x\,(\sqrt{x^3}-1)\sqrt{x^4-\sqrt{x^5}}}$$
$$= 0 \quad \text{however, } P_3(x) = -10\sqrt{x^3}-4x^3+5 \neq 0$$

This third-order polynomial with radical terms is always negative for $x \geq 1$. (Note that $P(1) = -9$ and it stays negative for any other $x > 1$.) By consequence, there are no inflection points.

8. *Summary of the important points, brunch #2:* (for $y > 0$ only)

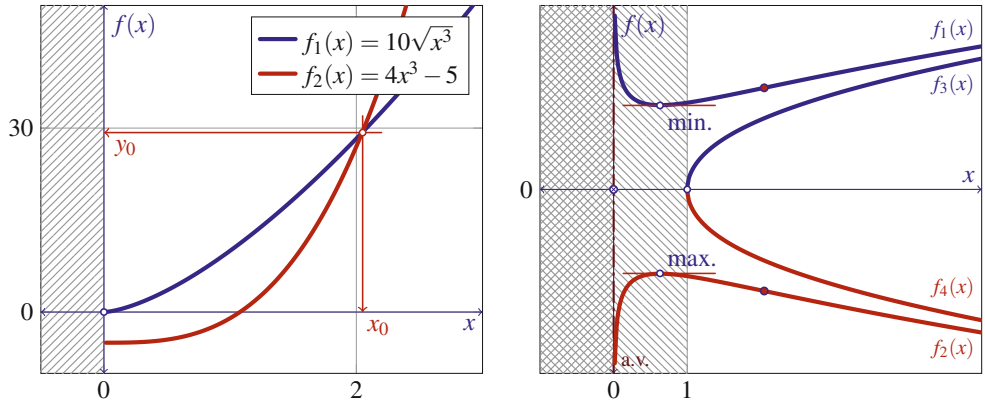


Fig. 3.52 Example P.3.60

defined:	$x \geq 1$
zeros:	$(x, y) = (1, 0)$
extremes:	(none, this branch of $f(x)$ always ascends $\nearrow$)
inflections:	(none, this branch of $f(x)$ is always convex $\cap$)

Graphical representation: graphical solution for the inflection point (x_0, y_0) is shown in Fig. 3.52 (left). Graph including all four versions of $f(x)$ in P.3.60 is shown in Fig. 3.52 (right).

3.61. Given radical function and its alternative forms

$$f(x) = \sqrt{x - \frac{1}{x}} = \sqrt{\frac{x^2 - 1}{x}} = \sqrt{\frac{(x-1)(x+1)}{x}}$$

1. *Domain of definition* $\mathbb{D}$: there are two functions that limit the domain of definition:
   - the the square root argument is a rational function; thus $x \neq 0$
   - the square root function itself; thus

$$x - \frac{1}{x} = \frac{x^2 - 1}{x} = \frac{(x-1)(x+1)}{x} \geq 0$$

which is satisfied when both numerator and denominator have the same sign (i.e., either both positive, or both negative), which can be summarized as

x		-1		0		1	
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$\sqrt{(x+1)(x-1)/x}$	n.d.	0	$+$	n.d.	n.d.	0	$+$
$f(x)$	n.d.	0	$+$	n.d.	n.d.	0	$+$

thus, in total $x \in \mathbb{R} | -1 \leq x < 0$ or, $x \geq 1$. In addition, $x = 0$ is a vertical asymptote.

2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = \sqrt{(-x) - \frac{1}{(-x)}} = \sqrt{-x + \frac{1}{x}} \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\sqrt{-x + \frac{1}{x}}\right) = \sqrt{x - \frac{1}{x}} \neq f(x) \quad \therefore \quad \text{(not odd)}$$

This radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt{\frac{(x-1)(x+1)}{x}} = 0 \quad \therefore \quad (x-1)(x+1) = 0$$

$$\therefore \quad x_1 = -1, \quad x_2 = 1$$

There are two first-order zeros, one at $(x, y) = (-1, 0)$ and one at $(x, y) = (1, 0)$. Having the form of $f(x) = \pm\sqrt{*}$, this function is positive at $y \geq 0$ side and negative at $y < 0$ side (its mirrored image).

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{0 - 1/0}$ is not defined.

5. *Limits:* there are two limits in total to be resolved (at $y \geq 0$ side).

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x^{+\infty} - 1/x^{0}} = +\infty$$

$$\underline{x \to \underset{\to}{0}:} \quad \lim_{x\to\underset{\to}{0}} f(x) = \lim_{x\to\underset{\to}{0}} \sqrt{x^{0} - 1/x^{+\infty}} = +\infty$$

6. *Oblique asymptote:* $(x \geq 1)$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{x - \frac{1}{x}}}{x} = \lim_{x\to\infty} \sqrt{\frac{x}{x^2}\frac{1}{x} - \frac{1}{x^2}\frac{1}{x}} = \lim_{x\to\infty} \sqrt{1/x^{0} - 1/x^{3}}^{0}$$

$$= 0$$

As $a = 0$ there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:}$$

$$f'(x) = \left(\sqrt{x - \frac{1}{x}}\right)' = \frac{1 + \frac{1}{x^2}}{2\sqrt{x - \frac{1}{x}}} = \frac{x^2 + 1}{2x^2\sqrt{x - \frac{1}{x}}}$$

$$\therefore \quad x^2 + 1 > 0 \quad \forall x$$

This function always ascends ↗ (at $y \geq 0$ side). Also,

$$\underline{f''(x) = 0:}$$

$$f''(x) = \left(\frac{x^2+1}{2x^2\sqrt{x-\frac{1}{x}}}\right)' = \frac{1}{2}\left(\frac{x^2+1}{x^2\sqrt{\frac{x^2-1}{x}}}\right)' = \frac{1}{2}\left(\frac{x^2+1}{\sqrt{x^5-x^3}}\right)'$$

$$= \frac{1}{2}\frac{2x\sqrt{x^5-x^3} - (x^2+1)\dfrac{5x^4-3x^2}{2\sqrt{x^5-x^3}}}{x^5-x^3}$$

$$= \frac{1}{4}\frac{4x\sqrt{x^5-x^3}\sqrt{x^5-x^3} - (x^2+1)(5x^4-3x^2)}{\sqrt{x^5-x^3}(x^5-x^3)}$$

$$= \frac{1}{4}\frac{4x^6-4x^4-5x^6+3x^4-5x^4+3x^2}{\sqrt{x^5-x^3}(x^5-x^3)} = \frac{1}{4}\frac{x^4+6x^2-3}{|x|\sqrt{x^3-x}\,(1-x^2)\,x}$$

$$= 0 \;\;\therefore\;\; x^4+6x^2-3 = 0 \quad \{x^2 = t \;\;\therefore\;\; t^2+6t-3=0\}$$

where this biquadratic equation by x may be solved directly as:

$$t_{1,2} = \frac{-6\pm\sqrt{36+12}}{2} = -3\pm 2\sqrt{3} \;\;\therefore\;\; x_{1,2,3,4} = \sqrt{t_{1,2}} = \pm\sqrt{-3\pm 2\sqrt{3}}$$

$$x_1 = -\sqrt{2\sqrt{3}-3}, \quad x_2 = \sqrt{2\sqrt{3}-3}, \quad \in \mathbb{R}$$

$$\text{and } x_{3,4} \in \mathbb{C}$$

Note that $\sqrt{2\sqrt{3}-3} \approx 0.681 < 1$, by consequence it is not in the domain of definition leaving only $x_1 = -\sqrt{2\sqrt{3}-3} \approx -0.681$, which is found in the $(-1, 0)$ interval. That gives

$$f(x_1) = \sqrt{\frac{\left(-\sqrt{2\sqrt{3}-3}\right)^2-1}{-\sqrt{2\sqrt{3}-3}}} = \sqrt{\frac{4-2\sqrt{3}}{\sqrt{2\sqrt{3}-3}}\;\frac{\sqrt{2\sqrt{3}-3}}{\sqrt{2\sqrt{3}-3}}}$$

$$= \sqrt{\frac{4-2\sqrt{3}}{2\sqrt{3}-3}\;\frac{2\sqrt{3}-3}{2\sqrt{3}-3}\;\sqrt{2\sqrt{3}-3}} = \sqrt{\frac{8\sqrt{3}+\cancel{12}-6\sqrt{3}-\cancel{12}}{3}\sqrt{2\sqrt{3}-3}}$$

$$= \sqrt{\frac{2\sqrt{3}}{3}\sqrt{2\sqrt{3}-3}} = \frac{\sqrt{6}}{3}\sqrt[4]{3(2\sqrt{3}-3)} = y_1 \approx 0.887$$

Sign of $f''(x)$ is then

x	-1		x_1		0		x_2		1	
$1/4$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\lvert x\rvert$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$N(f''(x))$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$1-x^2$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$
$\sqrt{x^3-x}$	0	$+$	$+$	$+$	$+$	n.d.	n.d.	n.d.	0	$+$
$f''(x)$	n.d.	$-$	0	$+$	n.d.	n.d.	n.d.	n.d.	n.d.	$-$
$f(x)$	0	$\cap$	y_1	$\cup$	n.d.	n.d.	n.d.	n.d.	0	$\cap$

Therefore, there is one inflection point within $-1 \leq x < 0$ interval at $(x, y) = (x_1, y_1)$ and one change of $f''(x)$ sign on each side of non-defined interval.

8. *Summary of the important points:* (for the positive $\sqrt{*}$ only)

$$\begin{aligned}
\underline{\text{defined:}} \quad & -1 \leq x < 0 \quad \text{or,} \quad x \geq 1 \\
\underline{\text{v. asymptotes:}} \quad & x = 0 \\
\underline{\text{zeros:}} \quad & (x, y) = (-1, 0) \\
& (x, y) = (1, 0) \\
\underline{\text{extremes:}} \quad & (\text{none}, f(x) \geq 0) \\
\underline{\text{inflections:}} \quad & (x, y) = \left(-\sqrt{2\sqrt{3}-3},\ \frac{\sqrt{6}}{3}\sqrt[4]{3(2\sqrt{3}-3)}\right)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.61 are summarized by $f(x)$ graph (see Fig. 3.53).

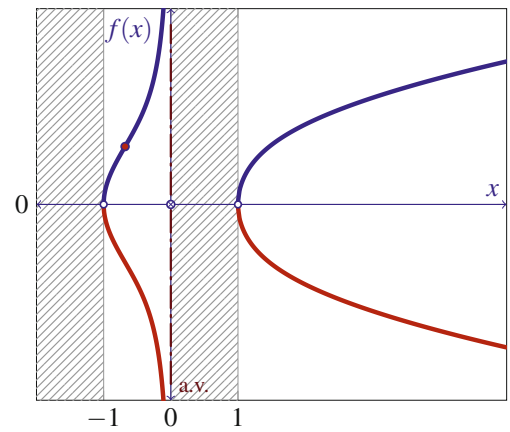


Fig. 3.53 Example P.3.61

3.62. Given radical function

$$f(x) = \sqrt{x^2 + 2x - 1}$$

1. *Domain of definition* $\mathbb{D}$: the square root argument must be non–negative, i.e.,

$$x^2 + 2x - 1 \geq 0 \;\; \therefore \;\; x_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$\therefore$$

$$x^2 + 2x - 1 = (x - x_1)(x - x_2) = \big(x - (-1 - \sqrt{2})\big)\big(x - (-1 + \sqrt{2})\big)$$

then (for $\sqrt{*} \geq 0$), $\big\{\text{if } AB > 0 \Rightarrow A > 0 \quad \text{and} \quad B > 0 \quad \text{or,} \quad A < 0 \quad \text{and} \quad B < 0\big\}$

x		x_1		x_2	
$x - x_1$	$-$	0	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	0	$+$
$\sqrt{(x - x_1)(x - x_2)}$	$+$	0	n.d.	0	$+$
$f(x)$	$+$	0	n.d.	0	$+$

thus, $x \in \mathbb{R} | x \notin (-1 - \sqrt{2}, -1 + \sqrt{2})$ interval.

2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = \sqrt{(-x)^2 + 2(-x) - 1} = \sqrt{x^2 - 2x - 1} \neq f(x) \;\; \therefore \;\; \text{(not even)}$$

$$\underline{\text{odd:}} \qquad -f(-x) = -\big(\sqrt{x^2 - 2x - 1}\big) \neq f(x) \;\; \therefore \;\; \text{(not odd)}$$

This radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \;\; \therefore \;\; \sqrt{x^2 + 2x - 1} = 0 \;\; \therefore \;\; x_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

There are two first-order zeros, at $(x, y) = (-1 - \sqrt{2}, 0)$ and at $(x, y) = (-1 + \sqrt{2}, 0)$. When defined, this radical function is $+\sqrt{*} \geq 0$ or, its mirrored image, $-\sqrt{*} \leq 0$.

4. *y-axis crossing point* $f(0)$: is not defined.

5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \sqrt{x^2 + 2x - 1} = \lim_{x \to +\infty} \sqrt{x^2 \left(1 + \frac{2}{x} - \frac{1}{x^2}\right)}$$

$$= \lim_{x \to +\infty} |x| \sqrt{1 + \cancelto{0}{2/x} - \cancelto{0}{1/x^2}} = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{x^2+2x-1} = \lim_{x\to-\infty} \sqrt{x^2\left(1+\frac{2}{x}-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to-\infty} |x| \sqrt{1+\cancelto{0}{2/x}-\cancelto{0}{1/x^2}} = +\infty$$

because $|x| = \sqrt{x^2}$ is always positive.

6. *Oblique asymptote:* $x \in \mathbb{R} | x \notin (-1-\sqrt{2}, -1+\sqrt{2})$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{x^2+2x-1}}{x} = \lim_{x\to\infty} \frac{|x|}{x} \sqrt{1+\cancelto{0}{2/x}-\cancelto{0}{1/x^2}} = \lim_{x\to\infty} \operatorname{sign}(x) = \pm 1$$

by consequence,

$\underline{\text{for } (a=+1):}$

$$b = \lim_{x\to\infty} \big(f(x) - ax\big) = \lim_{x\to\infty} \left(\sqrt{x^2+2x-1} - x\right) \quad \frac{\sqrt{x^2+2x-1}+x}{\sqrt{x^2+2x-1}+x}$$

$$\big\{(a-b)(a+b) = a^2-b^2\big\}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+2x-1-\cancel{x^2}}{\sqrt{x^2+2x-1}+x} = \lim_{x\to\infty} \frac{x\left(2-\cancelto{0}{(1/x)}\right)}{|x|\sqrt{1+\cancelto{0}{2/x}-\cancelto{0}{1/x^2}}+x}$$

$$\big\{x \geq 0 \quad \therefore \quad |x| = x, \quad |-x| = x\big\}$$

$$= \begin{cases} \lim_{x\to+\infty} \dfrac{\cancel{2x}}{\cancel{x+x}}\,1 = 1 \\ \\ \lim_{x\to-\infty} \dfrac{2x}{\cancel{x}-\cancel{x}\,0} = \dfrac{\infty}{0} \quad \therefore \quad \text{(not defined)} \end{cases}$$

$\underline{\text{for } (a=-1):}$

$$b = \lim_{x\to\infty} \big(f(x) + ax\big) = \lim_{x\to\infty} \left(\sqrt{x^2+2x-1} + x\right) \quad \frac{\sqrt{x^2+2x-1}-x}{\sqrt{x^2+2x-1}-x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}+2x-1-\cancel{x^2}}{\sqrt{x^2+2x-1}-x} = \lim_{x\to\infty} \frac{x\left(2-\cancelto{0}{(1/x)}\right)}{|x|\sqrt{1+\cancelto{0}{2/x}-\cancelto{0}{1/x^2}}-x}$$

$$= \begin{cases} \lim_{x\to+\infty} \dfrac{2x}{\cancel{x}-\cancel{x}\,0} = \dfrac{\infty}{0} \quad \therefore \quad \text{(not defined)} \\ \\ \lim_{x\to-\infty} \dfrac{\cancel{2(-x)}}{\cancel{x-(-x)}} \quad \dfrac{-2x}{+2x} = -1 \end{cases}$$

In conclusion, oblique asymptote(s) is(are):

$$y_{aa} = x + 1 \quad (x \to +\infty)$$
$$y_{aa} = -(x + 1) \quad (x \to -\infty)$$

7. *Critical points:*

$$f'(x) = 0 : x \in \mathbb{R} | x \notin (-1 - \sqrt{2}, -1 + \sqrt{2})$$

$$f'(x) = \left(\sqrt{x^2 + 2x - 1}\right)' = \frac{2x + 2\,1}{2\sqrt{x^2 + 2x - 1}} = \frac{x + 1}{\sqrt{x^2 + 2x - 1}} = 0$$

$$\therefore \quad x + 1 = 0 \quad \therefore \quad x = -1 \quad \text{but,} \quad x \in \mathbb{R} | x \notin (-1 - \sqrt{2}, -1 + \sqrt{2})$$

so that $f'(x)$ sign is

x		x_1		-1		x_2	
$x + 1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$\sqrt{x^2 + 2x - 1}$	$+$	0	n.d.	n.d.	n.d.	0	$+$
$f'(x)$	$-$	n.d.	n.d.	n.d.	n.d.	n.d.	$+$
$f(x)$	$\searrow$	0	n.d.	n.d.	n.d.	0	$\nearrow$

In conclusion, as $f'(x)$ changes its sign within two separate intervals, there are no extreme points. Also,

$$f''(x) = 0 : x \in \mathbb{R} | x \notin (-1 - \sqrt{2}, -1 + \sqrt{2})$$

$$f''(x) = \left(\frac{x + 1}{\sqrt{x^2 + 2x - 1}}\right)' = \frac{\sqrt{x^2 + 2x - 1} - \frac{(x + 1)(2x + 2\,1)}{2\sqrt{x^2 + 2x - 1}}}{(x^2 + 2x - 1)}$$

$$= \frac{(x^2 + 2x - 1) - (x + 1)^2}{(x^2 + 2x - 1)\sqrt{x^2 + 2x - 1}} = \frac{x^2 + 2x - 1 - x^2 - 2x - 1}{(x^2 + 2x - 1)\sqrt{x^2 + 2x - 1}}$$

$$= \frac{-2}{(x^2 + 2x - 1)\sqrt{x^2 + 2x - 1}} < 0$$

that is to say, $f(x)$ is always convex $\cap$ (for $\sqrt{*} \geq 0$).

8. *Summary of the important points:* (for the positive $\sqrt{*}$ only)

defined: $x \leq -1 - \sqrt{2}$ or, $x \geq -1 + \sqrt{2}$

a.a. asymptotes: $y = (x + 1)$

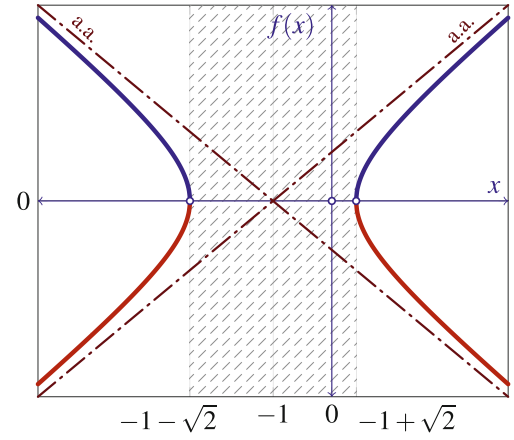
$y = -(x + 1)$

zeros: $(x, y) = (-1 + \sqrt{2}, 0)$

$(x, y) = (-1 - \sqrt{2}, 0)$

extremes: (none, $f(x) \geq 0$)

Fig. 3.54 Example P.3.62



inflections: (none, $f(x)$ is always convex $\cap$)

9. *Graphical representation:* results of the complete function $f(x) = \pm\sqrt{*}$ analysis in P.3.62 are summarized by $f(x)$ graph in Fig. 3.54.

3.63. Given radical function

$$f(x) = \sqrt{x^2 + 2x + 1} = \sqrt{(x+1)^2} = \left\{\sqrt{*^2} = |*|\right\} = |x+1|$$

it is worth remembering that square root of a quadratic argument is its absolute value (see P.3.50). Therefore, this function's form and analysis are the same $|x|$ delayed by "1" (see Fig. 3.55).

1. *Domain of definition* $\mathbb{D}$: an absolute function is defined for all real arguments, $(x \in \mathbb{R})$.
2. *Function's parity:*

$$\text{even: } f(-x) = \sqrt{(-x+1)^2} = \sqrt{x^2 - 2x + 1} \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\text{odd: } -f(-x) = -\sqrt{x^2 - 2x + 1} \neq f(x) \quad \therefore \quad \text{(not odd)}$$

3. *Function's sign:* $f(x) = 0 \quad \therefore \quad x + 1 = 0 \quad \therefore \quad x = -1$, which is its first-order zero. As absolute function of a real argument $|*|$ is always positive, then it must be that $f(x) = \sqrt{(x+1)^2} \geq 0$.
4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{(0+1)^2} = 1$.
5. *Limits:* there is one zero (at $x = -1$); therefore, there are four limits in total to resolve.

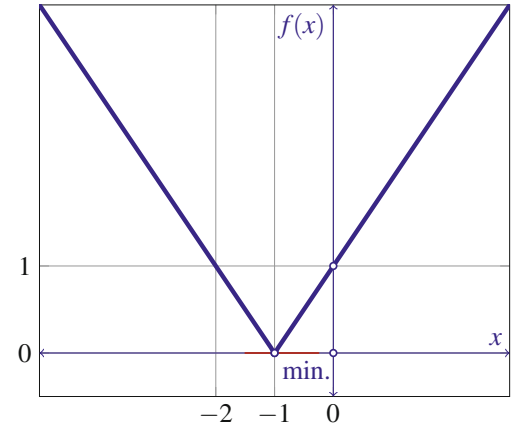
$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} |x+1| = +\infty$$

$$x \to -\infty: \quad \lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} |x+1| = +\infty$$

$$x \to -1: \quad \lim_{\substack{x\to-1\\ \twoheadrightarrow}} f(x) = \lim_{\substack{x\to-1\\ \twoheadrightarrow}} |x+1| = 0\downarrow$$

$$x \to -1: \quad \lim_{\substack{x\to-1\\ \twoheadleftarrow}} f(x) = \lim_{\substack{x\to-1\\ \twoheadrightarrow}} |x+1| = 0\downarrow$$

Fig. 3.55 Example P.3.63



6. *Oblique asymptote:*

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{|x+1|}{x}$$

$$= \begin{cases} \underline{x+1 \geq 0:} & \lim_{x\to\pm\infty} \dfrac{x+1}{x} = \lim\limits_{x\to\pm\infty} (1+1/x) = \pm 1 \\ \underline{x+1 < 0:} & \lim_{x\to\pm\infty} \dfrac{-(x+1)}{x} = \lim\limits_{x\to\pm\infty} (-1-1/x) = \pm 1 \end{cases}$$

As a is not defined, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x)=0:} \quad f'(x) = \left(\sqrt{(x+1)^2}\right)' = \frac{2(x+1)}{2\sqrt{(x+1)^2}} = \left\{ \text{sign}\,(*) = \frac{*}{|*|} = \frac{|*|}{*} \right\}$$

$$= \frac{x+1}{|x+1|} = \frac{|x+1|}{x+1} \quad \therefore \quad x+1=0 \quad \therefore \quad x=-1$$

Even though $f'(-1) = 0/0$ is not defined there is one minimum point at $(x, y) = (-1, 0)$ because $f(x)$ is continuous.

and,

$$\underline{f''(x)=0:} \quad f''(x) = \big(\text{sign}\,(x+1)\big)' = (\pm 1)' = 0\,; \quad (x \neq -1)$$

In conclusion, there are no extreme points, and this function is neither concave nor convex.

8. *Summary of the important points:*

y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
zero:	$(x, y) = (-1, 0)$
extreme:	$(x, y) = (-1, 0)$ (min)
inflection:	(none, $f''(x)$ not defined at $x = -1$)

3.64. Given radical function,

$$f(x) = \sqrt{x^3 + 2x^2 + 1}$$

1. *Domain of definition* $\mathbb{D}$: the square root argument must be non-negative, i.e.,

$$x^3 + 2x^2 + 1 \geq 0$$

that is to say, first, it is necessary to solve the third-order polynomial equation,

$$x^3 + 2x^2 + 1 = 0 \quad \text{or, equivalently}$$
$$(x - x_1)(x - x_2)(x - x_3) = 0$$

where (x_1, x_2, x_3) are the three zeros of this polynomial. Being the third-order polynomial, at least one of the three zeros must be real. Having said that, this particular third-order polynomial cannot be factorized using algebraic identities. Therefore, numerical and/or graphical methods must be employed.

Graphical method: this cubic equation may be rearranged into basic cubic and quadratic functions that intersect as

$$x^3 + 2x^2 + 1 = 0 \;\; \therefore \;\; x^3 = -2x^2 - 1$$

The quadratic polynomial on the right side is evidently always negative, and it crosses y-axis at $y = f(0) = -2(0)^2 - 1 = -1$. At the same time, all three real zeros of a simple cubic equation $x^3 = 0$ are at $x_{1,2,3} = 0$. Therefore, these two polynomials must intersect somewhere in the negative region, i.e., for $x < 0$ (see Fig. 3.56 (left)). The first guess may be, for example, that the only real zero is in proximity of $x_1 \approx -2$, where $(-2)^3 = -8$ and $-(-2)^2 = -4$ so that an iterative method should converge rapidly.

Numerical method: after taking the initial guess $x_0 = -2$, the iterative Newton-Raphson method gives

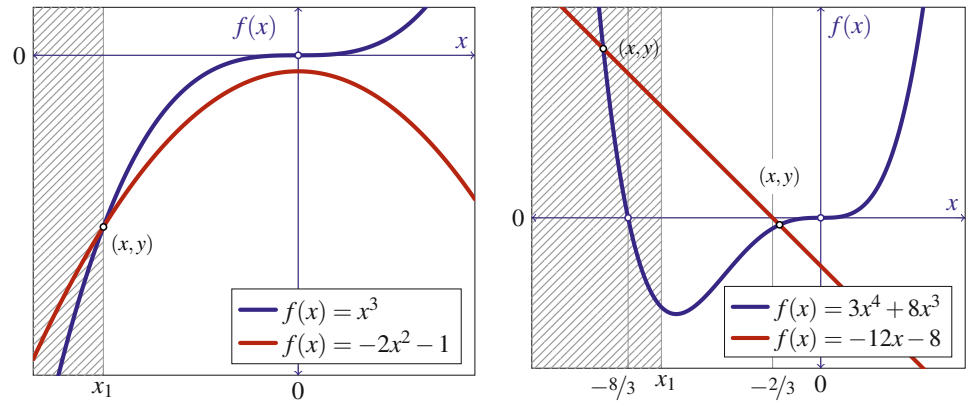


Fig. 3.56 Example P.3.64

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)} \quad \text{where,} \quad f(x) = x^3 + 2x^2 + 1$$
$$f'(x) = 3x^2 + 4x = x(3x + 4)$$

so that,

n	x	$f(x)$	$f'(x)$
0	-2.000	1.000	4.000
1	-2.2071	-0.265	6.187
2	-2.2056	-8.675×10^{-3}	5.785
3	-2.2056	-1.039×10^{-5}	5.771

and in $n = 3$ iterations, to the third decimal place $x_1 \approx -2.206$. To illustrate the resulting root,

$$P_3(-2.206) = x_1^3 + 2x_1^2 + 1 \approx -1.039 \times 10^{-5}$$

That is to say, the only real zero of $f(x)$ equals $x_1 \approx -2.206$. Now,

$$x^3 + 2x^2 + 1 = (x - x_1)(ax^2 + bc + c) = 0$$

where two remaining second-order polynomial zeros $P_2(x) = ax^2 + bc + c$ are complex, which is another way of saying that $P_2(x)$ is always either positive or negative. By consequence, the overall third-order polynomial's sign must follow its $(x - x_1)$ term's sign, as

x		x_1	
$x - x_1$	$-$	0	$+$
$x^3 + 2x^2 + 1$	$-$	0	$+$
$\sqrt{x^3 + 2x^2 + 1}$	n.d.	0	$+$
$f(x)$	n.d.	0	$+$

There is one zero at $(x, y) = (x_1, 0)$, and in total $x \in \mathbb{R} | x \geq x_1 \approx -2.206$.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt{(-x)^3 + 2(-x)^2 + 1} = \sqrt{-x^3 + 2x^2 + 1} \neq f(x) \quad \therefore \quad \text{(not even)}$$
$$\underline{\text{odd:}} \quad -f(-x) = -\left(\sqrt{-x^3 + 2x^2 + 1}\right) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

This radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt{x^3 + 2x^2 + 1} = 0 \quad \therefore \quad x_1 \approx -2.206 \quad z_{2,3} \in \mathbb{C}$$

Therefore, for $x \in \mathbb{R} | x \geq x_1$, and $f(x) \geq 0$ for "$+\sqrt{*}$." The horizontally mirrored image corresponds to "$-\sqrt{*}$" where $f(x) \leq 0$.

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt{(0)^3 + 2(0)^2 + 1} = \pm 1$

5. *Limits:* there is one limit in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x^3+2x^2+1} = \lim_{x\to+\infty} \sqrt{x^2\left(x+2+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to+\infty} \cancel{|x|}^{+\infty} \sqrt{x^{\infty} + 2 + \cancel{1/x^2}^{0}} = +\infty$$

because $|x| = \sqrt{x^2}$ is always positive. The mirrored image of $f(x)$ then tends to $-\infty$.

6. *Oblique asymptote:* $x \in \mathbb{R} | x \geq x_1$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{x^3+2x^2+1}}{x} = \lim_{x\to\infty} \sqrt{\frac{x^3}{x^2} + 2 + \frac{1}{x^2}^{0}} = \infty$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \; x \in \mathbb{R} | x \geq x_1$$

$$f'(x) = \left(\sqrt{x^3+2x^2+1}\right)' = \frac{3x^2+4x}{2\sqrt{x^3+2x^2+1}} = 0$$

$$\therefore \; 3x^2+4x = x(3x+4) = 0 \; \therefore \; \begin{cases} 3x+4=0 & \therefore \; x_2 = -\frac{4}{3} \\ x = 0; & \therefore \; x_3 = 0 \end{cases}$$

Note that $x_2 = -4/3 \approx -1.333 > x_1 \approx -2.206$, thus inside the definition region, so that $f'(x)$ sign is (only for the positive side of $\sqrt{*}$)

x	x_1		x_2		x_3	
x	$-$	$-$	$-$	$-$	0	$+$
$3x+4$	$-$	$-$	0	$+$	$+$	$+$
$2\sqrt{x^3+2x^2+1}$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	n.d	$+$	0	$-$	0	$+$
$f(x)$	0	$\nearrow$	y_2	$\searrow$	y_3	$\nearrow$

where y_2 $f(-4/3) \approx 1.48$, and $f(0) = 1$. There is one maximum at $(x_2, y_2) = (-4/3, 1.48)$ and one minimum at $(x_3, y_3) = (0, 1)$. Also,

$$\underline{f''(x) = 0:} \; x \in \mathbb{R} | x \geq x_1$$

$$f''(x) = \left(\frac{3x^2+4x}{2\sqrt{x^3+2x^2+1}}\right)' = \frac{1}{2} \frac{(6x+4)\sqrt{x^3+2x^2+1} - \frac{(3x^2+4x)(3x^2+4x)}{2\sqrt{x^3+2x^2+1}}}{(x^3+2x^2+1)}$$

$$= \frac{1}{4}\frac{(6x+4)\,2\,(x^3+2x^2+1)-(3x^2+4x)^2}{\sqrt{x^3+2x^2+1}\,(x^3+2x^2+1)}$$
$$= \frac{1}{4}\frac{3x^4+8x^3+12x+8}{\sqrt{x^3+2x^2+1}\,(x^3+2x^2+1)}$$
$$= 0 \quad \therefore \quad 3x^4+8x^3+12x+8=0$$

This fourth-order polynomial cannot be factorized by using the algebraic identities, so the solution is found by graphical/numerical methods.

Graphical method: first,

$$3x^4+8x^3+12x+8=0 \quad \therefore \quad 3x^4+8x^3=-12x-8$$
$$x^3(3x+8)=-4(3x+2)$$

On the left side, the simple cubic equation $x^3 = 0$ has third (odd)-order zero at $x_{1,2,3} = 0$. In addition, the $3x + 8$ term crosses the horizontal axis at first (odd)-order zero $x = -8/3$. It may be deduced $\lim_{x\to\pm\infty} x^3(3x+8) = (\pm\infty)(\pm\infty) = +\infty$, then $P_4(x) = 3x^4 + 8x^3$ is as in Fig. 3.56 (right). Note that this polynomial must change its sign after each zero. At the same time, the linear term $-12x - 8 = -4(3x + 2)$ has first (odd)-order zero at $x = -2/3$, Fig. 3.56 (right).

Evidently, there could be only two real zeros of $3x^4 + 8x^3 + 12x + 8 = 0$, while the other two solutions are therefore complex. One real solution is close to $x = -8/3$, and the other is close to $x = -2/3$. However, $x = -8/3 \notin \mathbb{D}$, which leaves only the second solution.

Numerical method: after taking the initial guess $x_0 = -2/3$, the iterative Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)} \qquad \text{where,} \quad (x) = 3x^4+8x^3+12x+8$$
$$f'(x) = 12x^3+24x^2+12$$

gives

n	x	$f(x)$	$f'(x)$
0	$-2/3$	-1.778	19.111
1	-0.574	-0.069	17.632
2	-0.570	-1.201×10^{-4}	17.571
3	-0.570	-3.656×10^{-10}	17.571

and in $n = 2$ iterations to the third decimal place $x \approx -0.570$. To illustrate the resulting root,

$$P_4(-0.570) = 3x^4+8x^3+12x+8 \approx -3.656 \times 10^{-10}$$

Summary: for $x \in \mathbb{R}|x \geq x_1$, the $f''(x)$ sign changes only at $x_4 \approx -0.570$, because $f''(x)$ denominator is positive within the given interval. Thus,

$$f''(x) = \frac{N(x)}{D(x)} = \frac{1}{4}\frac{3x^4 + 8x^3 + 12x + 8}{\sqrt{x^3 + 2x^2 + 1}\,(x^3 + 2x^2 + 1)}$$

x	x_1		x_4	
$1/4$	$+$	$+$	$+$	$+$
$N(x)$	$-$	$-$	0	$+$
$D(x)$	0	$+$	$+$	$+$
$f''(x)$	n.d.	$-$	0	$+$
$f(x)$	0	$\cap$	y_4	$\cup$

Where,

$$y_4 = f(x_4) \approx \sqrt{(-0.570)^3 + 2(-0.570)^2 + 1} \approx 1.21$$

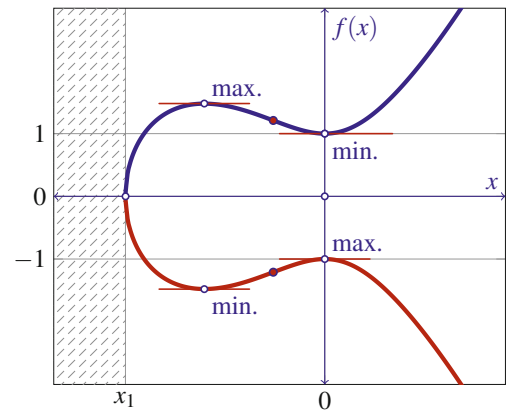
There is one inflection point at $(x, y) = (x_4, y_4)$.

8. *Summary of the important points:* (for the positive $\sqrt{*}$ only)

$$\begin{aligned}
\underline{\text{defined:}} \quad & x \geq -2.206 \\
\underline{\text{y-axis crossing point } f(0)\text{:}} \quad & (x, y) = (0, 1) \\
\underline{\text{zero:}} \quad & (x, y) = (-2.206, 0) \\
\underline{\text{extremes:}} \quad & (x, y) = \left(-\frac{4}{3}, 1.48\right) \quad \text{(max)} \\
& (x, y) = (0, 1) \quad \text{(min)} \\
\underline{\text{inflection:}} \quad & (x, y) = (-0.570, 1.21)
\end{aligned}$$

Graphical representation: results of the analysis for function in P.3.64 are summarized by $f(x)$ graph (see Fig. 3.57).

Fig. 3.57 Example P.3.64



3.65. Given radical function

$$f(x) = \sqrt{x^3 + 2x^2 - 1}$$

1. *Domain of definition* $\mathbb{D}$: the square (even) root argument must be non-negative, i.e.,

$$x^3 + 2x^2 - 1 \geq 0 \;\; \therefore \;\; x^3 + 2x^2 - 1 = (x - x_1)(x - x_2)(x - x_3) \geq 0$$

where roots of this cubic polynomial may be determined as follows. Evidently, first root may be deduced by inspection as $x_1 = -1$, because $f(-1) = 0$. By long division, it follows that

$$\begin{array}{rrrcrcl} (x^3 & +2x^2 & -1) & : & (x+1) & = & x^2 + x - 1 \\ \underline{x^3} & \underline{+x^2} & & & & & \\ & x^2 & -1 & & & & \\ & \underline{x^2} & \underline{+x} & & & & \\ & & -x - 1 & & & & \\ & & \underline{-x - 1} & & & & \\ & & 0 & & & & \end{array}$$

Roots of the remaining quadratic polynomial are derived as

$$x^2 + x - 1 = 0 \;\; \therefore \;\; x_{2,3} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

A recapitulation: there are three first-order zeros,

$$x_2 = \frac{-1 - \sqrt{5}}{2}, \;\; x_1 = -1, \;\; x_3 = \frac{-1 + \sqrt{5}}{2}$$

Thus $f(x)$ signs are (assuming the "$+\sqrt{\cdot}$" side only)

x		x_2		x_1		x_3	
$x - x_2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - x_1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x^3 + 2x^2 - 1$	$-$	0	$+$	0	$-$	0	$+$
$\sqrt{x^3 + 2x^2 - 1}$	n.d.	0	$+$	0	n.d.	0	$+$
$f(x)$	n.d.	0	$+$	0	n.d.	0	$+$

In summary, $x \in \mathbb{R} | x \in [x_2\,, x_1\,]$ and $x \in [x_3\,, +\infty]$.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt{(-x)^3 + 2(-x)^2 - 1} = \sqrt{-x^3 + 2x^2 - 1} \neq f(x) \;\; \therefore \;\; \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\sqrt{-x^3 + 2x^2 - 1}\right) \neq f(x) \;\; \therefore \;\; \text{(not odd)}$$

This radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x)=0:}\quad \therefore\ \sqrt{x^3+2x^2-1}=0\ \ \therefore\ \ x_1=\frac{-1-\sqrt{5}}{2},\ \ x_2=-1,\ \ x_3=\frac{-1+\sqrt{5}}{2}$$

Therefore, for $x\in\mathbb{R}|x\in[x_2\,,x_1\,]$ and $x\in[x_3\,,+\infty]$, function is either positive or equal zero (when defined), i.e., $f(x)\geq 0$ for "$+\sqrt{\cdot}$." The horizontally mirrored image corresponds to "$-\sqrt{\cdot}$" where $f(x)\leq 0$.

4. *y-axis crossing point* $f(0)$: $f(0)=\sqrt{(0)^3+2(0)^2-1}$ $\therefore$ not defined.
5. *Limits:* there is one limit in total to be resolved,

$$\underline{x\to+\infty:}\quad \lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}\sqrt{x^3+2x^2-1}=\lim_{x\to+\infty}\sqrt{x^2\left(x+2-\frac{1}{x^2}\right)}$$

$$=\lim_{x\to+\infty}|x|\sqrt{x+2-1/x^2}\ \ (x\to+\infty,\ 1/x^2\to 0)=+\infty$$

because $|x|=\sqrt{x^2}$ is positive. The horizontally mirrored image of $f(x)$ then tends to $-\infty$.

6. *Oblique asymptote:* $x\in\mathbb{R}|x\in[x_2\,,x_1\,]$ and, $x\in[x_3\,,+\infty]$

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\sqrt{x^3+2x^2-1}}{x}=\lim_{x\to\infty}\sqrt{\frac{x^3}{x^2}+2-\frac{1}{x^2}}\ \ (1/x^2\to 0)=\lim_{x\to\infty}\sqrt{x+2}=\infty$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x)=0:}\ x\in\mathbb{R}|x\in[x_2\,,x_1\,]\quad\text{and,}\quad x\in[x_3\,,+\infty]$$

$$f'(x)=\left(\sqrt{x^3+2x^2+1}\right)'=\frac{1}{2}\,\frac{3x^2+4x}{\sqrt{x^3+2x^2-1}}=0$$

$$\therefore\ 3x^2+4x=x(3x+4)=0\ \ \therefore\ \begin{cases}x & =0;\ \ x\notin\mathbb{D}\\ 3x+4 & =0\ \ \therefore\ \ x_4=-\dfrac{4}{3}\end{cases}$$

$$\therefore\ y_4=f(-4/3)=\sqrt{\left(-\frac{4}{3}\right)^3+2\left(-\frac{4}{3}\right)^2-1}=\sqrt{\frac{5}{27}}$$

$x=0$ is excluded as $f(0)$ is not defined, thus $f'(x)$ signs are (assuming the "$+\sqrt{\cdot}$" side only)

x	x_2		x_4		x_1	x_3	
$1/2$	+	+	+	+	+	+	+
x	−	−	−	−	+	+	+
$3x+4$	−	−	0	+	+	+	+
$\sqrt{x^3+2x^2-1}$	0	+	+	+	+	0	+
$f'(x)$	n.d.	+	0	−	+	n.d.	+
$f(x)$	0	↗	y_4	↘	↗	0	↗

There is one local maximum at $(x_4, y_4) = (-4/3, \sqrt{5/27})$. By symmetry, for $y = -\sqrt{\cdot}$, there is one local minimum at $(x, y) = (-4/3, -\sqrt{5/27})$.

Also,

$$f''(x) = 0 : x \in \mathbb{R} | x \in [x_2\,, x_2\,] \quad \text{and,} \quad x \in [x_3\,, +\infty]$$

$$\begin{aligned} f''(x) &= \left(\frac{3x^2+4x}{2\sqrt{x^3+2x^2-1}}\right)' = \frac{1}{2}\frac{(6x+4)\sqrt{x^3+2x^2-1} - \dfrac{(3x^2+4x)(3x^2+4x)}{2\sqrt{x^3+2x^2-1}}}{(x^3+2x^2-1)} \\ &= \frac{1}{4}\frac{(6x+4)\,2\,(x^3+2x^2-1) - (3x^2+4x)^2}{\sqrt{x^3+2x^2-1}\,(x^3+2x^2-1)} \\ &= \frac{1}{4}\frac{3x^4+8x^3-12x-8}{\sqrt{(x^3+2x^2-1)^3}} \\ &= 0 \quad \therefore \quad 3x^4+8x^3-12x-8=0 \end{aligned}$$

Solutions of this fourth-order polynomial equation can be found by Newton-Raphson method, after evaluating

$$3x^4 + 8x^3 = 12x + 8 \quad \text{or,}$$
$$x^3(3x+8) = 4(3x+2)$$

A rough sketch of intersects $x^3(3x+8) = 12x+8$ (Fig. 3.58 (left)) shows that there are only two real domain solutions to the fourth-order equation $3x^4 + 8x^3 + 12x + 8 = 0$, while the other two solutions are therefore in the complex domain. One of the two intersect points is obviously on the positive side, so, for example, the first guess for initial value could be $x_0 = 1$. On the negative side, the intersect point is inferior to the $x \in [x_2\,, x_1\,]$ interval, thus outside of the definition region.

For example, by taking the initial value $x_0 = -2$, numerical iterative method produces:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{where,} \quad f(x) = 3x^4 + 8x^3 - 12x - 8$$
$$f'(x) = 12x^3 + 24x^2 - 12$$

Taking the initial value $x_0 = -2$,

n	x	$f(x)$	$f'(x)$
0	−2.000	0.000	−12.000
1	−2.000	0.000	−12.000

this time, by chance, the initial value $x_0 = -2$ chosen is in fact the exact solution found in the first iteration, i.e.,

$$f(-2) = 3x^4 + 8x^3 - 12x - 8 = 0$$

However, as already estimated, this solution is indeed outside of the function's definition interval $[x_2\,, x_1\,]$, thus not valid.

Taking the initial value $x_0 = 1$ gives

n	x	$f(x)$	$f'(x)$
0	1.000	−9.000	24.000
1	1.375	7.020	64.570
2	1.266	0.761	50.848
3	1.251	0.013	49.089
4	1.251	4.211×10^{-6}	49.058

in $n = 3$ iterations, to the third decimal place $x \approx 1.251$. To illustrate the resulting root,

$$f(1.251) = 3x^4 + 8x^3 - 12x - 8 \approx 4.211 \times 10^{-6}$$

Summary: for $x \in \mathbb{R} | x \in [x_2\,, x_1\,]$ and, $x \in [x_3\,, +\infty]$, the $f''(x)$ sign could possibly change only at one of $f''(x)$ numerator's zeros $x_5 \approx 1.251$ (because its denominator, where defined, is positive). Thus,

$$f''(x) = \frac{N(x)}{D(x)} = \frac{1}{4}\,\frac{3x^4 + 8x^3 - 12x - 8}{\sqrt{(x^3 + 2x^2 - 1)^3}}$$

x	x_2		x_1	x_3		x_5	
1/4	+	+	+	+	+	+	+
$N(x)$	−	−	−	−	−	0	+
$D(x)$	0	+	0	0	+	+	+
$f''(x)$	n.d.	−	n.d.	n.d.	−	0	+
$f(x)$	0	∩	0	0	∩	y_5	∪

where,

$$y_5 = f(x_5) \approx \sqrt{(1.251)^3 + 2(1.251)^2 - 1} \approx 2.022$$

There is one inflection point at $(x, y) = (x_5, y_5)$.

8. *Summary of the important points:*

$$\underline{\text{defined:}} \quad x \in \left[\frac{-1-\sqrt{5}}{2}, -1\right] \quad \text{and,} \quad x \in \left[\frac{-1+\sqrt{5}}{2}, +\infty\right]$$

$$\underline{\text{zeros:}} \quad (x, y) = \left(\frac{-1-\sqrt{5}}{2}, 0\right)$$

$$(x, y) = (-1, 0)$$

$$(x, y) = \left(\frac{-1+\sqrt{5}}{2}, 0\right)$$

$$\underline{\text{extreme:}} \quad (x, y) = \left(-1+\sqrt{2}, \sqrt{5/27}\right) \quad \text{(max)}$$

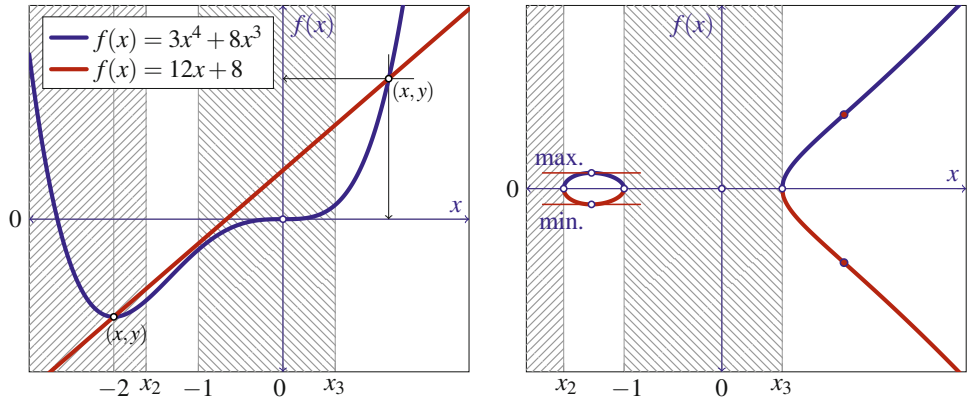


Fig. 3.58 Example P.3.65

$$\underline{\text{inflection:}} \quad (x,\, y) = (1.251,\, 2.022)$$

9. *Graphical representation:* graphical method used to estimate roots of the fourth-order polynomial equation $P_4(x) = 3x^4 + 8x^3 - 12x - 8 = 0$ (see Fig. 3.58 (left)) and the complete $f(x)$ graph (see Fig. 3.58 (right). Note very non-dramatic change of $f(x)$ at its inflection point.

3.66. Given radical function

$$f(x) = \sqrt[3]{x^3 + 3x^2 + 3x + 1} = \sqrt[3]{(x+1)^3} = x + 1$$

This example illustrates that it is always a good idea to verify if a long polynomial is in fact a developed version of some of the basic algebraic polynomial identities: the difference of two squares, a binomial expansion, etc. To recognize the well-known patterns in algebra is very important skill, here, by recalling that the cube power of a binomial (with the help of Pascal's triangle) is

$$(a+b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

shows that it corresponds exactly to the polynomial argument of the cube root in this example. Then

$$\sqrt[3]{x^3} = \left((x+1)^3\right)^{1/3} = (x+1)^{\cancel{3}\,\cancel{1/3}} = x + 1, \quad (x \in \mathbb{R})$$

For sake of tutorial's completeness, complete analysis of this linear function is as follows.

1. *Domain of definition* $\mathbb{D}$: a third (odd)-order radical function with polynomial argument is defined for $\{x \in \mathbb{R}\}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = -x + 1 \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -(-x+1) = x - 1 \neq f(x) \quad \therefore \quad \text{(not odd)}$$

This radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad x + 1 = 0 \quad \therefore \quad x = -1$$

Even though there is a third-order polynomial in the original form of the example (thus, there should be three zeros), due to the cube root, the final form is reduced to linear function (first-order polynomial). By consequence, there is only one first-order zero (i.e., where $f(x)$ does change its sign). Simply put: if $x < 1$, then $f(x) < 0$, and if $x > 1$, then $f(x) > 0$.

4. *y-axis crossing point* $f(0)$: $f(0) = 0 + 1 = 1$
5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x + 1 = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x + 1 = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} \left(1 + \cancelto{0}{\frac{1}{x}}\right) = 1$$

and,

$$b = \lim_{x\to\infty} \big(f(x) - ax\big) = \lim_{x\to\infty} \big(\cancel{x} + 1 - \cancel{x}\big) = 1$$

therefore, oblique asymptote is $y_{\mathtt{aa}} = x + 1$. Note: this is a trivial case when $f(x)$ is at the same time its own oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (x+1)' = 1 \neq 0$$

thus, there are no extreme points, as $f'(x) = 1 > 0$ this function always ascends ↗. Also,

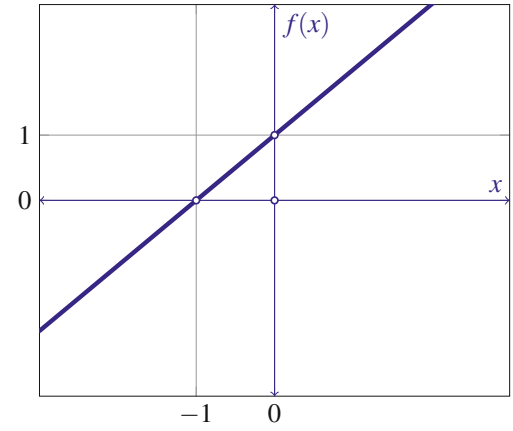
$$\underline{f''(x) = 0:} \quad f''(x) = (1)' = 0$$

8. *Summary of the important points:*

zero:	$(x, y) = (-1, 0)$
y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
a.a. asymptote:	$y = x + 1$ (the trivial case)
extremes:	(none, $f(x)$ always ascends ↗)
inflections:	(none)

9. *Graphical representation:* the calculated results are sufficient to plot $f(x)$ graph (see Fig. 3.59).

Fig. 3.59 Example P.3.66



3.67. Given radical function,

$$f(x) = \sqrt[3]{x^3 + 3x^2 - 3x - 1}$$

1. *Domain of definition* $\mathbb{D}$: a third (odd)-order radical function with polynomial argument is defined for $\{x \in \mathbb{R}\}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sqrt{(-x)^3 + 3(-x)^2 - 3(-x) - 1} = \sqrt{-x^3 + 3x^2 + 3x - 1} \neq f(x)$$

$$\therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(\sqrt{-x^3 + 3x^2 + 3x - 1}\right) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, this radical function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad \sqrt[3]{x^3 + 3x^2 - 3x - 1} = 0 \quad \therefore \quad x^3 + 3x^2 - 3x - 1 = 0$$

$$\therefore \quad (x - x_1)(x - x_2)(x - x_3) = 0$$

because a third-order polynomial must have three zeros, (x_1, x_2, x_3). First root may be deduced simply by inspection as $x_1 = 1$, because $f(1) = 0$. Then by long division, it follows that

$$\begin{array}{rrrrcrcl}
(x^3 & +3x^2 & -3x & -1) & : & (x-1) & = & x^2 + 4x + 1 \\
\underline{x^3} & \underline{-x^2} & & & & & & \\
 & 4x^2 & -3x & & & & & \\
 & \underline{4x^2} & \underline{-4x} & & & & & \\
 & & x & -1 & & & & \\
 & & \underline{x} & \underline{-1} & & & & \\
 & & 0 & 0 & & & &
\end{array}$$

Roots of the remaining quadratic polynomial are derived as

$$x^2 + 4x + 1 = 0 \quad \therefore \quad x_{2,3} = \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$$

A recapitulation: there are three first-order zeros.

$$x_1 = 1, \quad x_2 = -2 - \sqrt{3}, \quad x_3 = -2 + \sqrt{3}$$

So that the $f(x)$ signs are

x		x_2		x_3		x_1	
$x - x_2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x - x_1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

There are three first-order zeros, at $(x, y) = (-2 - \sqrt{3}, 0)$, $(x, y) = (-2 + \sqrt{3}, 0)$, and $(x, y) = (1, 0)$; thus, $f(x)$ changes its sign after crossing each of its odd zeros.

4. *y-axis crossing point* $f(0)$: $f(0) = \sqrt[3]{(0)^3 + 3(0)^2 - 3(0) - 1} = -1$
5. *Limits:* there are two limits in total to be resolved,

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt[3]{x^3 + 3x^2 - 3x - 1}$$

$$= \lim_{x\to+\infty} x\sqrt[3]{1 + \frac{3}{x} - \frac{3}{x^2} - \frac{1}{x^3}} = \lim_{x\to+\infty} x = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt[3]{x^3 + 3x^2 - 3x - 1}$$

$$= \lim_{x\to-\infty} x\sqrt[3]{1 + \frac{3}{x} - \frac{3}{x^2} - \frac{1}{x^3}} = \lim_{x\to-\infty} x = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt[3]{x^3 + 3x^2 - 3x - 1}}{x} = \lim_{x\to\infty} \sqrt[3]{\frac{x^3}{x^3} + 3\frac{x^2}{x^3} - 3\frac{x}{x^3} - \frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \sqrt[3]{1 + \frac{3}{x^2} - \frac{3}{x^2} - \frac{1}{x^3}} = 1$$

and,

$$b = \lim_{x\to\infty} \left(f(x) - ax\right) = \lim_{x\to\infty} \left(\sqrt[3]{x^3 + 3x^2 - 3x - 1} - x\right) = \{(\infty - \infty)\}$$

$$= \lim_{x\to\infty} \left[x\left(\sqrt[3]{1 + 3/x - 3/x^2 - 1/x^3} - 1\right)\right] = \{(\infty \cdot 0)\}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{1 + 3/x - 3/x^2 - 1/x^3} - 1}{1/x} = \left\{\left(\frac{0}{0}\right)\right\}$$

$$\overset{\text{l'H}}{=} \lim_{x\to-\infty} \frac{\frac{1}{3}\dfrac{-3/x^2+6/x^3+3/x^4}{\sqrt[3]{\left(1+3/x-3/x^2-1/x^3\right)^4}}}{-1/x^2} = \frac{1}{3}\lim_{x\to-\infty}\frac{3-\cancel{6/x}^{\,0}-\cancel{3/x^2}^{\,0}}{\sqrt[3]{\left(1+\cancel{3/x}^{\,0}-\cancel{3/x^2}^{\,0}-\cancel{1/x^3}^{\,0}\right)^4}}$$

$$= 1$$

therefore, oblique asymptote is $y_{\text{aa}} = x + 1$.

7. *Critical points:*

$\underline{f'(x) = 0:}$

$$f'(x) = \left(\sqrt[3]{x^3+3x^2-3x-1}\right)' = \left((x^3+3x^2-3x-1)^{1/3}\right)'$$

$$= \frac{1}{3}(x^3+3x^2-3x-1)^{-2/3}\ (3x^2+6x-3) = \frac{1}{\cancel{3}}\frac{\cancel{3}(x^2+2x-1)}{\sqrt[3]{(x^3+3x^2-3x-1)^2}} = 0$$

$$\therefore\ x^2+2x-1=0\ \ \therefore\ x_{1,2} = \frac{-2\pm\sqrt{4+4}}{2}$$

$$x_1 = -1-\sqrt{2},\quad x_2 = -1+\sqrt{2}$$

$$\therefore\ y_1 = f(x_1) = \sqrt{\left(-1-\sqrt{2}\right)^3+3\left(-1-\sqrt{2}\right)^2-3\left(-1-\sqrt{2}\right)-1}$$

$$= \sqrt[3]{-7-5\sqrt{2}+3(3+2\sqrt{2})+3+3\sqrt{2}-1}$$

$$= \sqrt[3]{4+4\sqrt{2}}$$

$$y_2 = f(x_2) = \sqrt{\left(-1+\sqrt{2}\right)^3+3\left(-1+\sqrt{2}\right)^2-3\left(-1+\sqrt{2}\right)-1}$$

$$= \sqrt[3]{-7+5\sqrt{2}+3(3-2\sqrt{2})+3-3\sqrt{2}-1}$$

$$= \sqrt[3]{4-4\sqrt{2}}$$

In addition, poles of $f'(x)$ are

$$\sqrt[3]{(x^3+3x^2-3x-1)^2} = 0\ \ \therefore\ x_3 = -2-\sqrt{3},\quad x_4 = -2+\sqrt{3},\quad x_5 = 1$$

In other words, $f'(x)$ is in the form of

$$f'(x) = \frac{(x-x_1)(x-x_2)}{\sqrt[3]{\left((x-x_3)(x-x_4)(x-x_5)\right)^2}} = \frac{N(x)}{D(x)}$$

where $D(x) \geq 0$, so that $f'(x)$ signs relative to its zeros (x_1, x_2) and poles (x_3, x_4, x_5) are

x		x_3		x_1		x_4		x_2		x_5	
$x-x_1$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x-x_2$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$D(x)$	$+$	0	$+$	$+$	$+$	0	$+$	$+$	$+$	0	$+$
$f'(x)$	$+$	n.d.	$+$	0	$-$	n.d.	$-$	0	$+$	n.d.	$+$
$f(x)$	$\nearrow$	0	$\nearrow$	y_1	$\searrow$	0	$\searrow$	y_2	$\nearrow$	0	$\nearrow$

There is one local maximum at (x_1, y_1) and one local minimum at (x_2, y_2). Note that, although $f'(x_3, x_4, x_5)$ are not defined, function $f(x)$ is continuous; thus, this function does not change its ascending/descending behaviour. Also,

$f''(x) = 0$:

$$
\begin{aligned}
f''(x) &= \left(\frac{x^2+2x-1}{\sqrt[3]{(x^3+3x^2-3x-1)^2}} \right)' \\
&= \frac{(2x+2)\sqrt[3]{(x^3+3x^2-3x-1)^2} - \dfrac{2(x^2+2x-1)\,\cancel{3}\,(x^2+2x-1)}{\cancel{3}\,\sqrt[3]{(x^3+3x^2-3x-1)}}}{\sqrt[3]{(x^3+3x^2-3x-1)^4}} \\
&= \frac{(2x+2)(x^3+3x^2-3x-1) - 2(x^2+2x-1)^2}{\sqrt[3]{(x^3+3x^2-3x-1)^5}} \\
&= \frac{-4(x^2+1)}{\sqrt[3]{(x^3+3x^2-3x-1)^5}} \\
&= 0 \quad \therefore \qquad \text{but,} \quad -4(x^2+1) < 0 \neq 0
\end{aligned}
$$

therefore, there are not zeros of $f''(x)$. However, there are three poles of $f''(x)$ that must be examined as well,

$$\sqrt[3]{(x^3+3x^2-3x-1)^2} = 0 \quad \therefore \quad x_1 = -2-\sqrt{3}, \quad x_2 = -2+\sqrt{3}, \quad x_3 = 1$$

that happen to be at the same location as the $f(x)$ zeros.

$$f''(x) = \frac{N(x)}{D(x)} = \frac{-4(x^2+1)}{\sqrt[3]{(x^3+3x^2-3x-1)^5}} = \frac{-4(x^2+1)}{\sqrt[3]{\big((x-x_1)(x-x_2)(x-x_3)\big)^5}}$$

where $N(x) < 0$, then $f''(x)$ sign is

x		x_2		x_3		x_1	
$N(x)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$x - x_2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x - x_1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	n.d.	$-$	n.d.	$+$	n.d.	$-$
$f(x)$	$\cup$	0	$\cap$	0	$\cup$	0	$\cap$

This function changes its convexity after passing through each of its $f''(x)$ poles.

8. *Summary of the important points:*

$$\begin{aligned}
\text{y-axis crossing point } f(0): &\quad (x, y) = (0, -1) \\
\text{a.a. asymptote:} &\quad y = x + 1 \\
\text{zeros:} &\quad (x, y) = (1, 0) \\
&\quad (x, y) = \left(-2 - \sqrt{3}, 0\right) \\
&\quad (x, y) = \left(-2 + \sqrt{3}, 0\right) \\
\text{extremes:} &\quad (x, y) = \left(-1 - \sqrt{2}, \sqrt[3]{4 + 4\sqrt{2}}\right) \quad \text{(max)} \\
&\quad (x, y) = \left(-1 + \sqrt{2}, \sqrt[3]{4 - 4\sqrt{2}}\right) \quad \text{(min)} \\
\text{inflections:} &\quad (x, y) = (1, 0) \\
&\quad (x, y) = \left(-2 - \sqrt{3}, 0\right) \\
&\quad (x, y) = \left(-2 + \sqrt{3}, 0\right)
\end{aligned}$$

9. *Graphical representation:* the calculated results are sufficient to plot $f(x)$ graph (see Fig. 3.60).

3.4 Exponential and Log Functions

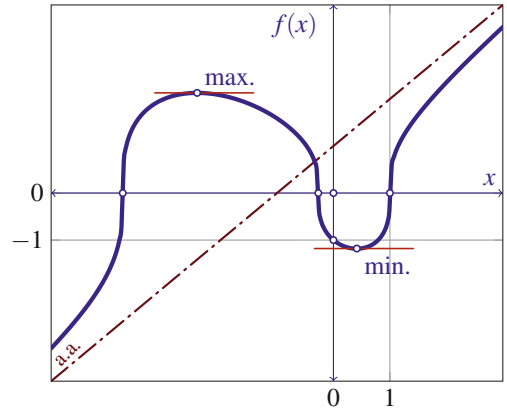
3.68. Given exponential function,

$$f(x) = f(x) = 2^x$$

1. *Domain of definition* $\mathbb{D}$: an exponential function is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\text{even:} \qquad f(-x) = 2^{-x} \neq f(x) \quad \therefore \quad \text{(not even)}$$

Fig. 3.60 Example P.3.67



$$\underline{\text{odd:}} \quad -f(-x) = -\left(2^{-x}\right) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

This exponential function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \text{however,} \quad 2^x > 0 \quad \text{i.e.} \quad 2^x \neq 0$$

Therefore, exponential function is always positive.

4. *y-axis crossing point* $f(0)$: $f(0) = 2^0 = 1$
5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} 2^x = 2^{+\infty} = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2^x = 2^{-\infty} = \frac{1}{2^{+\infty}} = 0\downarrow$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2^x}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2^x \ln 2}{1} = \infty$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(2^x\right)' = 2^x \ln 2 > 0 \quad \therefore \quad f'(x) \neq 0$$

therefore, $f(x)$ always ascends $\nearrow$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(2^x \ln 2\right)' = 2^x \ln^2 2 > 0 \quad \therefore \quad f''(x) \neq 0$$

therefore, $f(x)$ is always concave $\cup$.

8. *Summary of the important points:*

$$\underline{\text{h. asymptote:}} \quad y = 0 \quad (x < 0)$$

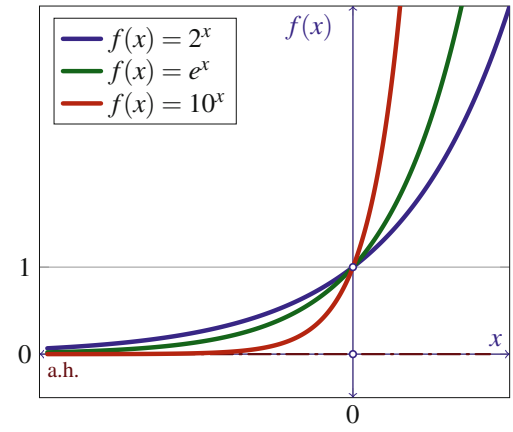


Fig. 3.61 Example P.3.68

y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
zers:	(none, $f(x)$ is always positive)
extremes:	(none, $f(x)$ always ascends $\nearrow$)
inflection:	(none, $f(x)$ is always concave $\cup$)

9. *Graphical representation:* results of the analysis for function in P.3.68 are summarized by $f(x)$ graph (see Fig. 3.61).

3.69. Given exponential function

$$f(x) = e^x$$

the overall analysis is identical to P.3.68, and the resulting graph is compared in Fig. 3.61.

3.70. Given exponential function

$$f(x) = 10^x$$

the overall analysis is identical to P.3.68, and the resulting graph is compared in Fig. 3.61.

3.71. Given rational function

$$f(x) = 2^{-x}$$

1. *Domain of definition* $\mathbb{D}$: an exponential function is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = 2^{-(-x)} = 2^x \neq f(x) \quad \therefore \quad \text{(not even)}$$
$$\text{odd:} \quad -f(-x) = -\left(2^x\right) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, this exponential function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \text{however,} \quad 2^{-x} = \frac{1}{2^x} > 0 \quad \text{i.e.,} \quad 2^{-x} \neq 0$$

Therefore, this exponential function is always positive.

4. *y-axis crossing point* $f(0)$: $f(0) = 2^0 = 1$
5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} 2^{-x} = \frac{1}{2^x} = 0\downarrow$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2^{-x} = 2^{-(-\infty)} = 2^{+\infty} = +\infty$$

There is a horizontal asymptote at $y = 0, \quad (x > 0)$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2^{-x}}{x} = \lim_{x\to\infty} \frac{1}{x\,2^x} = 0$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(2^{-x}\right)' = -2^{-x}\,\ln 2 < 0 \quad \therefore \quad f'(x) \neq 0$$

therefore, $f(x)$ always descends $\searrow$.

Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-2^{-x}\,\ln 2\right)' = 2^{-x}\,\ln^2 2 > 0 \quad \therefore \quad f''(x) \neq 0$$

therefore, $f(x)$ is always concave $\cup$.

8. *Summary of the important points:*

h. asymptote:	$y = 0 \quad (x > 0)$
y-axis crossing point $f(0)$:	$(x, y) = (0, 1)$
zeros:	(none, $f(x)$ is always positive)
extremes:	(none, $f(x)$ always descends $\searrow$)
inflection:	(none, $f(x)$ is always concave $\cup$)

9. *Graphical representation:* see Fig. 3.62.

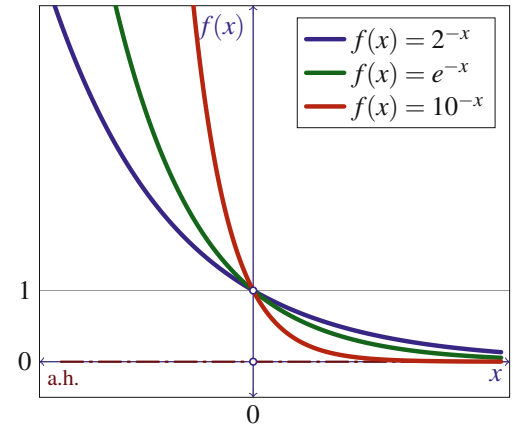


Fig. 3.62 Example P.3.71

3.72. Given rational function

$$f(x) = e^{-x}$$

the overall analysis is identical to P.3.71, and the resulting graph is compared in Fig. 3.62.

3.73. Given exponential function

$$f(x) = 10^{-x}$$

the overall analysis is identical to P.3.71, and the resulting graph is compared in Fig. 3.62.

3.74. Given exponential function

$$f(x) = 2^x - 2$$

1. *Domain of definition* $\mathbb{D}$: an exponential function is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = 2^{(-x)} - 2 \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(2^{(-x)} - 2\right) = -2^{(-x)} + 2 \neq f(x) \quad \therefore \quad \text{(not odd)}$$

This exponential function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad 2^x - 2 = 0 \quad \therefore \quad 2^x = 2$$

$$\therefore \quad \log_2\left(2^x\right) = \log_2 2^1 \quad \therefore \quad x = 1$$

Therefore,

$$\underline{x < 1:} \quad \therefore \quad 2^x < 2 \quad \therefore \quad 2^x - 2 < 0$$

$$\underline{x = 1:} \quad \therefore \quad 2^1 = 2 \quad \therefore \quad 2^x - 2 = 0$$
$$\underline{x > 1:} \quad \therefore \quad 2^x > 2 \quad \therefore \quad 2^x - 2 > 0$$

There is one first-order zero at $(x, y) = (1, 0)$.

4. *y-axis crossing point* $f(0)$: $f(0) = 2^0 - 2 = -1$
5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(2^x - 2\right) = \lim_{x\to+\infty} \left(2^{x \to +\infty} - 2\right) = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(2^x - 2\right) = 2^{(-\infty)\to 0} - 2 = -2\downarrow \quad (x < 0)$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2^x - 2}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2^x \ln 2}{1} = \infty$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(2^x - 2\right)' = 2^x \ln 2 > 0 \quad \therefore \quad f'(x) \neq 0$$

therefore, $f(x)$ always ascends $\nearrow$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(2^x \ln 2\right)' = 2^x \ln^2 2 > 0 \quad \therefore \quad f''(x) \neq 0$$

therefore, $f(x)$ is always concave $\cup$.

8. *Summary of the important points:*

y-axis crossing point $f(0)$:	$(x, y) = (0, -1)$
h. asymptote:	$y = -2 \quad (x < 0)$
zero:	$(x, y) = (1, 0)$
extremes:	(none, $f(x)$ always ascends $\nearrow$)
inflection:	(none, $f(x)$ is always concave $\cup$)

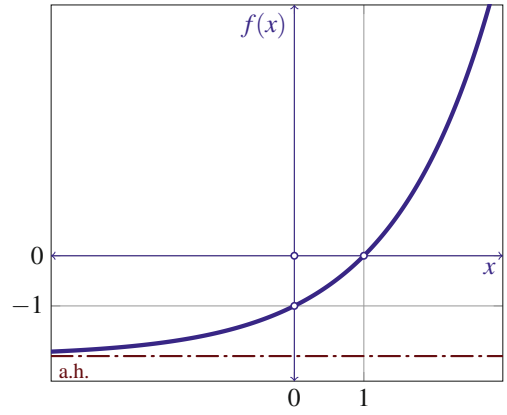
9. *Graphical representation:* results of the analysis for function in P.3.74 are summarized by $f(x)$ graph (see Fig. 3.63).

3.75. Given exponential function

$$f(x) = 3^x - 3^{-x}$$

1. *Domain of definition* $\mathbb{D}$: an exponential function is defined for $x \in \mathbb{R}$, because this function is a sum of two exponential functions each defined for all $x \in \mathbb{R}$.

Fig. 3.63 Example P.3.74



2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = 3^{-x} - 3^{x} \neq f(x) \;\; \therefore \;\; \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(3^{-x} - 3^{x}\right) = -3^{-x} + 3^{x} \neq f(x) \;\; \therefore \;\; \text{(not odd)}$$

This exponential function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; 3^{x} - 3^{-x} = 0 \;\; \therefore \;\; 3^{x}\,3^{x} = 3^{-x}\,3^{x}$$

$$\therefore \;\; 3^{2x} = 1 \;\; \left\{a^{0} = 1\right\} \;\; \therefore \;\; 2x = 0 \;\; \therefore \;\; x = 0$$

Therefore,

$$\underline{x < 0:} \quad \therefore \;\; 3^{x} < 3^{-x} \;\; \therefore \;\; 3^{x} - 3^{-x} < 0$$

$$\underline{x = 0:} \quad \therefore \;\; 3^{0} = 3^{0} \;\; \therefore \;\; 3^{0} - 3^{0} = 0$$

$$\underline{x > 0:} \quad \therefore \;\; 3^{x} > 3^{-x} \;\; \therefore \;\; 3^{x} - 3^{-x} > 0$$

There is one first-order zero at $(x, y) = (0, 0)$.

4. *y-axis crossing point* $f(0)$: $f(0) = 3^{0} - 3^{0} = 0$
5. *Limits:* there are two limits in total to be resolved.

$$\underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(3^{x \to +\infty} - 3^{-x \to 0}\right) = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(3^{x \to 0} - 3^{-x \to +\infty}\right) = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{3^x - 3^{-x}}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\ln 3 \left(3^x + 3^{-x}\right)}{1} = \infty$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(3^x - 3^{-x}\right)' = \ln 3 \left(3^x + 3^{-x}\right) > 0 \quad \therefore \quad f'(x) \neq 0$$

therefore, $f(x)$ always ascends $\nearrow$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\ln 3 \left(3^x + 3^{-x}\right)\right)' = \ln^2 3 \left(3^x - 3^{-x}\right)$$
$$\therefore \quad 3^x - 3^{-x} = 0 \quad \therefore \quad x = 0$$

therefore, $(\ln^2 3 > 0)$

$$\underline{x < 0:} \quad \therefore \quad 3^x < 3^{-x} \quad \therefore \quad 3^x - 3^{-x} < 0 \quad \therefore \quad f(x) \quad \text{is convex} \quad \cap$$
$$\underline{x = 0:} \quad \therefore \quad 3^0 = 3^0 \quad \therefore \quad 3^0 - 3^0 = 0$$
$$\underline{x > 0:} \quad \therefore \quad 3^x > 3^{-x} \quad \therefore \quad 3^x - 3^{-x} > 0 \quad \therefore \quad f(x) \quad \text{is concave} \quad \cup$$

There is inflection point at $(x, y) = (0, 0)$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{y-axis crossing point } f(0):} &\quad (x, y) = (0, 0) \\
\underline{\text{zero:}} &\quad (x, y) = (0, 0) \\
\underline{\text{extremes:}} &\quad (\text{none}, f(x) \text{ always ascends } \nearrow) \\
\underline{\text{inflection:}} &\quad (x, y) = (0, 0)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.75 are summarized by $f(x)$ graph (see Fig. 3.64).

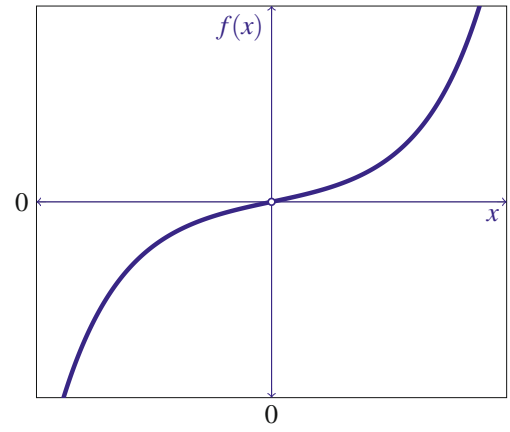
3.76. Given absolute function

$$f(x) = \left|5^{|x|} - 2\right|$$

1. *Domain of definition* $\mathbb{D}$: because $f(x)$ is a composition of absolute and exponential functions, each is defined for all $x \in \mathbb{R}$, and this function itself is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \left|5^{|-x|} - 2\right| = \left|5^{|x|} - 2\right| = f(x) \quad \therefore \quad (\text{even})$$

Fig. 3.64 Example P.3.75



$$\underline{\text{odd:}} \quad -f(-x) = -\left(\left|5^{|x|}-2\right|\right) = -\left|5^{|x|}-2\right| \neq f(x) \quad \therefore \quad \text{(not odd)}$$

This absolute/exponential function is even.

3. *Function's sign:*

$$\underline{f(x)=0:} \quad \therefore \quad \left|5^{|x|}-2\right| = 0 \quad \therefore \quad 5^{|x|} = 2$$

$$\therefore \quad \log_5\left(5^{|x|}\right) = \log_5 2 \quad \left\{\log_a b = \frac{\ln b}{\ln a}\right\}$$

$$\therefore \quad |x| = \frac{\ln 2}{\ln 5} \quad \therefore \quad x_{1,2} = \pm\frac{\ln 2}{\ln 5}$$

That is to say, there are two first-order symmetrical zeros. However, by definition $|\cdot| \geq 0$, that is to say $\left|5^{|x|}-2\right| \geq 0$. By consequence, this function does not cross the horizontal axis.

4. *y-axis crossing point* $f(0)$: $f(0) = \left|5^{|0|}-2\right| = |-1| = 1$
5. *Limits:* being even function, i.e., symmetric relative to the vertical axis, both positive and negative side limits are equal.

$$\underline{x \to \pm\infty:} \quad \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left|5^{|x|}-2\right| = \pm\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{\left|5^{|x|}-2\right|}{x}$$

$$\{\text{even function, so first for } (x>0)\}$$

$$= \lim_{x\to\infty}\left(\frac{5^x}{x} - \frac{\overset{0}{\not 2}}{\not x}\right) = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty}\left(\frac{5^x \ln 5}{1}\right) = \infty$$

and limit for ($x < 0$) also tends to infinity (even function), so, by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left|5^{|x|} - 2\right|'$$

$$\left\{|x| = \sqrt{x^2} = \left(x^2\right)^{1/2}\right\}$$

$$\left\{|x|' = \left[\left(x^2\right)^{1/2}\right]' = \frac{1}{2}(x^2)^{-1/2}\ 2x\,x' \quad \therefore \quad x' = \frac{x}{|x|} \equiv \frac{|x|}{x} = \operatorname{sign}(x)\right\}$$

$$= \frac{\left|5^{|x|} - 2\right|\ 5^{|x|}\ \ln 5}{\left(5^{|x|} - 2\right)} \frac{|x|}{x}$$

$$\therefore \quad \left|5^{|x|} - 2\right|\ 5^{|x|}\ \ln 5\,|x| = 0 \quad \text{(note: } x \neq 0\text{)}$$

$$\therefore \quad \left|5^{|x|} - 2\right|\ |x| = 0 \quad \text{because both:} \quad 5^{|x|} > 0 \quad \text{and} \quad \ln 5 > 0 \quad \text{(i.e. } \neq 0\text{)}$$

These two absolute functions may be decomposed as:

$$\underline{(5^{|x|} - 2) \geq 0 \quad \text{and,} \quad x > 0:} \quad \therefore \quad (5^x - 2)x = 0 \quad \therefore \quad 5^x = 2 \quad \therefore \quad x_1 = \log_5 2 = \frac{\ln 2}{\ln 5}$$

therefore, for the reason of even function's symmetry

$$x_2 = -\log_5 2 = -\frac{\ln 2}{\ln 5}$$

Note that $f'(0) = 0/0$ is not defined. However, as $f(0) = 1$, this function is continuous. Thus,

x		x_2		0		x_1	
$\ln 5$	+	+	+	+	+	+	+
$\lvert x\rvert$	+	+	+	0	+	+	+
$5^{\lvert x\rvert}$	+	+	+	+	+	+	+
$\left\lvert 5^{\lvert x\rvert} - 2\right\rvert$	+	0	+	+	+	0	+
$5^{\lvert x\rvert} - 2$	+	0	−	−	−	0	+
x	−	−	−	0	+	+	+
$f'(x)$	−	n.d.	+	n.d.	−	n.d.	+
$f(x)$	↘	0	↗	1	↘	0	↗

There are two minimums, $(x, y) = (x_2, 0)$ and $(x, y) = (x_1, 0)$, as well as one maximum at $(x, y) = (0, 1)$. $f'(0, x_{1,2})$ are not defined, but extremes exist because the function itself is continuous. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{\left|5^{|x|} - 2\right|\ 5^{|x|}\ \ln 5}{\left(5^{|x|} - 2\right)} \frac{|x|}{x}\right)'$$

{derivative of a rational function plus the chain rule}

$$= \ln^2(5)\, 5^{|x|}\, \frac{\left|5^{|x|}-2\right|}{5^{|x|}-2}\, \frac{x^2}{x^2} = 0$$

$$\therefore \quad 5^{|x|}\, \left|5^{|x|}-2\right| = 0, \quad (x \neq 0)$$

$$\therefore \quad x_{1,2} = \log_5 2 = \pm\frac{\ln 2}{\ln 5}$$

$f''(0) = 0/0$ is not defined, inflection points exist (the function itself is continuous). Thus:

x		x_2		0		x_1	
$5^{\|x\|}$	+	+	+	+	+	+	+
$\ln^2 5$	+	+	+	+	+	+	+
x^2	+	+	+	0	+	+	+
x^2	+	+	+	0	+	+	+
$\left\|5^{\|x\|}-2\right\|$	+	0	+	+	+	0	+
$5^{\|x\|}-2$	+	0	−	−	−	0	+
$f''(x)$	+	n.d.	−	n.d.	−	n.d.	+
$f(x)$	∪	0	∩	1	∩	0	∪

Even though second derivatives are not defined, there are three inflection points at $(x, y) = (x_2, 0)$, $(x, y) = (0, 1)$, and $(x, y) = (x_1, 0)$.

8. *Summary of the important points:*

parity: (even)

y-axis crossing point $f(0)$: $(x, y) = (0, 1)$

zeros: $(x, y) = (\ln 2/\ln 5, 0)$

$(x, y) = (-\ln 2/\ln 5, 0)$

extremes: $(x, y) = (-\ln 2/\ln 5, 0)$ (min)

$(x, y) = (0, 1)$ (max)

$(x, y) = (\ln 2/\ln 5, 0)$ (min)

inflection: $(x, y) = (-\ln 2/\ln 5, 0)$

$(x, y) = (0, 1)$

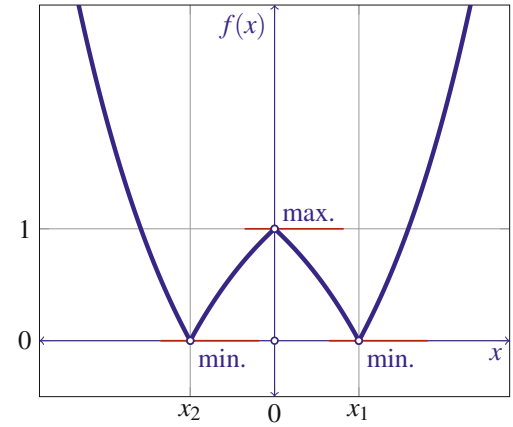
$(x, y) = (\ln 2/\ln 5, 0)$

9. *Graphical representation:* results of the analysis for function in P.3.76 are summarized by $f(x)$ graph (see Fig. 3.65).

3.77. Given a simple natural logarithm function,

$$f(x) = \ln x$$

Fig. 3.65 Example P.3.76



1. *Domain of definition* $\mathbb{D}$: by definition, $x \in \mathbb{R} | x > 0$.
2. *Function's parity:*

$$\begin{aligned} \underline{\text{even:}} \quad & f(-x) = \ln(-x) \quad \text{(not defined)} \\ \underline{\text{odd:}} \quad & -f(-x) = -(\ln(-x)) \quad \text{(not defined)} \end{aligned}$$

   This logarithmic function is neither odd nor even.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \ \ln x = 0 \ \therefore \ e^{\ln x} = e^{0} \ \therefore \ x = 1$$

   where if $0 < x < 1$, then $\ln x < 0$, and for $x > 1$, logarithmic function changes its sign, $\ln x > 0$.
4. *y-axis crossing point* $f(0)$: $f(0) = \ln 0$ (not defined)
5. *Limits:* there are two limits in total to resolve.

$$\begin{aligned} \underline{x \to +\infty:} \quad & \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \ln x = +\infty \\ \underline{x \to \underset{\twoheadleftarrow}{0}:} \quad & \lim_{x \to \underset{\twoheadleftarrow}{0}} f(x) = \lim_{x \to \underset{\twoheadleftarrow}{0}} \ln x = -\infty \end{aligned}$$

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\ln x}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{1}{x} = 0$$

   by consequence, there is no oblique asymptote.
7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (\ln x)' = \frac{1}{x} > 0, \quad (x > 0) \ \ \text{(i.e. } \neq 0)$$

   Therefore, a logarithmic function always ascends ↗. Also,

$$f''(x) = 0: \quad f''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0$$

Therefore, a logarithmic function is convex ∩.

8. *Summary of the important points:*

defined:	$x > 0$
v. asymptote:	$x = 0$
zero:	$(x, y) = (1, 0)$
extremes:	(none), $f(x)$ always ascends ↗
inflection:	(none), $f(x)$ is always convex ∩

9. *Graphical representation:* results of the analysis for function in P.3.77 are summarized by $f(x)$ graph (see Fig. 3.66).

3.78. Given logarithmic function,

$$f(x) = \log_2 x$$

Analysis of this function is identical to P.3.77, and the two functions are compared in Fig. 3.66.

3.79. Given composite absolute logarithmic function,

$$f(x) = \left|\log_2 x\right| \equiv \left|\frac{\ln(x)}{\ln(2)}\right| = \frac{1}{\ln(2)} \quad |\ln(x)|$$

1. *Domain of definition* $\mathbb{D}$: by definition, a logarithmic function is defined for strictly positive arguments, i.e., $x \in \mathbb{R} | x > 0$.
2. *Function's parity:*

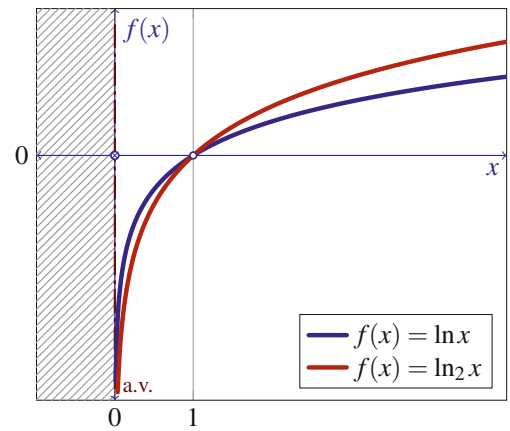


Fig. 3.66 Example P.3.77

$$\underline{\text{even:}} \quad f(-x) = \left|\log_2(-x)\right| \quad \text{(not defined)}$$
$$\underline{\text{odd:}} \quad -f(-x) = -\left|\log_2(-x)\right| \quad \text{(not defined)}$$

In conclusion, a logarithmic function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad \log_2 x = 0 \quad \therefore \quad 2^{\log_2 x} = 2^0 \overset{1}{} \quad \therefore \quad x = 1$$

or, equivalently,

$$\underline{f(x) = 0:} \quad \therefore \quad \frac{1}{\ln(2)} \left|\ln(x)\right| \quad \therefore \quad \ln x = 0 \quad \therefore \quad e^{\ln x} = e^0 \overset{1}{} \quad \therefore \quad x = 1$$

There is one first-order zero at $(x, y) = (1, 0)$. By itself, a function crosses the horizontal axes at odd order zeros. However, this is composite absolute function that is positive for all arguments, that is to say, does not cross the horizontal axes. Thus $\left|\log_2 x\right| \geq 0$ for all $x > 0$.

4. *y-axis crossing point* $f(0)$: $f(0) = \left|\log_2 0\right|$ (not defined)
5. *Limits:* there are two limits in total to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left|\log_2 x\right| = +\infty$$
$$\underline{x \underset{\leftarrow}{\to} 0:} \quad \lim_{x\underset{\leftarrow}{\to} 0} f(x) = \lim_{x\underset{\leftarrow}{\to} 0} \left|\log_2 x\right| = |-\infty| = +\infty$$

6. *Oblique asymptote:* note that an absolute function is either positive or equal zero. Thus, given $x > 1$, it is true for all log functions that $|\log(x)| \equiv \log(x)$. Therefore,

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\left|\log_2 x\right|}{x} \equiv \lim_{x\to\infty} \frac{\log_2 x}{x} = \left(\frac{\infty}{\infty}\right) \left\{\log_b x \equiv \frac{\ln(x)}{\ln(b)}\right\}$$
$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\left(\dfrac{\ln x}{\ln 2}\right)'}{x'} = \lim_{x\to\infty} \frac{1}{\ln 2}\frac{1}{x} = 0$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left|\log_2 x\right|' = \left|\frac{\ln x}{\ln 2}\right|'$$
$$= \frac{1}{\ln 2} |\ln x|' \left\{\text{composite function: } |f(x)|' = \frac{f(x)}{|f(x)|} f'(x)\right\}$$
$$= \frac{1}{\ln 2} \frac{\ln x}{x\,|\ln x|} \equiv \frac{1}{\ln 2} \frac{|\ln x|}{x\,\ln x} = 0 \quad \therefore \quad \ln x = 0 \quad \therefore \quad x = 1$$

note, however, there is also $|\ln x|$ term in the denominator, which results in division by 0 if $x = 1$. That is to say, $f'(1) = 0/0$ is not defined. Thus,

x	0		1	
$\ln 2$	+	+	+	+
x	0	+	+	+
$\ln x$	n.d.	−	0	+
$\lvert\ln x\rvert$	n.d.	+	0	+
$f'(x)$	n.d.	−	n.d	+
$f(x)$	n.d.	↘	0	↗

This $f(x)$ is an example of a function that, even though its first derivative is not defined at $x = 1$, the function itself is continuous, i.e., defined at $f(1)$, and thus has minimum point at $(x, y) = (1, 0)$.

$$f''(x) = 0: \quad f''(x) = \left(\frac{1}{\ln 2}\,\frac{|\ln x|}{x\,\ln x}\right)' = \frac{1}{\ln 2}\;\frac{x\,\ln x\,\dfrac{\frac{1}{x}\ln x}{|\ln x|} - |\ln x|\left(\ln x + \frac{1}{x}\,x\right)}{x^2\,(\ln x)^2}$$

$$= \frac{1}{\ln 2}\;\frac{\ln x\,\dfrac{\ln x}{|\ln x|} - |\ln x|\,\ln x - |\ln x|}{x^2\,(\ln x)^2}$$

$$\equiv \frac{1}{\ln 2}\;\frac{\ln x\,\dfrac{|\ln x|}{\ln x} - |\ln x|\,\ln x - |\ln x|}{x^2\,(\ln x)^2} = \frac{1}{\ln 2}\;\frac{-|\ln x|}{x^2\,\ln x}$$

note: $f''(1) = 0/0$ is not defined. Then,

x	0		1	
$-(1/\ln 2)$	−	−	−	−
$\ln x$	n.d.	−	0	+
$\lvert\ln x\rvert$	n.d.	+	0	+
x^2	0	+	+	+
$f''(x)$	n.d.	+	n.d	−
$f(x)$	n.d.	∪	0	∩

This $f(x)$ is an example of a function that, even though its second derivative is not defined at $x = 1$, the function itself is continuous, i.e., defined at $f(1) = 0$, thus $f(x)$ changes from concave ∪ to convex ∩ at $(x, y) = (1, 0)$.

8. *Summary of the important points:*

$$\begin{aligned} \text{defined:} \quad & x > 0 \\ \text{v. asymptote:} \quad & x = 0 \\ \text{zero:} \quad & (x, y) = (1, 0) \end{aligned}$$

Fig. 3.67 Example P.3.79

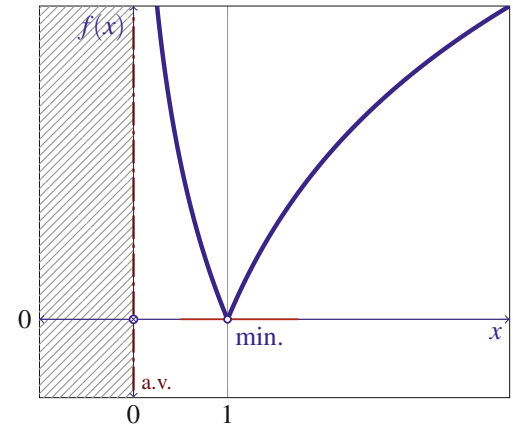
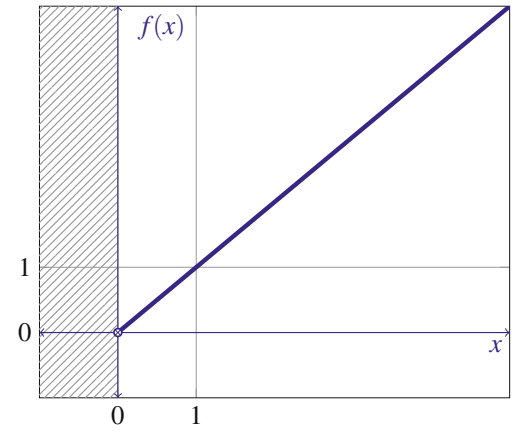


Fig. 3.68 Example P.3.80



$$\underline{\text{extremes:}} \quad (x, y) = (1, 0) \quad \text{(min)}$$
$$\underline{\text{inflection:}} \quad (x, y) = (1, 0)$$

9. *Graphical representation:* see Fig. 3.67.

3.80. Given exponential/log function,

$$f(x) = e^{\ln x}$$

a simple transformation uncovers true form of this function as

$$f(x) = e^{\ln x} = x$$

that is to say a linear function. However, logarithmic function is defined only for $x > 0$; therefore, domain of $f(x)$ definition is $x \in \mathbb{R} | x > 0$ as summarized by $f(x)$ graph (see Fig. 3.68).

3.81. Given logarithmic/exponential function,

$$f(x) = \ln(e^x)$$

a simple transformation uncovers true form of this function as

$$f(x) = \ln(e^x) = x$$

that is to say, it is a simple linear function. Note that the argument of logarithmic function is defined for all x because e^x is defined for all x, and more importantly, it is *always positive*, even when $x < 0$. Therefore, domain of $f(x) = \ln\big(\exp(x)\big) = x$ is $x \in \mathbb{R}$.

3.82. Given logarithmic composite function,

$$f(x) = \ln(x^2 - x) = \ln\Big(x(x-1)\Big)$$

1. *Domain of definition* $\mathbb{D}$: by definition, argument of logarithmic function must be strictly positive, that is to say

$$x(x-1) > 0 \;\; \therefore \;\; \begin{cases} \underline{x > 0 \,\&\, x - 1 > 0:} & \therefore \;\; x > 0 \,\&\, x > 1 \;\; \therefore \;\; x > 1 \\ \underline{x < 0 \,\&\, x - 1 < 0:} & \therefore \;\; x < 0 \,\&\, x < 1 \;\; \therefore \;\; x < 0 \end{cases}$$

(recall that by saying $AB > 0$ it is implied: $A > 0$ and $B > 0$, or, $A < 0$ and $B < 0$ at the same time).

In summary, domain of definition $\mathbb{D}$ is $x \in \mathbb{R} | x < 0, x > 1$. In addition, $x \neq 0$ and $x \neq 1$; therefore, these two points are where two vertical asymptotes are placed.

2. *Function's parity:*

$$\begin{aligned} &\underline{\text{even:}} \quad f(-x) = \ln((-x)^2 - (-x)) = \ln(x^2 + x) \neq f(x) \quad \text{(not even)} \\ &\underline{\text{odd:}} \quad -f(-x) = -\ln(x^2 + x) \neq f(x) \quad \text{(not odd)} \end{aligned}$$

In conclusion, this function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; x^2 - x = 1 \;\; \therefore \; x^2 - x - 1 = 0 \;\; \therefore \;\; x_{1,2} = \frac{1 \pm \sqrt{1+4}}{2}$$

$$x_1 = \frac{1 - \sqrt{5}}{2}, \quad x_2 = \frac{1 + \sqrt{5}}{2}$$

so that, sign of $f(x)$ is determined by the intervals where $(x^2 - x) <> 1$, because $\ln x < 0$ if $0 < x < 1$, and $\ln x > 0$ if $x > 1$, i.e.,

x		x_1		0		1		x_2	
$x - x_1$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x^2 - x - 1$	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	n.d.	n.d.	n.d.	$-$	0	$+$

There are two first-order zeros, one at $(x, y) = (x_1, 0)$ and one at $(x, y) = (x_2, 0)$.

4. *y-axis crossing point* $f(0)$: $f(0) = \ln(0)$ (not defined)
5. *Limits:* there are four limits in total to resolve,

$$x \to -\infty: \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \ln(x^2 - x) = \lim_{x\to-\infty} \ln\left[x^2\left(1 - \cancel{1/x}^{\,0}\right)\right] = +\infty$$

$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln(x^2 - x) = \lim_{x\to+\infty} \ln\left[x^2\left(1 - \cancel{1/x}^{\,0}\right)\right] = +\infty$$

$$x \to 0_{\to}: \quad \lim_{x\to 0_{\to}} f(x) = \lim_{x\to 0_{\to}} \ln(\cancel{x^2 - x}^{\,0}) = -\infty$$

$$x \to 1_{\leftarrow}: \quad \lim_{x\to 1_{\leftarrow}} f(x) = \lim_{x\to 1_{\leftarrow}} \ln(\cancel{x^2 - x}^{\,0}) = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln(x^2 - x)}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2x - 1}{x^2 - x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{2}{2x - 1}$$
$$= 0$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = \left(\ln(x^2 - x)\right)' = \frac{2x - 1}{x^2 - x} = 0 \;\; \therefore \;\; 2x - 1 = 0 \;\; \therefore \;\; x = \frac{1}{2}$$

but $f(x)$ is not defined in the $(0, 1)$ interval. Then,

x		0		$1/2$		1	
$2x - 1$	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	n.d.	$+$	0	$-$	n.d.	$+$
$f(x)$	$\searrow$	n.d.	n.d.	n.d.	n.d.	n.d.	$\nearrow$

That is to say, this type of logarithmic function does not have extremes.
Also,

$$f''(x) = 0: \quad f''(x) = \left(\frac{2x-1}{x^2-x}\right)' = \frac{2(x^2-x)-(2x-1)^2}{x^2(x-1)^2} = -\frac{2x^2-2x+1}{x^2(x-1)^2} = 0$$
$$\therefore \quad 2x^2-2x+1=0 \quad \text{however,} \quad 2x^2-2x+1>0$$

that is to say, $f''(x) < 0$, and by consequence, $f(x)$ is always convex $\cap$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} \quad & x<0, \quad x>1 \\
\underline{\text{v. asymptotes:}} \quad & x=0 \\
& x=1 \\
\underline{\text{zeros:}} \quad & (x,y)=\left(\frac{1-\sqrt{5}}{2},0\right) \\
& (x,y)=\left(\frac{1+\sqrt{5}}{2},0\right) \\
\underline{\text{extremes:}} \quad & \text{(none)} \\
\underline{\text{inflection:}} \quad & \text{(none, } f(x) \text{ is always convex } \cap)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.82 are summarized by $f(x)$ graph (see Fig. 3.69).

3.83. Given exponential/quadratic function,

$$f(x) = e^{2x} - 2e^x + 1$$

1. *Domain of definition* $\mathbb{D}$: exponential function is defined for all x; thus, domain of definition $\mathbb{D}$ is $x \in \mathbb{R}$.
2. *Function's parity:*

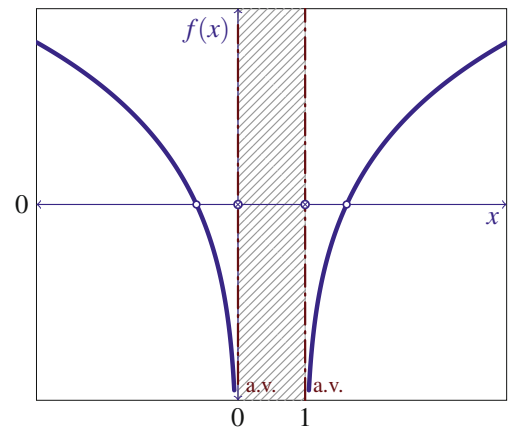


Fig. 3.69 Example P.3.82

$$\underline{\text{even:}} \quad f(-x) = e^{2(-x)} - 2e^{(-x)} + 1 = e^{-2x} - 2e^{-x} + 1 \neq f(x) \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -(e^{-2x} - 2e^{-x} + 1) \neq f(x) \quad \text{(not odd)}$$

In conclusion, this function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad e^{2x} - 2e^{x} + 1 = \left(e^{x}\right)^2 - 2\left(e^{x}\right) + 1 = \left(e^{x} - 1\right)^2 = 0 \quad \therefore \quad e^{x} = 1 \quad \therefore \quad x_{1,2} = 0$$

However,

$$\left(e^{x} - 1\right)^2 \geq 0 \quad \therefore \quad f(x) \geq 0$$

There is one second-order zero at $(x, y) = (0, 0)$; otherwise, $f(x)$ is always positive.

4. *y-axis crossing point* $f(0)$: $f(0) = e^{2(0)} - 2e^{0} + 1 = 0$
5. *Limits:* there are two limits in total to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} e^{2x} - 2e^{x} + 1 = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \overset{0}{\cancel{e^{2x}}} - \overset{0}{\cancel{2e^{x}}} + 1 = 1$$

That is to say, for $x < 0$, there is horizontal asymptote at $y = 1$.

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{e^{2x} - 2e^{x} + 1}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{2e^{2x} - 2e^{x}}{1} = \infty$$

By consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(e^{2x} - 2e^{x} + 1\right)' = 2e^{2x} - 2e^{x} = 0 \quad \therefore \quad \cancel{2}e^{\cancel{2}x} = \cancel{2}e^{x}\cancel{1} \quad \therefore \quad e^{x} = 1$$

$$\therefore \quad x = 0$$

Therefore,

$$f'(x) = 2e^{2x} - 2e^{x} = 2e^{x}\left(e^{x} - 1\right)$$

x		0	
2	+	+	+
e^{x}	+	+	+
$e^{x} - 1$	−	0	+
$f'(x)$	−	0	+
$f(x)$	↘	0	↗

There is one minimum point at $(x, y) = (0, 0)$. Also,

$$f''(x) = 0: \quad f''(x) = \left(2e^{2x} - 2e^{x}\right)' = 4e^{2x} - 2e^{x} = 0 \;\; \therefore \;\; 2e^{2x} = e^{x} \;\; \therefore \;\; 2e^{x} = 1$$

$$\therefore \;\; e^{x} = \frac{1}{2} \;\; \therefore \;\; x = \ln\frac{1}{2}$$

$$\text{and, } f\left(\tfrac{1}{2}\right) = e^{2\ln 1/2} - 2e^{\ln 1/2} + 1 = \left(\frac{1}{2}\right)^{2} - 2\frac{1}{2} + 1 = \frac{1}{4}$$

Therefore, as $f''(x) = 2e^{x}\,(2e^{x} - 1)$

x		$\ln(1/2)$	
2	+	+	+
e^{x}	+	+	+
$2e^{x} - 1$	−	0	+
$f''(x)$	−	0	+
$f(x)$	∩	1/4	∪

There is one inflection point at $(x, y) = (\ln(1/2), 1/4)$.

8. *Summary of the important points:*

h. asymptote: $y = 1 \quad (x < 0)$

y-axis crossing point $f(0)$: $(x, y) = (0, 0)$

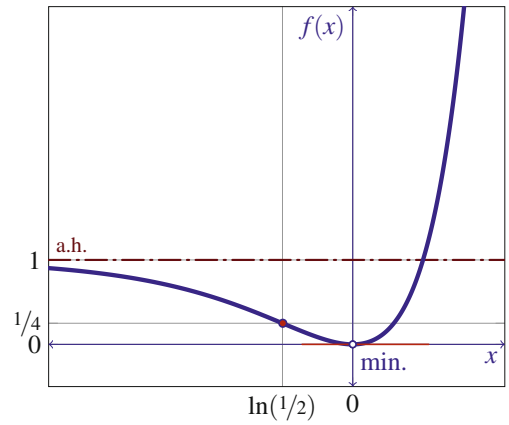
zero: $(x, y) = (0, 0)$

extremes: $(x, y) = (0, 0) \quad (\text{min})$

inflection: $(x, y) = \left(\ln(1/2), 1/4\right)$

9. *Graphical representation:* results of the analysis for function in P.3.83 are summarized by $f(x)$ graph (see Fig. 3.70).

Fig. 3.70 Example P.3.83



3.84. Given exponential composite function,

$$f(x) = 1 + e^{-x^2+x} = 1 + e^{x(1-x)}$$

1. *Domain of definition* $\mathbb{D}$: an exponential function is defined for all x; thus, domain of definition $\mathbb{D}$ is $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = 1 + e^{-(-x)^2+(-x)} = 1 + e^{-x^2-x} \neq f(x) \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(1 + e^{-x^2-x}\right) = -1 - e^{-x^2-x} \neq f(x) \quad \text{(not odd)}$$

   In conclusion, this function is neither odd nor even.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad 1 + e^{-x^2+x} > 0$$

   which is to say that this function does not have zeros, and it is always positive.
4. *y-axis crossing point* $f(0)$: $f(0) = 1 + \cancelto{1}{e^{0(1-0)}} = 2$
5. *Limits:* there are two limits in total to resolve.

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(1 + e^{x(1-x)}\right) = 1 + e^{-\infty(+\infty)} = 1 + \cancelto{0}{e^{-\infty}} = 1$$

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(1 + e^{x(1-x)}\right) = 1 + e^{+\infty(-\infty)} = 1 + \cancelto{0}{e^{-\infty}} = 1$$

   Therefore, there is one horizontal asymptote at $y = 1$.
6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1 + e^{x(1-x)}}{x} = \lim_{x\to\infty} \left(\cancelto{0}{\frac{1}{x}} + \cancelto{0}{e^{x(1-x)}} \cdot \cancelto{0}{\frac{1}{x}} \right) = 0$$

   By consequence, there is no oblique asymptote.
7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(1 + e^{-x^2+x}\right)' = (-2x+1)e^{-x^2+x} = 0 \quad \therefore \quad -2x + 1 = 0$$

$$\therefore \quad x_1 = \frac{1}{2}$$

$$f(1/2) = 1 + e^{-(1/2)^2+(1/2)} = 1 + \sqrt[4]{e} \approx 2.284 = y_1$$

   Then,

x		x_1	
$1-2x$	$+$	0	$-$
$\exp(-x^2+x)$	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	y_1	$\searrow$

There is one maximum point $(x, y) = (x_1, y_1)$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left((-2x+1)e^{-x^2+x}\right)' = -2e^{-x^2+x} + (-2x+1)^2 e^{-x^2+x}$$

$$= (4x^2 - 4x - 1)e^{-x^2+x} \quad \therefore \quad P(x) = 4x^2 - 4x - 1 = 0$$

$$\therefore \quad x_{1,2} = \frac{4 \pm \sqrt{32}}{8} \quad \therefore \quad x_1 = \frac{1-\sqrt{2}}{2}, \; x_2 = \frac{1+\sqrt{2}}{2}$$

$$\text{and, } f(x_{1,2}) = 1 + e^{-(x_{2,3})^2 + x_{2,3}} \approx 1.779 = y_{1,2}$$

Then, knowing that $P(x) = (x - x_1)(x - x_2)$,

x		x_1		x_2	
$x - x_2$	$-$	0	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	0	$+$
$\exp(-x^2+x)$	$+$	$+$	$+$	$+$	$+$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$	y_1	$\cap$	y_2	$\cup$

There are two inflection points, one at $(x, y) = (x_1, y_1)$ and the second at $(x, y) = (x_1, y_2)$.

8. *Summary of the important points:*

y-axis crossing point $f(0)$: $(x, y) = (0, 2)$

h. asymptote: $y = 1$

zeros: (none, $f(x)$ is always positive)

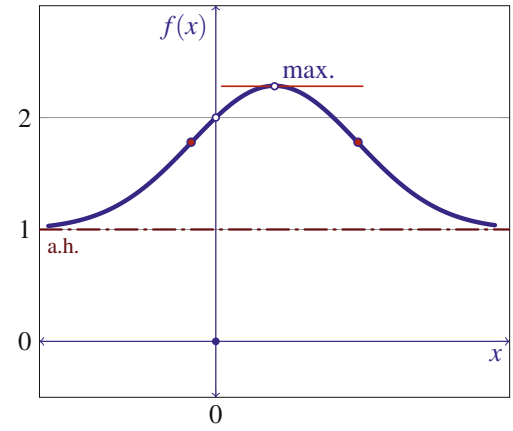
extremes: $(x, y) = \left(1/2, 1 + \sqrt[4]{e}\right)$ (max)

inflection: $(x, y) = \left(\left(1-\sqrt{2}\right)/2, 1.779\right)$

$(x, y) = \left(\left(1+\sqrt{2}\right)/2, 1.779\right)$

9. *Graphical representation:* results of the analysis for function in P.3.84 are summarized by $f(x)$ graph (see Fig. 3.71).

Fig. 3.71 Example P.3.84



3.85. Given exponential/log function,

$$f(x) = e^x \ln(2x+1)$$

1. *Domain of definition* $\mathbb{D}$: exponential function is defined for all x, while logarithmic function is defined for all $x > 0$, that is to say, $2x + 1 > 0 \;\;\therefore\;\; x > -1/2$; thus domain of definition $\mathbb{D}$ is $x \in \mathbb{R} | x > -1/2$.
2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = e^{-x} \ln(2(-x)+1) = \frac{\ln(1-2x)}{e^x} \neq f(x) \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\frac{\ln(1-2x)}{e^x} \neq f(x) \quad \text{(not odd)}$$

   In conclusion, this function is neither odd nor even.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; e^x \ln(2x+1) = 0 \;\; \therefore \;\; 2x+1 = 1 \;\; \therefore \;\; x_1 = 0$$

   Then,

x	$-1/2$		0	
e^x	$+$	$+$	1	$+$
$\ln(2x+1)$	n.d.	$-$	0	$+$
$f(x)$	n.d.	$-$	0	$+$

   There is one first-order zero at $(x, y) = (0, 0)$.
4. *y-axis crossing point* $f(0)$: $f(0) = e^0 \ln(2(0)+1) = 0$
5. *Limits:* there are two limits in total to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^x \ln(2x+1) = +\infty$$

$$x \to -1/2^{-}: \quad \lim_{x\to -1/2^{-}} f(x) = \lim_{x\to -1/2^{-}} e^x \ln(2x+1) = \lim_{x\to -1/2^{-}} \frac{\ln(2x+1)}{\sqrt{e}} \overset{-\infty}{} = -\infty$$

Reminder: Even though on the right side this function tends to $-\infty$, limit does not exist because on the left side, this function is not defined and, by consequence, the two limits are not equal – that is to say, by definition, this limit does not exist.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{e^x \ln(2x+1)}{x} = \left(\frac{\infty}{\infty}\right)$$
$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \left(e^x \ln(2x+1) \overset{\infty}{} + e^x \frac{2}{2x+1} \overset{0}{} \right) = \infty$$

By consequence, there is no oblique asymptote.

7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = \left(e^x \ln(2x+1)\right)' = e^x \ln(2x+1) + e^x \frac{2}{2x+1}$$
$$= e^x \frac{(2x+1)\ln(2x+1)+2}{2x+1} \quad \therefore \quad N(x) = (2x+1)\ln(2x+1)+2 = 0$$

Sign and zero(s) of the last transcendental equation $N(x)$ can be deduced, for example, as follows. Given that $f(x)$ is defined in $\mathbb{D}: x \in \mathbb{R} | x > -1/2$ and it has only one zero at $x_1 = 0$, then for $x > 0$, it must be $N(x) > 0$. In addition, $N(0) = 2 > 0$. What is left to be resolved is the $N(x)$ sign within $(-1/2, 0)$ interval. Again, $(2x+1)$ term is positive; however, $\ln(2x+1)$ is negative. Therefore, $(2x+1)\ \ln(2x+1) < 0$, which must be then compared with the $'+2'$ term.

It is sufficient to compare the minimum of $N(x)$ relative to $+2$. (Note: it is not difficult to verify that indeed its extreme is 'min' not 'max').

$$N'(x) = \Big((2x+1)\ln(2x+1)\Big)' = 2\ln(2x+1) + 2\frac{2x+1}{2x+1} = 0$$
$$\therefore \quad 1 + \ln(2x+1) = 0 \quad \therefore \quad \ln(2x+1) = -1 \quad \therefore \quad 2x+1 = \frac{1}{e}$$
$$\therefore \quad x = \frac{1-e}{2e} \quad \text{and,}$$
$$(2x+1)\ \ln(2x+1) = \left(2\frac{1-e}{2e} + 1\right) \ln\left(2\frac{1-e}{2e} + 1\right)$$
$$= \left(\frac{1}{e} - 1 + 1\right) \ln\left(\frac{1}{e} - 1 + 1\right) = -\frac{1}{e}$$

Therefore

$$2 - \frac{1}{e} > 0 \quad \text{because,} \quad e > 2$$

which leads to conclusion that $(2x+1)\ln(2x+1)+2 > 0$ for all x within domain $\mathbb{D} : x \in \mathbb{R} | x > -1/2$. Consequently, $f(x)$ has no critical points, and it always ascends $\nearrow$, because $f'(x) > 0$.

Also,

$$\begin{aligned} \underline{f''(x) = 0:} \quad f''(x) &= \left(\frac{e^x\,(2x+1)\ln(2x+1) + 2e^x}{2x+1} \right)' \\ &= \left(\frac{2xe^x\,\ln(2x+1) + e^x\,\ln(2x+1) + 2e^x}{2x+1} \right)' \\ &= \cdots = e^x\,\frac{4x^2\,\ln(2x+1) + 4x\,\ln(2x+1) + \ln(2x+1) + 8x}{(2x+1)^2} = 0 \\ \therefore \quad P(x) &= 4x^2\,\ln(2x+1) + 4x\,\ln(2x+1) + \ln(2x+1) + 8x \end{aligned}$$

Sign and zero(s) of the last transcendental equation $P(x)$ can be determined by Newton–Raphson method. However, in this particular case, an intuition and a simple inspection reveals that $P(0) = 0$, which is to say that possible inflection point should be at $x = 0$. Therefore, it is sufficient to verify $f''(x)$ sign for one negative and one positive value. For $x \in (-1/2, 0) \Rightarrow f''(x) < 0$ and therefore $f(x)$ is convex $\cap$, while $x > 0 \Rightarrow f''(x) > 0$ and therefore $f(x)$ is concave $\cup$. In conclusion, there is one inflection point at $(x, y) = (0, 0)$.

8. *Summary of the important points:*

defined:	$x > -1/2$
v. asymptote:	$x = -1/2$
y-axis crossing point $f(0)$:	$(x, y) = (0, 0)$
zero:	$(x, y) = (0, 0)$
extremes:	(none, $f(x)$ always ascends $\nearrow$)
inflection:	$(x, y) = (0, 0)$

9. *Graphical representation:* results of the analysis for function in P.3.85 are summarized by $f(x)$ graph (see Fig. 3.72).

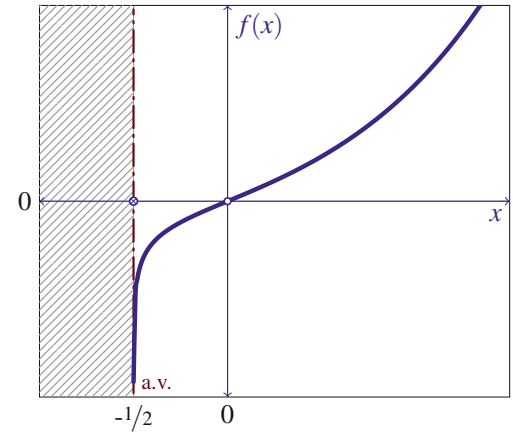
3.86. Given logarithmic function,

$$f(x) = x\ln^2(x^2 + x)$$

1. *Domain of definition* $\mathbb{D}$: logarithmic function is defined for all $x > 0$; thus, domain of definition is limited to

$$x^2 + x > 0 \;\; \therefore \;\; x(x+1) > 0 \;\; \therefore \;\; \begin{cases} x > 0 & \text{and,} \quad x > -1 \;\; \therefore \;\; x > 0 \\ x < 0 & \text{and,} \quad x < -1 \;\; \therefore \;\; x < -1 \end{cases}$$

Fig. 3.72 Example P.3.85



In summary, $\mathbb{D}$ is $x \in \mathbb{R} | x < -1 \quad$ and, $\quad x > 0$. In other words, $x \notin [-1, 0]$ interval.

2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = (-x)\ln^2((-x)^2 + (-x)) = -x\ln^2(x^2 - x) \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -(-x\ln^2(x^2 - x)) = x\ln^2(x^2 - x) \quad \text{(not odd)}$$

In conclusion, this function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \quad x\ln^2(x^2 + x) = 0 \quad \therefore \quad \begin{cases} \underline{x = 0}\,; & f(0) \quad \text{not defined} \\ \underline{x^2 + x = 1}\,; & x^2 + x - 1 = 0 \quad \therefore \\ & x_{1,2} = \dfrac{-1 - \sqrt{5}}{2} = x_{01} \\ & x_{3,4} = \dfrac{-1 + \sqrt{5}}{2} = x_{02} \end{cases}$$

Note: both zeros are double (even) because there are two logarithms in the "$\ln(\cdot)\ln(\cdot)$" product. Then,

x		x_{01}		-1		0		x_{02}	
x	$-$	$-$	$-$	-1	n.d.	0	$+$	$+$	$+$
$\ln^2(x^2 + x)$	$+$	0	$+$	n.d.	n.d.	n.d.	$+$	0	$+$
$f(x)$	$-$	0	$-$	n.d.	n.d.	n.d.	$+$	0	$+$

There are two second-order zeros, one at $(x, y) = (x_{01}, 0)$ and one at $(x, y) = (x_{02}, 0)$

4. *y-axis crossing point* $f(0)$: $f(0) = (0)\ln^2\left((0)^2 + (0)\right)$ (not defined)
5. *Limits:* there are four limits in total to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\ln^2(x^2 + x) = (+\infty)(+\infty) = +\infty$$

$$x \to -\infty: \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x\ln^2(x^2+x) = (-\infty)(+\infty) = -\infty$$

$$x \underset{\twoheadleftarrow}{\to} 0: \quad \lim_{x\to 0\atop\twoheadleftarrow} f(x) = \lim_{x\to 0\atop\twoheadleftarrow} x\ln^2(x^2+x) = (0\cdot\infty) = \lim_{x\to 0\atop\twoheadleftarrow} \frac{\ln^2(x^2+x)}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x\to 0\atop\twoheadleftarrow} \frac{\frac{2(2x+1)\ln(x^2+x)}{x(x+1)}}{-\frac{1}{x^2}} = -2\lim_{x\to 0\atop\twoheadleftarrow} \frac{x(2x+1)\ln(x^2+x)}{x+1} = (0\cdot\infty)$$

$$= -2\lim_{x\to 0\atop\twoheadleftarrow} \frac{\ln(x^2+x)}{\frac{x+1}{2x^2+x}} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} 2\lim_{x\to 0\atop\twoheadleftarrow} \frac{\frac{2x+1}{x(x+1)}}{\frac{2x^2+4x+1}{x^2(2x+1)^2}}$$

$$= 2\lim_{x\to 0\atop\twoheadleftarrow} \frac{x(2x+1)^3}{(x+1)(2x^2+4x+1)} = \frac{0\downarrow\ (1)^3}{(1)(1)} = 0\downarrow$$

$$x \underset{\twoheadrightarrow}{\to} -1: \quad \lim_{x\to -1\atop\twoheadrightarrow} f(x) = \lim_{x\to -1\atop\twoheadrightarrow} x\ln^2(x^2+x) = (-1)(+\infty) = -\infty$$

Both of the last two limits are one-sided; therefore, by definition, limits do not exist in $x = -1$ and $x = 0$ points.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x\ln^2(x^2+x)}{x} = \lim_{x\to\infty} \ln^2(x^2+x) = \infty$$

By consequence, there is no oblique asymptote.

7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = \left(x\ln^2(x^2+x)\right)' = \ln^2(x^2+x) + x\,\frac{2(2x+1)\ln(x^2+x)}{x^2+x}$$

$$= \frac{(x^2+x)\ln^2(x^2+x) + 2x(2x+1)\ln(x^2+x)}{x(x+1)}$$

$$= \ln(x^2+x)\ \frac{(x+1)\ln(x^2+x) + 2(2x+1)}{x+1} = 0$$

$$\therefore\ x^2+x = 1 \quad \text{or,} \quad (x+1)\ln(x^2+x) + 2(2x+1) = 0$$

where equation $x^2 + x = 1$ was already used to deliver zeros $x_{01,02}$, and the remaining transcendental equation may be solved as follows. Function $f(x)$ is positive within (0, 1) interval, its zero is at $x = 1$, and it tends to zero for $x \to 0$. Therefore, there must be one extreme (max.) point within this interval. By taking, for example, the initial value $x_0 = 0.5$ Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$$

where,

$$N(x_0) = (x+1)\ln(x^2+x) + 2(2x+1)$$

$$N'(x_0) = \ln(x^2+x) + \cancel{(x+1)}\frac{2x+1}{x\cancel{(x+1)}} + 4 = \frac{x\ln(x^2+x)+6x+1}{x}$$

results in

n	x_0	$f(x_0)$	$f'(x_0)$
0	0.500	3.568	7.712
1	0.037	−1.224	29.556
2	0.079	−0.345	16.237
3	0.100	−0.028	13.794
4	0.102	-1.831×10^{-4}	13.616
5	0.102	-7.718×10^{-9}	

There are three first-order zeros of $f'(x)$: $x_1 = (-1-\sqrt{5})/(2)$, $x_2 \approx 0.102$, $x_3 = (-1+\sqrt{5})/(2)$ where x_3 is associated with $N(x)$ function, so that $f(x_3) \approx 0.487 = y_3$, and

x		x_1		0		x_2		x_3	
$\ln(x^2+x)$	+	0	−	n.d.	−	−	−	0	+
$N(x)$	−	−	−	n.d.	−	0	+	+	+
$x+1$	−	−	−	1	+	+	+	+	+
$f'(x)$	+	0	−	n.d.	+	0	−	0	+
$f(x)$	↗	0	↘	n.d.	↗	y_2	↘	0	↗

There are two maximum points at $(x, y) = (x_1, 0)$ and $(x, y) = (x_2, y_2)$, as well as one minimum point at $(x, y) = (x_3, 0)$ Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\ln^2(x^2+x) + \frac{2(2x+1)\ln(x^2+x)}{x+1}\right)'$$

$$= 2\frac{(2x+1)\ln(x^2+x)}{x(x+1)} + 2\frac{4x(x+1)+1+x\ln(x^2+x)}{x(x+1)^2}$$

$$= 2\frac{(2x+1)(x+1)\ln(x^2+x)+4x(x+1)+1+x\ln(x^2+x)}{x(x+1)^2}$$

$$= 2\frac{(2x^2+4x+1)\ln(x^2+x)+(2x+1)^2}{x(x+1)^2}$$

That is to say,

$$P(x) = (2x^2+4x+1)\ln(x^2+x) + (2x+1)^2 = 0$$

By inspection of $f(x)$ and $f'(x)$ signs, it can be deduced that there cannot be change of sign of $f''(x)$ for $x < -1$; that is because there is only one extreme in that interval. On the other hand, for $x > 0$, function is continuous, and it has one maximum followed by one minimum point, which is possible if there is one inflection point. Thus, it is sufficient to search $P(x) = 0$ only in $x > 0$ interval, for example, $x_0 = 0.5$, so that Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$$

where,

$$P(x_0) = (2x^2 + 4x + 1)\ln(x^2 + x) + (2x + 1)^2$$

$$P'(x_0) = 4(x + 1)\ln(x^2 + x) + \frac{(2x^2 + 4x + 1)(2x + 1)}{x^2 + x} + 4(2x + 1)$$

results in

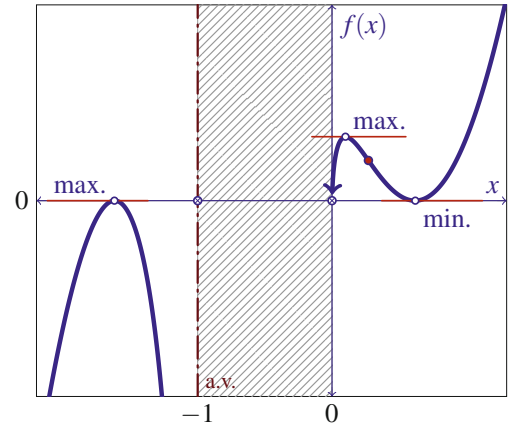
n	x_0	$P(x_0)$	$P'(x_0)$
0	0.500	2.993	15.607
1	0.308	0.412	11.426
2	0.272	0.012	10.757
3	0.271	1.142×10^{-5}	10.737
4	0.271	9.900×10^{-12}	10.737

and $f(0.271) \approx 0.308$, that is to say, there is one inflection point at $(x_4, y_4) \approx (0.271, 0.308)$. Then, $f''(x) < 0$ (i.e., $\cap$) for $x < x_4$, and $f''(x) > 0$ (i.e., $\cup$) for $x > x_4$

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} \quad & x < -1 \quad \text{and,} \quad x > 0 \\
\underline{\text{v. asymptote:}} \quad & x = -1 \\
\underline{\text{zeros:}} \quad & (x, y) = \left(\frac{-1 - \sqrt{5}}{2}, 0\right) \quad \text{(double)} \\
& (x, y) = \left(\frac{-1 + \sqrt{5}}{2}, 0\right) \quad \text{(double)} \\
\underline{\text{extremes:}} \quad & (x, y) = \left(\frac{-1 - \sqrt{5}}{2}, 0\right) \\
& (x, y) = \left(\frac{-1 + \sqrt{5}}{2}, 0\right) \\
& (x, y) = (0.102, 0) \\
\underline{\text{inflection:}} \quad & (x, y) = (0.271, 0.308)
\end{aligned}$$

Fig. 3.73 Example P.3.86



9. *Graphical representation:* results of the analysis for function in P.3.86 are summarized by $f(x)$ graph (see Fig. 3.73).

3.87. Given an arbitrary base logarithmic function

$$f(x) = \log_{(x/2)}(x) = \left\{ \log_a b = \frac{\ln b}{\ln a} \right\} = \frac{\ln(x)}{\ln\left(\frac{x}{2}\right)}$$

1. *Domain of definition* $\mathbb{D}$: both logarithmic functions $\ln(x)$ and $\ln(x/2)$ are defined for all $x > 0$. At the same time, $\ln(x/2)$ is denominator of $f(x)$, therefore $\ln(x/2) \neq 0$, i.e., $x/2 \neq 1$, i.e., $x \neq 2$. That is to say there is vertical asymptote at $x = 2$.
   In summary, $\mathbb{D} : x \in \mathbb{R} | x \neq 2, x > 0$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \log_{(-x/2)}(-x) \neq f(x) \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\log_{(-x/2)}(-x) \neq f(x) \quad \text{(not odd)}$$

   In conclusion, this function is neither odd nor even.
3. *Function's sign:* numerator and denominator signs are determined separately, as

x	0		1		2	
$\ln(x)$	n.d.	−	0	+	+	+
$\ln(x/2)$	n.d.	−	−	−	0	+
$f(x)$	n.d.	+	0	−	n.d.	+

   There is one first-order zero at $(x, y) = (1, 0)$. In addition, this function changes its sign around the vertical asymptote $x = 2$.
4. *y-axis crossing point* $f(0)$:

$$f(0) = \log_{(0/2)}(0) \quad \therefore \quad \text{(not defined)}$$

5. *Limits:* there are four limits in total to resolve.

$$\underline{x \to \underset{\leftarrow}{0}:} \quad \lim_{x\to \underset{\leftarrow}{0}} f(x) = \lim_{x\to \underset{\leftarrow}{0}} \frac{\ln(x)}{\ln(x/2)} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to \underset{\leftarrow}{0}} \frac{\frac{1}{x}}{\frac{1/2}{x/2}} = \lim_{x\to \underset{\leftarrow}{0}} 1 = 1$$

$$\underline{x \to \underset{\rightarrow}{2}:} \quad \lim_{x\to \underset{\rightarrow}{2}} f(x) = \lim_{x\to \underset{\rightarrow}{2}} \frac{\ln(x)}{\ln(x/2)} = \frac{\ln(2)}{\lim_{x\to \underset{\rightarrow}{2}} \ln(x/2)} = \frac{\ln(2)}{0} = -\infty$$

$$\underline{x \to \underset{\leftarrow}{2}:} \quad \lim_{x\to \underset{\leftarrow}{2}} f(x) = \lim_{x\to \underset{\leftarrow}{2}} \frac{\ln(x)}{\ln(x/2)} = \frac{\ln(2)}{\lim_{x\to \underset{\leftarrow}{2}} \ln(x/2)} = \frac{\ln(2)}{0} = +\infty$$

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln(x)}{\ln(x/2)} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1/2}{x/2}} = \lim_{x\to+\infty} 1 = 1$$

Even though on the positive side this function tends to one, limit does not exist because on the negative side, this function is not defined, and by consequence, the negative side limit does not exist either. For the same reason, the horizontal asymptote $y = 1$ exists only on the positive side, i.e., $x >$, while it does not exist for $x \leq 0$.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln(x)}{x \ln\left(\frac{x}{2}\right)} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{1 + \ln(x/2)} = \lim_{x\to\infty} \frac{1}{x(1 + \ln(x/2))}$$

$$= 0$$

By consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{\ln(x)}{\ln(x/2)}\right)' = \frac{(1/x)\ln(x/2) - \ln(x)\frac{1/2}{x/2}}{\left(\ln(x/2)\right)^2} = \frac{\ln(x/2) - \ln(x)}{x\left(\ln(x/2)\right)^2} = 0$$

$$\therefore \quad \text{however, } \ln(x/2) - \ln(x) \neq 0 \quad \text{because,} \quad \ln(x) > \ln(x/2), \quad \forall x > 0$$

$$\therefore \quad \ln(x/2) - \ln(x) < 0, \quad \forall x > 0$$

Then,

x	0		1		2	
$\ln(x/2) - \ln(x)$	n.d.	$-$	$-$	$-$	$-$	$-$
x	0	$+$	$+$	$+$	$+$	$+$
$(\ln(x/2))^2$	n.d.	$+$	$+$	$+$	0	$+$
$f'(x)$	n.d.	$-$	$-$	$-$	n.d.	$-$
$f(x)$	n.d.	$\searrow$	0	$\searrow$	n.d.	$\searrow$

There are no extreme points and $f(x)$ always descends $\searrow$. Also,

$$f''(x) = 0: \quad f''(x) = \left(\frac{\ln(x/2) - \ln(x)}{x \left(\ln(x/2)\right)^2} \right)'$$

$$= \frac{-\left(\ln(x/2) - \ln(x)\right)\left(\left(\ln(x/2)\right)^2 + 2\cancel{x}\ln(x/2)\frac{1/\cancel{2}}{\cancel{x/2}}\right)}{\left(x\left(\ln(x/2)\right)^2\right)^2}$$

$$= \frac{\left(\ln(x) - \ln(x/2)\right)\left(\ln(x/2) + 2\right)}{x^2\left(\ln(x/2)\right)^3} = 0$$

$$\therefore \quad \ln(x/2) + 2 = 0 \quad \therefore \quad \ln(x/2) = -2 \quad \therefore \quad \frac{x}{2} = \frac{1}{e^2}$$

$$\therefore \quad x_2 = \frac{2}{e^2}$$

because $\ln(x) - \ln(x/2) > 0, \forall x > 0$. In addition, $f(x_2)$ is found as

$$f(x_2) = \frac{\ln\left(2/e^2\right)}{\ln\left(\frac{\cancel{2}\,1/e^2}{\cancel{2}}\right)} = \left\{ \ln e = 1; \quad \log\frac{a}{b} = \log a - \log b; \quad \log a^b = b\log a \right\}$$

$$= \frac{\ln 2 - \ln e^2}{\cancel{\ln 1}^{\,0} - \ln e^2} = \frac{2 - \ln 2}{2} = y_2$$

Then,

x	0		x_2		2	
$\ln(x/2) + 2$	n.d.	$-$	0	$+$	$+$	$+$
$\ln(x) - \ln(x/2)$	n.d.	$+$	$+$	$+$	$+$	$+$
x^2	0	$+$	$+$	$+$	$+$	$+$
$(\ln(x/2))^3$	n.d.	$-$	$-$	$-$	0	$+$
$f''(x)$	n.d.	$+$	0	$-$	n.d.	$+$
$f(x)$	n.d.	$\cup$	y_2	$\cap$	n.d.	$\cup$

There is one first-order inflection point at $(x, y) = (x_2, y_2)$.

8. *Summary of the important points:*

$$\text{defined:} \quad x \in \mathbb{R} | x \neq 2, x > 0$$

$$\text{h. asymptote:} \quad y = 1 \quad (x > 0)$$

$$\text{v. asymptote:} \quad x = 2$$

$$\begin{aligned} \underline{\text{zeros:}} \quad & (x, y) = (1, 0) \\ \underline{\text{extremes :}} \quad & \text{(none) } f(x) \text{ always descends } \searrow \\ \underline{\text{inflection:}} \quad & (x, y) = \left(\frac{2}{e^2}, \frac{2 - \ln 2}{2} \right) \end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.87 are summarized by $f(x)$ graph (see Fig. 3.74).

3.88. Given logarithmic function, done as follows.

$$f(x) = x \ln^2 x$$

1. *Domain of definition* $\mathbb{D}$: logarithmic function is defined for all $x > 0$; thus domain of definition $\mathbb{D}$ is $x \in \mathbb{R} | x > 0$.
2. *Function's parity:*

$$\begin{aligned} \underline{\text{even:}} \quad & f(-x) = (-x) \ln^2(-x) \quad \text{(not defined)} \\ \underline{\text{odd:}} \quad & -f(-x) = -(-x) \ln^2(-x) \quad \text{(not defined)} \end{aligned}$$

   In conclusion, this function is neither odd nor even.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \; x \ln^2 x = 0 \quad \therefore \; x = 0 \notin \mathbb{D}, \quad \text{and,} \quad x_{1,2} = 1$$

   Then,

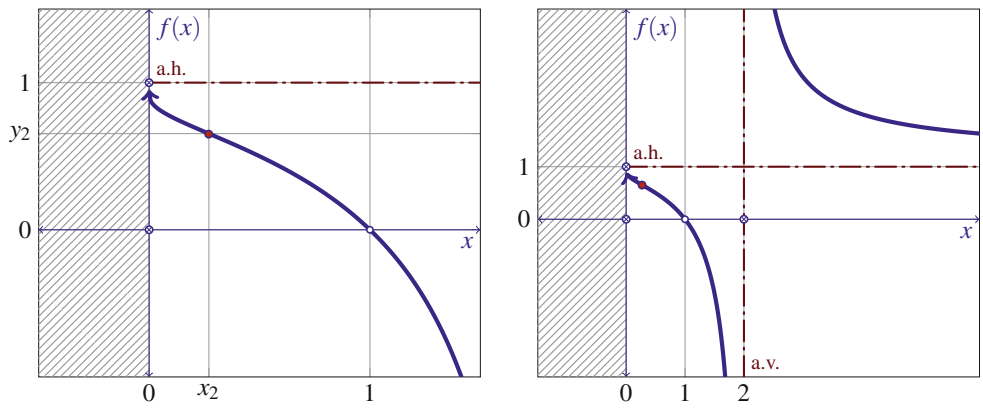


Fig. 3.74 Example P.3.87

x	0		1	
x	0	+	1	+
$\ln^2 x$	n.d.	+	0	+
$f(x)$	n.d.	+	0	+

There is one second-order zero at $(x, y) = (1, 0)$.

4. *y-axis crossing point* $f(0)$: $f(0) = (0)\ln^2(0)$ (not defined)
5. *Limits:* there are two limits in total to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\ln^2 x = +\infty$$

$$\underline{x \to \underset{\leftarrow}{0}:} \quad \lim_{x\to\underset{\leftarrow}{0}} f(x) = \lim_{x\to\underset{\leftarrow}{0}} x\ln^2 x = (0\cdot\infty) = \lim_{x\to\underset{\leftarrow}{0}} \frac{\ln^2 x}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x\to\underset{\leftarrow}{0}} \frac{\frac{2\ln x}{x}}{-\frac{1}{x^2}} = \lim_{x\to\underset{\leftarrow}{0}} -2x\ln x = (0\cdot\infty) = -2\lim_{x\to\underset{\leftarrow}{0}} \frac{\ln x}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} -2\lim_{x\to\underset{\leftarrow}{0}} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -2\lim_{x\to\underset{\leftarrow}{0}} (-x) = 0$$

Even though $x \to \underset{\leftarrow}{0}$ function tends to zero, limit does not exist because function is not defined for $x \leq 0$, and by definition, this limit does not exist.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x\ln^2 x}{x} = \lim_{x\to\infty} \ln^2 x = \infty$$

By consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(x\ln^2 x\right)' = \ln^2 x + \cancel{x}\frac{2\ln x}{\cancel{x}} = (2 + \ln x)\ln x = 0$$

$$\therefore \quad 2 + \ln x = 0 \quad \text{or,} \quad \ln x = 0$$

$$\therefore \quad \begin{cases} \ln x & = -2 \quad \therefore \quad x_1 = e^{-2} = \dfrac{1}{e^2} \\ \ln x & = 0 \quad \therefore \quad x_2 = 1 \end{cases}$$

$$\text{and, } f(x_1) = \frac{1}{e^2}\left(\ln\frac{1}{e^2}\right)^2 = \frac{1}{e^2}(-2)^2 = \frac{4}{e^2} = y_1$$

$$f(x_2) = 1\ln^2 1 = 0 = y_2$$

Then,

x	0		x_1		x_2	
$2+\ln x$	n.d.	$-$	0	$+$	$+$	$+$
$\ln x$	n.d.	$-$	$-$	$-$	0	$+$
$f'(x)$	n.d.	$+$	0	$-$	0	$+$
$f(x)$	n.d.	$\nearrow$	y_1	$\searrow$	y_2	$\nearrow$

There is one maximum point at $(x, y) = (1/e^2, 4/e^2)$ and one minimum point at $(x, y) = (1, 0)$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\ln^2 x + 2\ln x\right)' = 2\,\frac{1+\ln x}{x} = 0$$

$$\therefore\ 1+\ln x = 0\ \ \therefore\ \ln x = -1\ \ \therefore\ x_1 = \frac{1}{e}$$

$$\text{and, } f(1/e) = \frac{1}{e}\left(\ln\frac{1}{e}\right)^2 = \frac{1}{e}(-1)^2 = \frac{1}{e} = y_1$$

Then,

x	0		x_1	
2	$+$	$+$	$+$	$+$
$1+\ln x$	n.d.	$-$	0	$+$
x	0	$+$	$+$	$+$
$f''(x)$	n.d.	$-$	0	$+$
$f(x)$	n.d.	$\cap$	y_1	$\cup$

There is one inflection point at $(x, y) = (1/e, 1/e)$.

8. *Summary of the important points:*

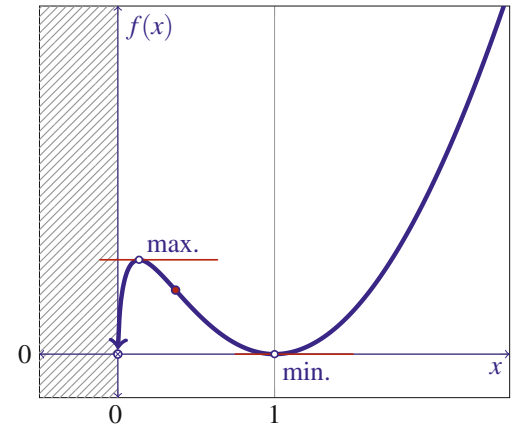
$$\begin{aligned}
\underline{\text{defined:}} &\quad x > 0 \\
\underline{\text{zero:}} &\quad (x, y) = (1, 0) \quad \text{(double)} \\
\underline{\text{extremes:}} &\quad (x, y) = \left(1/e^2, 4/e^2\right) \quad \text{(max)} \\
&\quad (x, y) = (1, 0) \quad \text{(min)} \\
\underline{\text{inflection:}} &\quad (x, y) = (1/e, 1/e)
\end{aligned}$$

9. *Graphical representation:* results of the analysis for function in P.3.88 are summarized by $f(x)$ graph (see Fig. 3.75).

3.89. Given composite logarithmic function,

$$f(x) = \frac{1}{x} + \ln(x^2 + 2x + 1) = \frac{1}{x} + \ln\left((x+1)^2\right)$$

Fig. 3.75 Example P.3.88



1. *Domain of definition* $\mathbb{D}$: denominator of a rational function must not equal zero; thus $x = 0$ is location of first vertical asymptote. Logarithmic function is defined for $x > 0$ and $x \neq 0$, that is to say, $x + 1 \neq 0 \;\; \therefore \;\; x \neq -1$, which is location of second vertical asymptote.
   In summary, domain of definition $\mathbb{D}$ is $x \in \mathbb{R} | x \neq 0, x \neq -1$.
2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = \frac{1}{(-x)} + \ln((-x)^2 + 2(-x) + 1) = -\frac{1}{x} + \ln(x^2 - 2x + 1) \neq f(x)$$

$$\underline{\text{odd:}} \quad -f(-x) = \frac{1}{x} - \ln(x^2 - 2x + 1) \neq f(x)$$

   In conclusion, this function is neither odd nor even.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \therefore \;\; \frac{1}{x} + \ln(x^2 + 2x + 1) = 0 \;\; \therefore \;\; \ln\left((x+1)^2\right) = -\frac{1}{x}$$

   where, it should be noted that

$$\ln\left((x+1)^2\right) = 0 \;\; \therefore \;\; (x+1)^2 = 1 \;\; \therefore \;\; x_1 = 0, x_2 = -2$$

   and, the $f(x) = 0$ transcendental equation may be solved by Newton–Raphson method as

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$$

   where,

$$f(x_0) = \frac{1}{x} + \ln(x^2 + 2x + 1)$$

$$f'(x_0) = \left(\frac{1}{x} + \ln(x_0^2 + 2x_0 + 1)\right)' = \frac{2x_0^2 - x_0 - 1}{x_0^2(x_0 + 1)}$$

By inspection of Fig. 3.76 (left), and by knowing the properties of log and inverse functions, obviously $x_0 < -2$. Taking, for example, the initial value $x_0 = -3$ results in

n	x_0	$f(x_0)$	$f'(x_0)$
0	−3.000	1.053	−1.111
1	−2.052	−0.385	−2.138
2	−2.233	−0.030	−1.799
3	−2.249	−0.000	−1.799
4	−2.249		

with this initial guess $x_0 = -2.249$, in only three iterations, the third decimal place is fixed. To illustrate this result,

$$f(-2.249) = \frac{1}{x} + \ln(x^2 + 2x + 1) \approx -8.84348 \times 10^{-9}$$

x		x_0		-1		0	
$1/x$	−	−	−	−	−	n.d.	+
$\ln[(x+1)^2]$	+	+	−	n.d.	−	0	+
$f(x)$	+	0	−	n.d.	−	n.d.	+

There is one first-order zero at $(x, y) = (x_0, 0)$. In this example, it is necessary to compare which of the two terms in $f(x) = 0$, i.e., $1/x$ or $\ln(\cdot)$, is greater because it is their sum (not product) that determines the sign of $f(x)$. A rough sketch as in Fig. 3.76 (left) helps visualize this relationship and subsequently determine the sign of $f(x)$ in each of the given intervals. Note that Fig. 3.76 (left) shows the relation $\ln\big((x+1)^2\big) \lessgtr -1/x$.

4. *y-axis crossing point* $f(0)$: $f(0) = (1/0) + \ln(0^2 + (0)x + 1)$ (not defined)
5. *Limits:* there are six limits in total to resolve.

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{1}{x} + \ln\left((x+1)^2\right)\right) = \lim_{x\to-\infty}\left(\cancel{(1/x)}^{\to 0} + \ln\left((x+1)^2\right)\right)$$

$$= \lim_{x\to-\infty}\left(\ln\left(+\infty\right)\right) = +\infty$$

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{1}{x} + \ln\left((x+1)^2\right)\right) = \lim_{x\to+\infty}\left(\cancel{(1/x)}^{\to 0} + \ln\left((x+1)^2\right)\right)$$

$$= \lim_{x\to+\infty}\left(\ln\left(+\infty\right)\right) = +\infty$$

$$\underline{\underline{x \to -1:}} \quad \lim_{x\to-1_{\to}} f(x) = \lim_{x\to-1_{\to}}\left(\cancel{(1/x)}^{\to -1} + \ln\left((x+1)^2\right)\right)$$

$$= \lim_{x \to -1^{+}} \left(\ln\left(0\right) \right) = -\infty$$

$$x \to -1^{-}: \quad \lim_{x \to -1^{-}} f(x) = \lim_{x \to -1^{-}} \left(\cancel{(1/x)^{-1}} + \ln\left((x+1)^2\right) \right)$$

$$= \lim_{x \to -1^{-}} \left(\ln\left(0\right) \right) = -\infty$$

$$x \to 0^{+}: \quad \lim_{x \to 0^{+}} f(x) = \lim_{x \to 0^{+}} \left(\frac{1}{x} + \ln\left(\cancel{(x+1)^2}^{\,0}\right) \right) = -\infty$$

$$x \to 0^{-}: \quad \lim_{x \to 0^{-}} f(x) = \lim_{x \to 0^{-}} \left(\frac{1}{x} + \ln\left(\cancel{(x+1)^2}^{\,0}\right) \right) = +\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{x} + \ln\left((x+1)^2\right)}{x} = \lim_{x \to \infty} \left(\frac{1}{x^2}^{\,0} + \frac{\ln\left((x+1)^2\right)}{x} \right) = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{2}{x+1} = 0$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = \left(\frac{1}{x} + \ln\left((x+1)^2\right) \right)' = \frac{2x^2 - x - 1}{x^2(x+1)} = 0 \;\; \therefore \;\; 2x^2 - x - 1 = 0$$

$$2x^2 - 2x + x - 1 = 0 \;\; \therefore \;\; 2x(x-1) + x - 1 = 0 \;\; \therefore \;\; (x-1)(2x+1) = 0$$

$$\therefore \;\; x_1 = -\frac{1}{2}, \; x_2 = 1 \quad \text{so that,}$$

$$f(1) = 1 + \ln 4, \quad \text{and,} \quad f(-1/2) = -2 + \ln \frac{1}{4}$$

Then,

x		-1		x_1		0		x_2	
$x - x_1$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
x^2	$+$	1	$+$	$+$	$+$	0	$+$	$+$	$+$
$x + 1$	$-$	0	$+$	$+$	$+$	1	$+$	$+$	$+$
$f'(x)$	$-$	n.d.	$+$	0	$-$	n.d.	$-$	0	$+$
$f(x)$	$\searrow$	n.d.	$\nearrow$	y_1	$\searrow$	n.d.	$\searrow$	y_2	$\nearrow$

There is one maximum point at $(x, y) = (x_1, y_1)$ and one minimum point at $(x, y) = (x_2, y_2)$ Also,

$$f''(x) = 0: \quad f''(x) = \left(\frac{2x^2 - x - 1}{x^2(x+1)}\right)' = -2\,\frac{x^3 - x^2 - 2x - 1}{x^3(x+1)^2}$$
$$\therefore \quad P(x) = x^3 - x^2 - 2x - 1 = 0$$

This third-order polynomial must have at least one real zero. Location of that zero may be narrowed to an interval by applying the intermediate value theorem, for example,

$$\begin{aligned}
&\vdots \\
x_0 = -1 &: P(x_0) = -1 \\
x_0 = \;\; 0 &: P(x_0) = -1 \\
x_0 = \;\; 1 &: P(x_0) = -3 \\
x_0 = \;\; 2 &: P(x_0) = -1 \\
x_0 = \;\; 3 &: P(x_0) = 11 \\
&\vdots
\end{aligned}$$

which is to say that one zero must be within (2, 3) interval where this polynomial (note: polynomial is a continuous function) changed its sign. Taking, for example, the initial value $x_0 = 2$ Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$$

results in

n	x_0	$f(x_0)$	$f'(x_0)$
0	2.000	-1	6.000
1	2.167	0.002	7.750
2	2.148	3.378×10^{-7}	7.545
3	2.148		

with this initial guess, $x_1 = 2.148$ is fixed to the third decimal place in only two iterations, as

$$f''(2.148) = -2\,\frac{x^3 - x^2 - 2x - 1}{x^3(x-1)^2} \approx 3.378 \times 10^{-7}$$
$$f(x_1) = f(2.148) = \frac{1}{2.148} + \ln\left((2.148+1)^2\right) \approx 2.759 = y_1$$

Then, as $P(x) = (x^2 + ax + b)(x - x_3)$ and $x^2 + ax + b > 0$ (check by long division, approx.)

x		-1		0		x_1	
-2	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$x - x_3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x^2 + ax + b$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
x^3	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x+1)^2$	$+$	0	$+$	$+$	$+$	$+$	$+$
$f''(x)$	$-$	n.d.	$-$	n.d.	$+$	0	$-$
$f(x)$	$\cap$	n.d.	$\cap$	n.d.	$\cup$	y_1	$\cap$

There is one inflection point at $(x, y) = (x_1, y_1)$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}} \quad & x \neq 0, x \neq -1 \\
\underline{\text{v. asymptote:}} \quad & x = 0, x = -1 \\
\underline{\text{zeros:}} \quad & (x, y) = (-2.249, 0) \\
\underline{\text{extremes:}} \quad & (x, y) = (-1/2, -2 + \ln(1/4)) \quad \text{(max)} \\
& (x, y) = (1, 1 + \ln(4)) \quad \text{(min)} \\
\underline{\text{inflection:}} \quad & (x, y) = (2.148, 2.759)
\end{aligned}$$

9. *Graphical representation:* see Fig. 3.76 graphical method to estimate x_0 (left) and complete function graph (right).

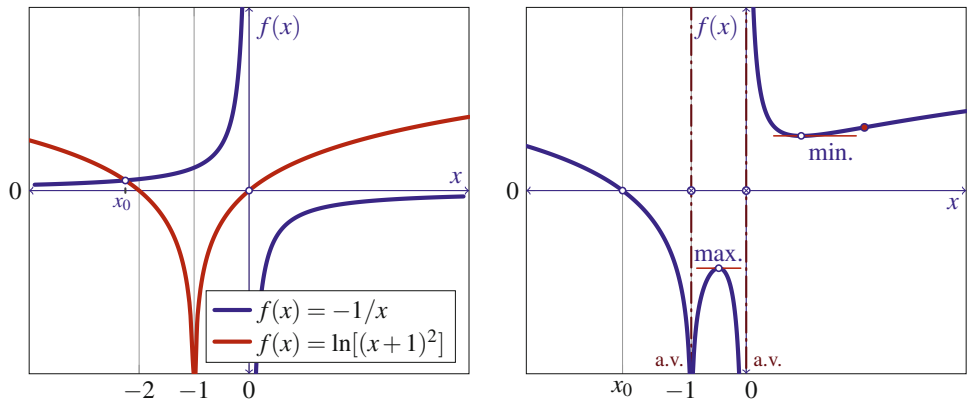


Fig. 3.76 Example P.3.89

3.5 Trigonometric Functions

3.90. Given trigonometric function,

$$f(x) = \sin\left(x + \frac{\pi}{2}\right) = \cos x$$

as per well-known basic relationship between $\sin x$ and $\cos x$ functions (if in doubt, visualize these two functions with the help of the unit circle: their rotating vectors are orthonormal, i.e., they always create $\pi/2$ angle).

1. *Domain of definition* $\mathbb{D}$: $\cos x$ function is defined for $x \in \mathbb{R}$.
2. *Periodicity:* by definition, $\cos(x + n\,2\pi) = \cos x$, $n = \pm 1, \pm 2, \ldots$. That is to say, the *shortest* period of this function is $T = 2\pi$. However, any other multiple of 2π, for example, 4π, 6π etc. also defines repetitive pattern thus can be declared as one period.
3. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \cos(-x) = \cos(x) = f(x) \quad \therefore \quad \text{(even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\Big(\cos(x)\Big) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

   In conclusion, function $\cos x$ is even but not odd.
4. *Function's sign:*

$$\underline{f(x) = 0:} \quad \cos x = 0 \quad \therefore \quad x = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

   Due to its periodicity, zeros of $\cos x$, starting with $\pi/2$, are evenly distributed at π intervals in both directions.

   As this function is even, so it is sufficient to look only at the positive side (the negative side is mirrored) as

x	0		$\pi/2$		$3\pi/2$		$5\pi/2$	$\cdots$
$\cos x$	1	$+$	0	$-$	0	$+$	0	$\cdots$
$f(x)$	1	$+$	0	$-$	0	$+$	0	$\cdots$

5. *Function's range:* amplitude of trigonometric functions $\sin x$ and $\cos x$ is limited to the range $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$. In other words, amplitudes of these two functions are $|\sin x| = 1$ and $|\cos x| = 1$.
6. *y-axis crossing point* $f(0)$: $f(0) = \cos 0 = 1$
7. *Limits:* limits $\lim_{\pm\infty} \sin x$ (as well as $\lim_{\pm\infty} \cos x$) are not defined; however, it is always contained within $[-1, 1]$ interval.
8. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = \left(\frac{\text{n.d.}}{\infty}\right)$$

This ambiguous form may be resolved by the help of the "the squeeze theorem" (aka the "sandwich theorem"), which in a not-too-rigorous interpretation may be used as follows.

Given that $x \to \infty$, denominator "x" never equals to zero; thus, there is no danger of division by zero. On the other hand, module of the numerator is $|\cos x| \leq 1$ (i.e., including $|\cos x| = 0$). Therefore, there are two possible extreme situations:

$$\frac{0}{x} \leq \lim_{x\to\infty} \frac{|\cos x|}{x} \leq \frac{1}{x}$$

(Note the form of this double inequality: left side, middle, right side – like a sandwich). As the left-side limit tends to the same value as the right side limit, then the middle limit is "pressed" between the two, and it must converge to the same value; otherwise, it is not converging. That is to say, at the two extreme instances:

$$\underline{|\cos x| \to 1:} \quad \lim_{x\to\infty} \frac{f(x)}{x} \leq \lim_{x\to\infty} \frac{|\cos x|}{x} \leq \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\underline{|\cos x| \to 0:} \quad \therefore \ (x \neq 0) \ \therefore \ x \to n\pi \ \ (n = \pm 1, \pm 2, \ldots)$$

$$\therefore \ \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{|\cos x|}{x} = \left(\frac{0}{n\pi}\right) = 0$$

Therefore, because the left-side limit equals to the right-side limit, it can be concluded that

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = 0$$

which is to say, there is no oblique asymptote.

9. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = (\cos x)' = -\sin x = 0 \ \therefore \ x = n\pi, \ \ n = 0, \pm 1, \pm 2, \ldots$$

Thus, (over one period)

x	0		π		2π	$\cdots$
-1	$-$	$-$	$-$	$-$	$-$	$\cdots$
$\sin x$	0	$+$	0	$-$	0	$\cdots$
$f'(x)$	0	$-$	$-$	$+$	0	$\cdots$
$f(x)$	1	$\searrow$	-1	$\nearrow$	1	$\cdots$

In conclusion, both extreme points (min and max) are periodic with the period $T = 2\pi$ as well. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = (-\sin x)' = -\cos x = 0 \ \therefore \ x = \frac{\pi}{2} + n\pi, \ \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Thus, (over one period)

x	0		$\pi/2$		$3\pi/2$	$\cdots$
-1	$-$	$-$	$-$	$-$	$-$	$\cdots$
$\cos x$	1	$+$	0	$-$	0	$\cdots$
$f''(x)$	-1	$-$	0	$+$	0	$\cdots$
$f(x)$	1	$\cap$	0	$\cup$	-1	$\cdots$

therefore, concave/convex intervals are also periodic with the period 2π, and they overlap with the function's zeros.

10. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{periodicity:}} \quad & T = \pi \\
\underline{\text{parity:}} \quad & \text{(even)} \\
\underline{\text{y–axis crossing:}} \quad & (x, y) = (0, 1) \\
\underline{\text{zeros:}} \quad & (x, y) = \left(\frac{\pi}{2} \pm n\pi, 0\right) \\
\underline{\text{extremes:}} \quad & (x, y) = (0, 1) \quad \text{(max)} \\
& (x, y) = \left(\pi, -1\right) \quad \text{(min)} \\
& (x, y) = \left(2\pi, 1\right) \quad \text{(max)} \\
& \cdots \\
\underline{\text{inflections:}} \quad & (x, y) = (\pm\pi/1, 0) \\
& (x, y) = (\pm 3\pi/2, 0) \\
& \cdots
\end{aligned}$$

11. *Graphical representation:* results of the analysis for function in P.3.90 are summarized by $f(x)$ graph (see Fig. 3.77). Note that $\sin x$ and $\cos x$ have identical forms, spatially separated by quarter of their period, $T/4$, in other words by $\pi/2$ difference in their arguments.

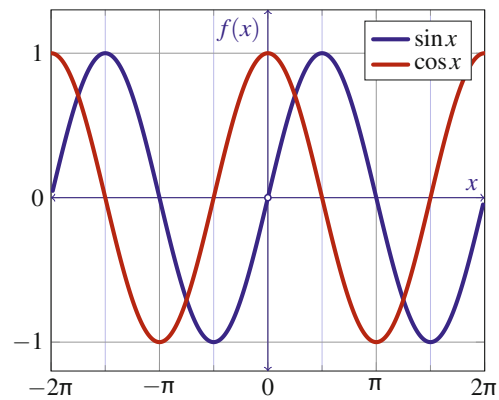


Fig. 3.77 Example P.3.90

3.91. Given function,

$$f(x) = \sin^2 x, \quad x \in [-\pi, \pi]$$

1. *Domain of definition* $\mathbb{D}$: $\sin x$ function is defined for $x \in \mathbb{R}$; therefore $\sin^2 x$ is also defined for $x \in \mathbb{R}$.
2. *Periodicity:* in general, first, product of two trigonometric should be transformed by using product-to-sum identities, in this case

$$\sin^2 x = \sin(x)\sin(x) = \frac{1}{2}\Big(\cos(x-x) - \cos(x+x)\Big) = \frac{1}{2}\Big(\cos(0) - \cos(2x)\Big)$$
$$= \frac{1}{2}\Big(1 - \cos(2x)\Big)$$

Thus, periodicity is determined as

$$\cos 2(x+T) = \cos 2x$$
$$\cos 2(x+T) - \cos 2x = 0$$
$$\left\{\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right\}$$
$$-2\sin\left(\frac{2(x+T)+2x}{2}\right)\sin\left(\frac{2(x+T)-2x}{2}\right) = 0$$
$$-2\sin\left(\frac{2(2x+T)}{2}\right)\sin\left(\frac{2T}{2}\right) = 0$$

First term in the last product can take non-zero values because it is $f(x)$; therefore, it is sufficient to find condition when the second term always equals zero, i.e.,

$$\sin T = 0 \quad \therefore \quad T = \pi$$

as the shortest non-zero period.

3. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = \Big(\sin(-x)\Big)^2 = (-\sin x)^2 = \sin^2(x) = f(x) \quad \therefore \quad \text{(even)}$$
$$\underline{\text{odd:}} \quad -f(-x) = -\sin^2(x) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $\sin^2 x$ is even but not odd.

4. *Function's sign:* a quadratic function is positive for all values of its argument; thus $(\sin x)^2 \geq 0$.

$$\underline{f(x) = 0:} \quad (\sin x)^2 = 0 \quad \therefore \quad x = n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Due to its periodicity, zeros of $\sin^2 x$, starting with $x = 0$, are evenly distributed at π intervals in both directions.

5. *Function's range:* amplitude of trigonometric functions $\sin x$ and $\cos x$ is limited to the range $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$. Therefore, $0 \leq \sin^2 x \leq 1$ for all values of x.

6. *y-axis crossing point* $f(0)$: $f(0) = \sin^2 0 = 0$
7. *Limits:* limits $\lim_{\pm\infty} \sin^2 x$ are not defined; however, the function's amplitude is contained within [0, 1] interval.
8. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sin^2 x}{x} = \left(\frac{\text{not defined}}{\infty}\right)$$

This ambiguous form may be resolved by the help of "the squeeze theorem" (aka the "sandwich theorem") that, in a not-too-rigorous interpretation, may be used as follows.

Given that $x \to \infty$, denominator "x" never equals to zero; thus there is no danger of division by zero. On the other hand, module of the numerator is $|\sin^2 x| \le 1$ (i.e., including $|\sin^2 x| = 0$). Therefore, there are two possible extreme situations:

$$\frac{0}{x} \le \lim_{x\to\infty} \frac{|\sin^2 x|}{x} \le \frac{1}{x}$$

That is to say, at the two extreme instances:

$$\underline{|\sin^2 x| \to 1:} \quad \lim_{x\to\infty} \frac{f(x)}{x} \le \lim_{x\to\infty} \frac{|\sin^2 x|}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\underline{|\sin^2 x| \to 0:} \quad \therefore \ (x \ne 0) \ \therefore \ x \to \frac{\pi}{2} + n\pi \ \ (n = \pm 1, \pm 2, \dots)$$

$$\therefore \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{|\sin^2 x|}{x} = \left(\frac{0}{\pi/2 + n\pi}\right) = 0$$

Therefore, because both the left and right side limits tend to zero, then it must be

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = 0$$

and, by consequence, there is no oblique asymptote.

9. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\sin^2 x\right)' = 2\sin x \cos x = 0$$

$$\therefore \ \sin x = 0 \ \therefore \ x = n\pi, \ \ n = 0, \pm 1, \pm 2, \dots$$

or,

$$\cos x = 0 \ \therefore \ x = \frac{\pi}{2} + n\pi, \ \ n = 0, \pm 1, \pm 2, \dots$$

Thus, (over one period)

x	0		$\pi/2$		π	$\cdots$
2	+	+	+	+	+	$\cdots$
$\sin x$	0	+	1	+	0	$\cdots$
$\cos x$	1	+	0	−	−1	$\cdots$
$f'(x)$	0	+	0	−	0	$\cdots$
$f(x)$	0	↗	1	↘	0	$\cdots$

In conclusion, both extreme points (min and max) are periodic with the period $T = \pi$ as well. Also,

$$f''(x) = 0: \quad f''(x) = (2\sin x \cos x)' = 2\left(\cos^2 x - \sin^2 x\right) = 0$$
$$\therefore \ \cos^2 x = \sin^2 x \ \therefore \ \cos x = \sin x \ \therefore \ x = \pm\frac{\pi}{4}$$

Periodicity of $f''(x)$ is

$$\cos^2(x + T_1) - \sin^2(x + T_2) = \cos^2 x - \sin^2 x$$

In general, function that consists of sum (and/or difference) of trigonometric functions is also periodic; its period may be found as the smallest common factor among individual periods of all trigonometric terms. In this case, both $\cos^2 x$ and $\sin^2 x$ are periodical with the shortest period $T = T_1 = T_2 = \pi$; by consequence, $f''(x)$ is also periodic with $T = \pi$.

In addition,

$$0 \le x \le \pi/4: \quad \cos^2 x > \sin^2 x \ \therefore \ \cos^2 x - \sin^2 x > 0$$
$$\pi/4 \le x \le 3\pi/4: \quad \cos^2 x < \sin^2 x \ \therefore \ \cos^2 x - \sin^2 x < 0$$
$$3\pi/4 \le x \le \pi: \quad \cos^2 x > \sin^2 x \ \therefore \ \cos^2 x - \sin^2 x > 0$$

Thus, (over one period)

x	0		$\pi/4$		$3\pi/4$		π	$\cdots$
$\cos^2 x - \sin^2 x$	1	+	0	−	0	+	1	$\cdots$
$f''(x)$	1	+	0	−	0	+	1	$\cdots$
$f(x)$	0	∪	1/2	∩	1/2	∪	0	$\cdots$

therefore, concave/convex intervals are also periodic with the period π.

10. *Summary of the important points:*

periodicity: $T = \pi$

parity: (even)

y–axis crossing: $(x, y) = (0, 0)$

zeros: $(x, y) = (\pm n\pi, 0)$

$$\begin{aligned}
&\underline{\text{extremes:}} && (x, y) = (0, 0) \quad \text{(min)} \\
&&& (x, y) = \big(\pi/2, 1\big) \quad \text{(max)} \\
&&& (x, y) = \big(\pi, 0\big) \quad \text{(min)} \\
&&& \dots \\
&\underline{\text{inflections:}} && (x, y) = (\pm\pi/4, 1/2) \\
&&& (x, y) = (\pm 3\pi/4, 1/2) \\
&&& \dots
\end{aligned}$$

11. *Graphical representation:* results of the analysis for function in P.3.91 are summarized by $f(x)$ graph (see Fig. 3.78).

3.92. Given function,

$$f(x) = \sin(\pi\sqrt{x}), \quad x \in [0, 2\pi]$$

1. *Domain of definition* $\mathbb{D}$: although $\sin x$ is defined for all $x \in \mathbb{R}$, square root function is defined only for $x \geq 0$; therefore $\sin(\pi\sqrt{x})$ is defined for $x \in \mathbb{R} | x \geq 0$.
2. *Periodicity:* argument of this composite trigonometric function contains square root, thus

$$\sin(\pi\sqrt{x+T}) \neq \sin(\pi\sqrt{x})$$

   for all values of x, i.e., there is no $T > 0$ that satisfies the periodicity equation.
3. *Function's parity:*

$$\begin{aligned}
&\underline{\text{even:}} && f(-x) = \sin(\pi\sqrt{-x}) \quad \therefore \quad \text{(not defined)} \\
&\underline{\text{odd:}} && -f(-x) = -\sin(\pi\sqrt{-x}) \quad \therefore \quad \text{(not defined)}
\end{aligned}$$

   In conclusion, function $\sin(\pi\sqrt{x})$ is neither even not odd.

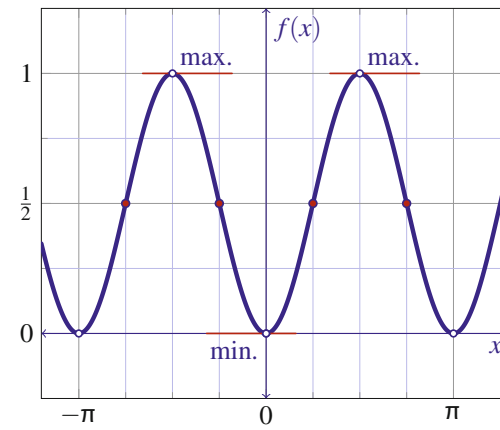


Fig. 3.78 Example P.3.91

4. *Function's sign:* within the interval $x \in [0, 2\pi]$, there are three zeros,

$$\underline{f(x) = 0:} \quad \sin(\pi\sqrt{x}) = 0 \;\; \therefore \;\; \pi\sqrt{x} = n\pi, \;\; n = 0, 1, 2$$

$$\therefore \begin{cases} \underline{n = 0} : & \pi\sqrt{x} = 0 \cdot \pi & \therefore \;\; x = 0 \\ \underline{n = 1} : & \cancel{\pi}\sqrt{x} = \cancel{\pi}\,1 & \therefore \;\; x = 1 \\ \underline{n = 2} : & \cancel{\pi}\sqrt{x} = 2\cancel{\pi} & \therefore \;\; x = 4 \end{cases}$$

Obviously, the zeros are not periodically distributed. Note that due to the $\sqrt{x}$ argument term, the distance between the subsequent zeros increases quadratically. Thus,

x	0		1		4		$\cdots$
$\sin(\pi\sqrt{x})$	0	$+$	0	$-$	0	$+$	$\cdots$
$f(x)$	0	$+$	0	$-$	0	$+$	$\cdots$

5. *Function's range:* amplitude of a composite trigonometric function $\sin f(x)$ is still limited to the range $-1 \le \sin f(x) \le 1$.
6. *y-axis crossing point* $f(0)$: $f(0) = \sin(\pi\sqrt{0}) = 0$
7. *Limits:* limits $\lim_{\pm\infty} \sin(\pi\sqrt{x})$ are not defined; however, the function's amplitude is contained within $[-1, 1]$ interval.
8. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sin(\pi\sqrt{x})}{x} = \left(\frac{\text{not defined}}{\infty}\right)$$

This ambiguous form may be resolved by the help of "the squeeze theorem" (aka the "sandwich theorem") that, in a not-too-rigorous interpretation, may be used as follows.

Given that $x \to \infty$, denominator "x" never equals to zero; thus, there is no danger of division by zero. On the other hand, module of the numerator is $|\sin(\pi\sqrt{x})| \le 1$ (i.e., including $|\sin(\pi\sqrt{x})| = 0$). Therefore, there are two possible extreme situations:

$$\frac{0}{x} \le \lim_{x\to\infty} \frac{|\sin(\pi\sqrt{x})|}{x} \le \frac{1}{x}$$

That is to say, at the two extreme instances:

$$\underline{|\sin(\pi\sqrt{x})| \to 1:} \quad \lim_{x\to\infty} \frac{f(x)}{x} \le \lim_{x\to\infty} \frac{|\sin(\pi\sqrt{x})|}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\underline{|\sin(\pi\sqrt{x})| \to 0:} \quad \therefore \;\; (x \neq 0) \;\; \therefore \;\; \cancel{\pi}\sqrt{x} \to n\cancel{\pi} \;\; (n = 1, 2, \ldots)$$

$$\therefore \;\; x \to n^2$$

$$\therefore \;\; \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{|\sin(\pi\sqrt{x})|}{x} = \left(\frac{0}{n^2}\right) = 0$$

Therefore, because both the left and right side limits tend to zero, then it must be

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = 0$$

and, by consequence, there is no oblique asymptote.

9. *Critical points:*

$$\underline{f'(x)=0:}\quad f'(x)=\left(\sin(\pi\sqrt{x})\right)'=\frac{\pi}{2\sqrt{x}}\cos(\pi\sqrt{x})=0$$

$$\therefore\ \cos(\pi\sqrt{x})=0\ \ \therefore\ \ \pi\sqrt{x}=\frac{\pi}{2}+n\pi,\quad n=0,1,2\quad (x\in[0,2\pi])$$

$$\therefore\ \begin{cases}\underline{n=0}: & \pi\sqrt{x} &= \dfrac{\pi\,1}{2} & \therefore\ x=\dfrac{1}{4}\\[2ex] \underline{n=1}: & \pi\sqrt{x} &= \dfrac{3\pi\,1}{2} & \therefore\ x=\dfrac{9}{4}\\[2ex] \underline{n=2}: & \pi\sqrt{x} &= \dfrac{5\pi\,1}{2} & \therefore\ x=\dfrac{25}{4}\end{cases}$$

Thus, $(x\in[0,2\pi])$

x	0		1/4		9/4		25/4		2π
$\cos(\pi\sqrt{x})$	1	+	0	−	0	+	0	−	−
$\pi/(2\sqrt{x})$	∞	+	+	+	+	+	+	+	+
$f'(x)$	∞	+	0	−	0	+	0	−	−
$f(x)$	0	↗	1	↘	−1	↗	1	↘	↘

Also,

$$\underline{f''(x)=0:}\quad f''(x)=\frac{\pi}{2}\left(\frac{\cos(\pi\sqrt{x})}{\sqrt{x}}\right)'=-\frac{\pi}{4}\,\frac{\pi\sqrt{x}\sin(\pi\sqrt{x})+\cos(\pi\sqrt{x})}{\sqrt{x^3}}=0$$

$$\therefore\ N(x)=\pi\sqrt{x}\sin(\pi\sqrt{x})+\cos(\pi\sqrt{x})=0,\quad (x\neq 0)$$

By taking, for example, the initial value $x_0=\pi/4$ (close to first zero of $f(x)$) Newton–Raphson method

$$x_{n+1}=x_n-\frac{f(x_0)}{f'(x_0)}$$

where,

$$N(x_0)=\pi\sqrt{x}\sin(\pi\sqrt{x})+\cos(\pi\sqrt{x})$$

$$N'(x_0)=\frac{\pi^2}{2}\cos(\pi\sqrt{x})$$

results in

n	x_0	$N(x_0)$	$N'(x_0)$
0	0.785	0.037	−4.623
1	0.793	-9.810×10^{-5}	−4.647
2	0.793	-6.524×10^{-10}	−4.647

and, with the initial value $x_0 = \pi$,

n	x_0	$f(x_0)$	$f'(x_0)$
0	3.142	−2.895	3.727
1	3.918	0.597	4.925
2	3.797	0.003	4.871
3	3.796	1.056×10^{-7}	4.870
4	3.796	1.332×10^{-15}	4.870

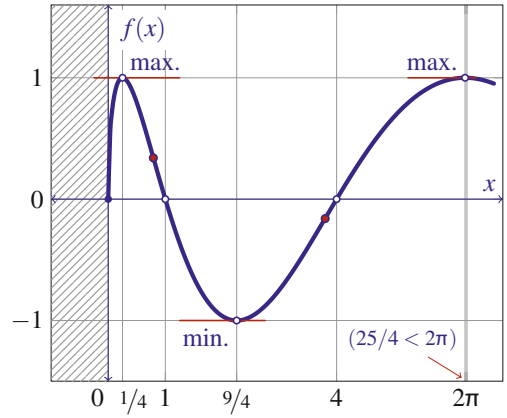
Thus, within interval ($x \in [0, 2\pi]$), there are two zeros of $f''(x)$, where $f(x)$ takes values: $(x_1, y_1) = (0.793, 0.337)$ and $(x_2, y_2) = (3.796, -0.162)$, so that

x	0		x_1		x_2		2π
$f''(x)$	$-\infty$	−	0	+	0	−	
$f(x)$	0	∩	y_1	∪	y_2	∩	

10. *Summary of the important points:* $x \in [0, \sqrt{2}]$

$$
\begin{aligned}
\underline{\text{defined:}} \quad & x \geq 0 \\
\underline{\text{y–axis crossing:}} \quad & (x, y) = (0, 0) \\
\underline{\text{zeros:}} \quad & (x, y) = (0, 0) \\
& (x, y) = (1, 0) \\
& (x, y) = (4, 0) \\
& \ldots \\
\underline{\text{extremes:}} \quad & (x, y) = (1/4, 1) \quad \text{(max)} \\
& (x, y) = \left(9/4, -1\right) \quad \text{(min)} \\
& (x, y) = \left(25/4, 1\right) \quad \text{(max)} \\
& \ldots \\
\underline{\text{inflections:}} \quad & (x, y) = (0.793, 0.337) \\
& (x, y) = (3.796, -0.162) \\
& \ldots
\end{aligned}
$$

Fig. 3.79 Example P.3.92



11. *Graphical representation:* results of the analysis for function in P.3.92 are summarized by $f(x)$ graph (see Fig. 3.79).

3.93. Given function

$$f(x) = \sin(\pi x^2), \quad x \in \left[-\sqrt{2}, \sqrt{2}\right]$$

1. *Domain of definition* $\mathbb{D}$: argument of this function, πx^2, is defined for all $x \in \mathbb{R}$; therefore, $\sin(\pi x^2)$ is also defined for $x \in \mathbb{R}$.
2. *Periodicity:* argument of this composite trigonometric function contains square; thus

$$\sin\left(\pi(x+T)^2\right) = \sin\left(\pi(x^2 + 2xT + T^2)\right) \neq \sin(\pi x^2)$$

for all values of x, i.e., there is no $T > 0$ that satisfies the periodicity equation.
3. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \sin\left(\pi(-x)^2\right) = \sin(\pi x^2) = f(x) \quad \therefore \quad \text{(even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\sin(\pi x^2) = \sin(-\pi x^2) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $\sin(\pi x^2)$ is even but not odd.
4. *Function's sign:* this function is even; thus it is sufficient to analyze within the interval $x \in [0, \sqrt{2}]$,

$$\underline{f(x) = 0:} \quad \sin(\pi x^2) = 0 \quad \therefore \quad \pi x^2 = n\pi, \quad n = 0, 1, 2$$

$$\therefore \quad \begin{cases} \underline{n=0} : & \pi x^2 = 0 \cdot \pi & \therefore \quad x_1 = 0 \\ \underline{n=1} : & \cancel{\pi} x^2 = \cancel{\pi} 1 & \therefore \quad x_2 = 1 \\ \underline{n=2} : & \cancel{\pi} x^2 = 2\cancel{\pi} & \therefore \quad x_3 = \sqrt{2} \end{cases}$$

Obviously, the zeros are not periodically distributed. Note that due to the x^2 argument term, the distance between the subsequent zeros decreases as per square root. Thus,

x	$\cdots$		$-\sqrt{2}$		-1		0		1		$\sqrt{2}$		$\cdots$
$\sin(\pi x^2)$		$+$	0	$-$	0	$+$	0	$+$	0	$-$	0	$+$	$\cdots$
$f(x)$		$+$	0	$-$	0	$+$	0	$+$	0	$-$	0	$+$	$\cdots$

5. *Function's range:* amplitude of a composite trigonometric function $\sin f(x)$ is still limited to the range $-1 \leq \sin f(x) \leq 1$.
6. *y-axis crossing point* $f(0)$: $f(0) = \sin(\pi \cdot 0^2) = 0$
7. *Limits:* limits $\lim_{\pm\infty} \sin(\pi x^2)$ are not defined; however, the function's amplitude is contained within $[-1, 1]$ interval.
8. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sin(\pi x^2)}{x} = \left(\frac{\text{not defined}}{\infty}\right)$$

This ambiguous form may be resolved by the help of "the squeeze theorem" (aka the "sandwich theorem") that, in a not-too-rigorous interpretation, may be used as follows.

Given that $x \to \infty$, denominator "x" never equals to zero; thus, there is no danger of division by zero. On the other hand, module of the numerator is $|\sin(\pi x^2)| \leq 1$ (i.e., including $|\sin(\pi x^2)| = 0$). Therefore, there are two possible extreme situations:

$$\frac{0}{x} \leq \lim_{x\to\infty} \frac{|\sin(\pi x^2)|}{x} \leq \frac{1}{x}$$

That is to say, at the two extreme instances:

$$\begin{aligned}
&\underline{|\sin(\pi x^2)| \to 1:} \quad \lim_{x\to\infty} \frac{f(x)}{x} \leq \lim_{x\to\infty} \frac{|\sin(\pi x^2)|}{x} \leq \lim_{x\to\infty} \frac{1}{x} = 0 \\
&\underline{|\sin(\pi x^2)| \to 0:} \quad \therefore \ (x \neq 0) \ \ \therefore \ \cancel{\pi} x^2 \to n\cancel{\pi} \ \ (n = 1, 2, \ldots) \\
&\qquad\qquad\qquad\quad \therefore \ x \to \sqrt{n} \\
&\qquad\qquad\qquad\quad \therefore \ \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{|\sin(\pi x^2)|}{x} = \left(\frac{0}{\sqrt{n}}\right) = 0
\end{aligned}$$

Therefore, because both the left- and right-side limits tend to zero, then it must be

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = 0$$

and, by consequence, there is no oblique asymptote.

9. *Critical points:*

$$\begin{aligned}
&\underline{f'(x) = 0:} \quad f'(x) = \left(\sin(\pi x^2)\right)' = 2\pi\, x\, \cos(\pi x^2) = 0 \\
&\qquad\qquad \therefore \ x\cos(\pi x^2) = 0 \ \ \therefore \ \ x_1 = 0 \ \text{ and,} \\
&\qquad\qquad \therefore \ \pi x^2 = \frac{\pi}{2} + n\pi, \quad n = 0, 1 \quad (x \in [0, 2\pi])
\end{aligned}$$

$$\therefore \begin{cases} \underline{n=0} : \cancel{\pi} x^2 = \dfrac{\cancel{\pi}\,1}{2} \;\; \therefore \;\; x_2 = \dfrac{1}{\sqrt{2}} \\ \\ \underline{n=1} : \cancel{\pi} x^2 = \dfrac{3\cancel{\pi}\,1}{2} \;\; \therefore \;\; x_3 = \sqrt{\dfrac{3}{2}} \end{cases}$$

where all maximum values equal to one and all minimum values equal to minus one. Thus, $\left(x \in [0, \sqrt{2}]\right)$

x	x_1		x_2		x_3		$\sqrt{2}$
x	0	+	+	+	+	+	+
$\cos(\pi x^2)$	1	+	0	−	0	+	+
$f'(x)$	0	+	0	−	0	+	+
$f(x)$	0	↗	1	↘	−1	↗	0

Also,

$$\underline{f''(x) = 0:} \quad f''(x) = 2\pi \left(x \cos(\pi x^2)\right)' = -2\pi \left(2\pi x^2 \sin(\pi x^2) - \cos(\pi x^2)\right) = 0$$

$$\therefore \;\; N(x) = 2\pi x^2 \sin(\pi x^2) - \cos(\pi x^2) = 0$$

By taking, for example, the initial value $x_0 = 0.5$ (close to midpoint between two zeros of $f(x)$) Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_0)}{f'(x_0)}$$

where,

$$N(x_0) = 2\pi x^2 \sin(\pi x^2) - \cos(\pi x^2)$$

$$N'(x_0) = 2\pi\, x \left(2\pi\, x^2 \cos(\pi x^2) + 3\, \sin(\pi x^2)\right)$$

results in

n	x_0	$N(x_0)$	$N'(x_0)$
0	0.500	0.404	10.154
1	0.460	0.035	8.384
2	0.456	0.389×10^{-4}	8.199
3	0.456	4.951×10^{-8}	8.197

and, with the initial value $x_0 = 1.0$ (close to the second zero in this interval),

n	x_0	$f(x_0)$	$f'(x_0)$
0	1.000	1.000	−39.478
1	1.025	−0.073	−45.105
2	1.024	-2.691×10^{-4}	−44.772
3	1.024	-3.747×10^{-9}	−44.770

Thus, within interval $\left(x \in [0, \sqrt{2}]\right)$, there are two zeros of $f''(x)$, where $f(x)$ takes values: $(x_4, y_4) = (0.456, 0.607)$ and $(x_5, y_5) = (1.024, -0.152)$, so that

x	0		x_4		x_5		$\sqrt{2}$
$f''(x)$	2π	+	0	−	0	+	+
$f(x)$	0	∪	y_4	∩	y_5	∪	0

10. *Summary of the important points:* $x \in [0, \sqrt{2}]$

parity: (even)

y–axis crossing: $(x, y) = (0, 0)$

zeros: ⋯

$(x, y) = (0, 0)$

$(x, y) = (1, 0)$

$(x, y) = \left(\sqrt{2}, 0\right)$

⋯ (cont.)

extremes: ⋯

$(x, y) = (0, 0)$ (min)

$(x, y) = \left(1/\sqrt{2}, 1\right)$ (max)

$(x, y) = \left(\sqrt{3/2}, -1\right)$ (min)

⋯

inflections: ⋯

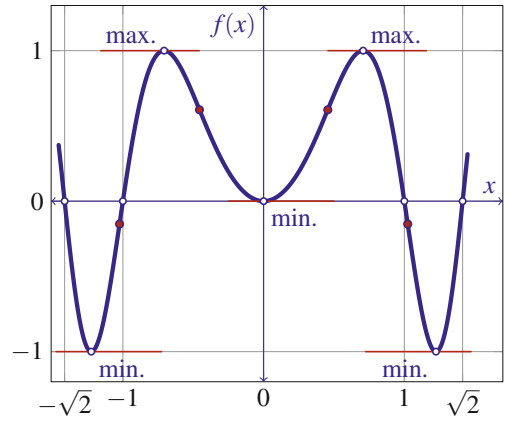
$(x, y) = (0.456, 0.016)$

$(x, y) = (1.024, -0.152)$

⋯

11. *Graphical representation:* results of the analysis for function in P.3.93 are summarized by $f(x)$ graph (see Fig. 3.80).

Fig. 3.80 Example P.3.93



3.94. Given function,

$$f(x) = e^x \sin(2\pi x), \quad x \in [-3, 3]$$

1. *Domain of definition* $\mathbb{D}$: this composed function $f(x)$ is product of exponential and sinusoid functions, each defined for $x \in \mathbb{R}$; by consequence, their product is also defined for $x \in \mathbb{R}$.
2. *Periodicity:* after multiplication by exponential function that is not periodic, the sinusoid function's amplitude changes in accordance. By consequence, there is no repeating pattern; therefore, this product is not periodic,

$$e^{(x+T)} \sin(2\pi(x+T)) = e^T e^x \sin(2\pi(x+T)) \neq e^x \sin(2\pi x)$$

for all values of x, i.e., there is no $T > 0$ that satisfies the periodicity equation.
3. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = e^{-x} \sin(2\pi(-x)) = -e^{-x} \sin(2\pi x) \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = e^{-x} \sin(2\pi x) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $e^x \sin(2\pi x)$ is neither even nor odd.
4. *Function's sign:* within the interval $x \in [-3, 3]$, and because $e^x > 0$ for all x,

$$\underline{f(x) = 0:} \quad e^x \sin(2\pi x) = 0 \quad \therefore \quad \sin(2\pi x) = 0 \quad \therefore \quad 2\pi x = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots, \pm 6$$

$$\therefore \quad x = \frac{n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots, \pm 6$$

Obviously, the zeros are periodically distributed and normalized relative to π. Given that $e^x > 0$, then

x	$\cdots$	$-3/2$		-1		$-1/2$		0		$1/2$		1		$3/2$	$\cdots$
$f(x)$	$\cdots$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$\cdots$

5. *Function's range:* on its own, amplitude of a trigonometric function $\sin f(x)$ is limited to the range $-1 \leq \sin f(x) \leq 1$. However, after the multiplication with exponential function, sinusoid amplitude follows its exponential "envelope" (see Fig. 3.81).
6. *y-axis crossing point* $f(0)$: $f(0) = e^0 \sin(2\pi \cdot 0) = 0$
7. *Limits:* for the reason that limits to infinity of a sinusoid are not defined, but its amplitude is limited to $|1|$, then only limits of exponential envelope may be determined, i.e., there are two limits in total to be resolved,

$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} e^x \underbrace{\sin(2\pi x)}_{\leq |1|} = +\infty$$

$$x \to -\infty: \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^x \underbrace{\sin(2\pi x)}_{\leq |1|} = 0$$

8. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{e^x \sin(2\pi x)}{x} = \left(\frac{\infty \times \text{n.d.}}{\infty}\right)$$

This ambiguous form is not defined; by consequence, there is no oblique asymptote. The envelope limit, however, is the same as in P.3.68.

9. *Critical points:* given that $e^x > 0$,

$$f'(x) = 0: \quad f'(x) = \left(e^x \sin(2\pi x)\right)' = e^x \left(2\pi \cos(2\pi x) + \sin(2\pi x)\right) = 0$$

$$\therefore \quad 2\pi \cos(2\pi x) + \sin(2\pi x) = 0$$

Note that, although periodic with $T = 1/2$, $f'(x)$ zeros are not exactly positioned on "nice" numbers. With the help of Newton-Raphson method and by sweeping the initial value $x_0 = [-2.5, -2.0, \ldots, 2.5, 3.0]$, these zeros are found (after rounding to three decimal places) at $x_i = [-2.725, -2.225, \ldots, 2.275, 2.775]$, where $f(x_i) = [0.065, -0.107, 0.176, -0.29, 0.478, -0.789, 1.3, -2.144, 3.535, -5.828, 9.608, -15.841]$. This example also illustrates that Newton-Raphson may be sometimes sensitive to the choice of initial value, even to the point of becoming unstable. For more on this issue, revisit the course lessons.

Thus, ($x \in [-3, 3]$)

x	-3		-2.72		-2.22		$\cdots$		2.27		2.77		3
$f'(x)$	0.31	+	0	−	0	+	$\cdots$	+	0	−	0	+	126
$f(x)$	0	↗	0.06	↘	−0.11	↗	$\cdots$	↗	9.61	↘	−15.8	↗	0

Also,

$$f''(x) = 0: \quad f''(x) = \left(e^x \left(2\pi \cos(2\pi x) + \sin(2\pi x)\right)\right)'$$

$$= -e^x \left(2\pi(4\pi^2 - 3)\cos(2\pi x) + (12\pi^2 - 1)\sin(2\pi x)\right) = 0$$

$$\therefore \quad N(x) = 2\pi(4\pi^2 - 3)\cos(2\pi x) + (12\pi^2 - 1)\sin(2\pi x) = 0$$

Similarly to the $f'(x)$ function, period of $f''(x)$ is $T = 1/2$ with the zeros that are not exactly positioned on "nice" numbers. With the help of Newton-Raphson method and by sweeping the initial value $x_0 = [-3.0, -2.5, \ldots, 2.0, 2.5]$, these zeros are found (after rounding to three decimal places) at $x_i = [-2.949, -2.449, \ldots, 2.050, 2.550]$, where $f(x_i) = [0.016, -0.027, 0.044, \ldots, -1.4630, 2.4121, -3.977]$.

Thus, (results where differences are after third decimal place are marked as '$\approx$'),

x	-3		-2.95		-2.45		$\cdots$		2.05		2.55		3
$f''(x)$	$0\downarrow$	$+$	0	$-$	0	$+$	$\cdots$	$+$	0	$-$	0	$+$	252
$f(x)$	0	$\cup$	0.01	$\cap$	-0.02	$\cup$	$\cdots$	$\cup$	2.41	$\cap$	-3.98	$\cup$	0

Within given interval, there are 12 inflection points.

10. *Summary of the important points:*

y-axis crossing: $(x, y) = (0, 0)$

zeros: $(x, y) = (\pm n/2, 0)$

extremes: $\cdots$

$(x, y) = (-2.725, 0.065)$ (min)

$(x, y) = (-2.225, , -0.107)$ (max)

$\cdots$

$(x, y) = (2.275, 9.608)$ (max)

$(x, y) = (2.775, -15.841)$ (max)

$\cdots$

inflections: $\cdots$

$(x, y) = (-2.949, 0.016)$

$(x, y) = (-2.449, -0.027)$

$\cdots$

$(x, y) = (2.050, 2.4121)$

$(x, y) = (2.550, -3.977)$

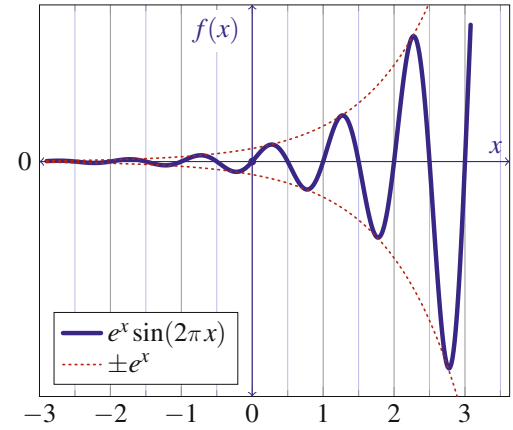
$\cdots$

11. *Graphical representation:* results of the analysis for function in P.3.94 are summarized by $f(x)$ graph (see Fig. 3.81).

3.95. Given function,

$$f(x) = \ln x \, \sin(4x), \quad x \in [0, 2\pi]$$

Fig. 3.81 Example P.3.94



1. *Domain of definition* $\mathbb{D}$: this composed function $f(x)$ is a product of logarithmic and sinusoid functions. Although a sinusoid function is defined for $x \in \mathbb{R}$, it is logarithmic function that is limited to $x > 0$ interval; by consequence, their product is also defined for $x \in \mathbb{R} | x > 0$.
2. *Periodicity:* after multiplication by logarithmic function that is not periodic, the sinusoid function's amplitude changes in accordance. By consequence, there is no repeating pattern; therefore, this product is not periodic,

$$\ln(x+T)\sin\big(4(x+T)\big) \neq \ln x \sin(4x)$$

for all values of x, i.e., there is no $T > 0$ that satisfies the periodicity equation.
3. *Function's parity:*

$$\begin{aligned} \underline{\text{even:}} \quad & f(-x) = \ln(-x)\sin(-4x) \quad \therefore \quad \text{(not defined)} \\ \underline{\text{odd:}} \quad & -f(-x) = -\ln(-x)\sin(-4x) \quad \therefore \quad \text{(not defined)} \end{aligned}$$

In conclusion, function $\ln x \sin(4x)$ is neither even nor odd.
4. *Function's sign:* $(x > 0)$

$$\underline{f(x) = 0:} \quad \ln x \sin(4x) = 0 \quad \therefore \quad \begin{cases} \ln x = 0 & \therefore \quad x_1 = 1 \\ \sin(4x) = 0 & \therefore \quad 4x = n\pi, \quad n = 1, 2, \ldots \\ & \therefore \quad x_i = n\dfrac{\pi}{4}, \quad n = 1, 2, \ldots \end{cases}$$

In this case, the sinusoid's zeros are periodically distributed by $\pi/4$, but there is an additional zero at $(1, 0)$ that is forced by the logarithmic function, then

x	0		$\pi/4$		1		$\pi/2$		$3\pi/4$		π		$5\pi/4$		$\cdots$
$\ln x$	n.d.	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0	$+$	$\cdots$
$\sin(4x)$	0	$+$	0	$-$	$-$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	$\cdots$
$f(x)$	n.d.	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$-$	$\cdots$

5. *Function's range:* on its own, amplitude of a trigonometric function $\sin f(x)$ is limited to the range $-1 \leq \sin f(x) \leq 1$. However, after the multiplication with logarithmic function, sinusoid amplitude follows its exponential "envelope" (see Fig. 3.82).
6. *y-axis crossing point* $f(0)$: $f(0) = \ln 0 \, \sin(4 \cdot 0) \quad \therefore$ (not defined)
7. *Limits:* for the reason that limits to infinity of a sinusoid are not defined but its amplitude is limited to $|1|$, then only limit of logarithmic envelope may be determined. However, when $x \to \underset{\leftarrow}{0}$, the sinusoid's value is well defined, and then l'Hôpital's rule may be used. That is to say,

$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln x \, \underbrace{\sin(4x)}_{\leq |1|} = +\infty$$

$$x \to \underset{\leftarrow}{0}: \quad \lim_{x\to \underset{\leftarrow}{0}} f(x) = \lim_{x\to \underset{\leftarrow}{0}} \ln x \sin(4x) = \lim_{x\to \underset{\leftarrow}{0}} \frac{\ln x}{\frac{1}{\sin(4x)}} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x\to \underset{\leftarrow}{0}} \frac{1/x}{-4\cos(4x)/\sin^2(4x)} = -\frac{1}{4} \lim_{x\to \underset{\leftarrow}{0}} \frac{\sin^2(4x)}{x \, \cos(4x)} = \left(\frac{0}{0}\right)$$

$$\overset{\text{l'H}}{=} -\frac{1}{4} \lim_{x\to \underset{\leftarrow}{0}} \frac{8\cos(4x)\sin(4x)}{\cos(4x) - 4x\sin(4x)} = -\frac{1}{4}\frac{(8 \cdot 1 \cdot 0)}{(1 - 4 \cdot 0 \cdot 0)}$$

$$= 0\uparrow$$

8. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln x \sin(4x)}{x} = \left(\frac{\infty \times \text{n.d.}}{\infty}\right)$$

This ambiguous form is not defined; by consequence, there is no oblique asymptote.

9. *Critical points:* given that $x > 0$,

$$f'(x) = 0: \quad f'(x) = \left(\ln x \sin(4x)\right)' = \frac{\sin(4x) + 4x \ln x \cos(4x)}{x} = 0$$

$$\therefore \quad \sin(4x) + 4x \ln x \cos(4x) = 0$$

Same as in problems P.3.93 and P.3.94, even though $f(x)$ zeros may be equidistant, its critical points are slightly shifted due to distortion of the sinusoid form and its ever-increasing amplitude.

With the help of Newton-Raphson method and by sweeping the initial value $x_0 = [0.25, 1.0, 1.5, 2.0, 3.0, 3.5, 4.5, 5.0, 6.0]$, these zeros are found (after rounding to three decimal places) at $x_i = [0.234, 0.886, 1.326, 2.008, 2.771, 3.548, 4.330, 5.13, 5.896]$, where $f(x_i) = [-1.169, 0.047, -0.234, 0.686, -1.015, 1.264, -1.464, 1.631, -1.774]$. This example also illustrates that Newton-Raphson is sensitive to the initial value and may become unstable. Thus, sometimes, it is necessary to use "trial and error method" to choose the initial values.

Thus, $(x > 0)$.

x	0		0.23		0.88		1.32		2		2.77		3.54		$\cdots$
$f'(x)$	n.d.	$-$	0	$-$	$-$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	$\cdots$
$f(x)$	n.d.	$\searrow$	-1.1	$\nearrow$	0.04	$\searrow$	-0.2	$\nearrow$	0.6	$\searrow$	-1	$\nearrow$	1.2	$\searrow$	$\cdots$

Also,

$$f''(x) = 0: \quad f''(x) = \left(\frac{\sin(4x) + 4x \ln x \cos(4x)}{x}\right)'$$
$$= \frac{8x\cos(4x) - (16x^2 \ln x + 1)\sin(4x)}{x^2} = 0$$
$$\therefore \quad N(x) = 8x\cos(4x) - (16x^2 \ln x + 1)\sin(4x) = 0$$

With the help of Newton-Raphson method and by sweeping the initial value $x_0 = [0.5, 1, 2, 2.5, 3, 4, 4.5, 5.5]$, these zeros are found (after rounding to three decimal places) at $x_i = [0.497, 1.1, 1.694, 2.413, 3.175, 3.95, 4.729, 5.511]$, where $f(x_i) = [-0.639, -0.091, 0.250, -0.199, 0.155, -0.126, 0.105, -0.090]$.

Thus, (with truncated numbers),

x	0		0.49		1.1		1.69		2.41		3.17		3.95	$\cdots$
$f''(x)$	n.d.	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	0	$\cdots$
$f(x)$	n.d.	$\cup$	-0.6	$\cap$	-0.1	$\cup$	0.2	$\cap$	-0.2	$\cup$	0.15	$\cap$	-0.1	$\cdots$

Within given interval, there are six inflection points.

10. *Summary of the important points:*

defined: $x > 0$

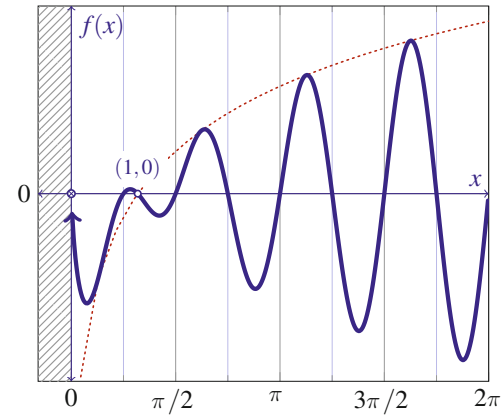
y-axis crossing: (not defined)

zeros: $(x, y) = (\pi/4, 0)$
$(x, y) = (1, 0)$
$(x, y) = (\pi/2, 0)$
$(x, y) = (3\pi/4, 0)$
$\cdots$

extremes: $\cdots$
$(x, y) = (0.234, -1.169)$ (min)
$(x, y) = (0.886, 0.047)$ (max)
$(x, y) = (1.326, -0.234)$ (min)
$(x, y) = (2.008, 0.686)$ (max)
$\cdots$ (cont.)

inflections: $(x, y) = (0.497, -0.639)$
$(x, y) = (1.1, -0.091)$
$(x, y) = (1.694, 0.250)$

Fig. 3.82 Example P.3.95



$$(x, y) = (2.413, -0.199)$$

$$\ldots$$

11. *Graphical representation:* see Fig. 3.82.

3.96. Given function,

$$f(x) = \cos^2 x - \cos x, \quad x \in [0, 2\pi]$$

1. *Domain of definition* $\mathbb{D}$: each of the two sinusoid functions are defined for $x \in \mathbb{R}$; therefore, $f(x)$ is also defined for $x \in \mathbb{R}$.
2. *Periodicity:* in general, first, product of two trigonometric should be transformed by using product-to-sum identities; in this case

$$\cos^2 x = \cos(x)\cos(x) = \frac{1}{2}\big(\cos(x - x) + \cos(x + x)\big) = \frac{1}{2}\big(\cos(0) + \cos(2x)\big)$$
$$= \frac{1}{2}\big(1 + \cos(2x)\big)$$

Thus, periodicity of $\cos(2x)$ term is determined as

$$\cos 2(x + T) = \cos 2x$$
$$\cos 2(x + T) - \cos 2x = 0$$
$$\left\{\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)\right\}$$
$$-2\sin\left(\frac{2(x + T) + 2x}{2}\right)\sin\left(\frac{2(x + T) - 2x}{2}\right) = 0$$

$$-2\sin\left(\frac{2(2x+T)}{2}\right)\sin\left(\frac{2T}{2}\right)=0$$

First term in the last product can take non-zero values because it is $f(x)$; therefore, it is sufficient to find condition when the second term always equals zero, i.e.,

$$\sin T=0 \;\;\therefore\;\; T=n\pi,\quad n=\pm1,\pm2,\pm3,\ldots$$

At the same time,

$$\cos(x+T)=\cos(x) \;\;\therefore\;\; T=n\,2\pi,\quad n=\pm1,\pm2,\pm3,\ldots$$

as two non-zero periods.

Sum (or, difference) of periodic functions is also periodic function with the period equal to smallest common factor of the individual periods. Thus, in this case, smallest common factor is $T=2\pi$.

3. *Function's parity:*

$$\underline{\text{even:}}\quad f(-x)=\big(\cos(-x)\big)^2-\cos(-x)=\cos^2x-\cos x=f(x)\;\;\therefore\;\;\text{(even)}$$

$$\underline{\text{odd:}}\quad -f(-x)=-\cos^2x+\cos x\neq f(x)\;\;\therefore\;\;\text{(not odd)}$$

In conclusion, function $\cos^2x-\cos x$ is even but not odd.

4. *Function's sign:*

$$\underline{f(x)=0:}\quad \cos^2x-\cos x=0\;\;\therefore\;\;\cos x(\cos x-1)=0\;\;\therefore$$

$$\begin{cases}\cos x=0 \;\;\therefore\;\; x=\dfrac{\pi}{2}+n\,\pi,\quad n=0,\pm1,\pm2,\pm3,\ldots\\ \cos x=1 \;\;\therefore\;\; x=n\,2\pi,\quad n=0,\pm1,\pm2,\pm3,\ldots\end{cases}$$

That is to say,

x	$\cdots$		$-\pi/2$		0		$\pi/2$		$3\pi/2$		2π		$5\pi/2$		$\cdots$
$\cos x$	$\cdots$	$-$	0	$+$	1	$+$	0	$-$	0	$+$	1	$+$	0	$-$	$\cdots$
$\cos x-1$	$\cdots$	$-$	$-$	$-$	0	$-$	$-$	$-$	$-$	$-$	0	$-$	$-$	$-$	$\cdots$
$f(x)$	$\cdots$	$+$	0	$-$	0	$-$	0	$+$	0	$-$	0	$-$	0	$+$	$\cdots$

5. *y-axis crossing point* $f(0)$: $f(0)=\cos^2 0-\cos 0=0$
6. *Limits:* limits $\lim_{\pm\infty}\cos x$ and $\lim_{\pm\infty}(\cos x-1)$ are not defined; therefore, limit of their product is also not defined.
7. *Oblique asymptote:*

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\cos^2x-\cos x}{x}=\left(\frac{\text{not defined}}{\infty}\right)$$

by consequence, there is no oblique asymptote.

8. *Critical points:*

$$\underline{f'(x)=0:}\quad f'(x)=\left(\cos^2 x-\cos x\right)'=\sin x-2\sin x\cos x=\sin x(1-2\cos x)=0$$

$$\therefore\ \begin{cases}\sin x=0 & \therefore\ \ x=n\pi,\quad n=0,\pm1,\pm2,\ldots\\ & \text{or,}\\ 1-2\cos x=0 & \therefore\ \ \cos x=\dfrac{1}{2}\ \ \therefore\ \ x=\arccos\dfrac{1}{2}\\ & \therefore\ \ x=\pm\pi/3+n\,2\pi,\quad n=0,\pm1,\pm2,\ldots\end{cases}$$

Thus, (positive side only)

x	0		$\pi/3$		π		$5\pi/3$		2π		$\cdots$
$\sin x$	0	+	$\sqrt{3}/2$	+	0	−	−	−	0	+	$\cdots$
$1-2\cos x$	−1	−	0	+	+	+	0	−	−	−	$\cdots$
$f'(x)$	0	−	0	+	0	−	0	+	0	−	$\cdots$
$f(x)$	0	↘	−1/4	↗	2	↘	−1/4	↗	0	↘	$\cdots$

Note that there are two levels of maximum points, local maximums $y=0$ and global maximums at $y=2$.

Also, aside from numerical Newton-Raphson method, it is possible to use the following analytical method to solve trigonometric equations (note negative "−4" factor of quadratic therm in the coming quadratic equation)

$$\underline{f''(x)=0:}\quad f''(x)=(\sin x-2\sin x\cos x)'=-4\cos^2 x+\cos x+2=0$$

$$\{t=\cos x\}$$

$$\therefore\ \ -4t^2+t+2=0\ \ \therefore\ \ t_{1,2}=\frac{\pm\sqrt{33}+1}{8}$$

$$\therefore\ \ x_{1,2}=\arccos\left(\frac{\pm\sqrt{33}+1}{8}\right)$$

$$\left\{\frac{\pi}{2}=\arcsin x+\arccos x\right\}$$

$$x_{1,2,3,4}=\pm\frac{n\pi}{2}\pm\arcsin\left(\frac{\pm\sqrt{33}-1}{8}\right)$$

where $f''(x)$ zeros x_{i} are periodic. Within $[0\le x\le 2\pi]$ period, there are four inflection points

$$f(x_1)=\cos^2\left[\frac{\pi}{2}+\arcsin\left(\frac{-\sqrt{33}-1}{8}\right)\right]-\cos\left[\frac{\pi}{2}+\arcsin\left(\frac{-\sqrt{33}-1}{8}\right)\right]$$

$$\left\{\begin{array}{l}\sin(-x) = -\sin x; \quad \arcsin(-x) = -\arcsin x; \\ \cos(x+y) = \cos x \cos y - \sin x \sin y; \\ \sin(\arcsin x) = x; \quad \cos(\arcsin x) = \sqrt{1-x^2}\end{array}\right\}$$

$$= \frac{21 - 5\sqrt{33}}{32}$$

$$f(x_2) = \cos^2\left[\frac{\pi}{2} + \arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right] - \cos\left[\frac{\pi}{2} + \arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right]$$

$$= \frac{21 + 5\sqrt{33}}{32}$$

$$f(x_3) = \cos^2\left[\frac{3\pi}{2} - \arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right] - \cos\left[\frac{3\pi}{2} - \arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right]$$

$$= \frac{21 + 5\sqrt{33}}{32}$$

$$f(x_4) = \cos^2\left[\frac{3\pi}{2} - \arcsin\left(\frac{-\sqrt{33}-1}{8}\right)\right] - \cos\left[\frac{3\pi}{2} - \arcsin\left(\frac{-\sqrt{33}-1}{8}\right)\right]$$

$$= \frac{21 - 5\sqrt{33}}{32}$$

Thus (over one period),

x	0		x_1		x_2		x_3		x_4		2π
-4	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$x - x_1$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$x - x_4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$
$f''(x)$	$-$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	$-$
$f(x)$	0	$\cap$	y_1	$\cup$	0	$\cap$	y_3	$\cup$	0	$\cap$	0

9. *Function's range:* $f(x)$ is limited to the range between its minimum and absolute maximum points (there are local maximums at $y = 0$ repeating at $n\,2\pi$).
10. *Summary of the important points:*

periodicity: $T = 2\pi$

parity: (even)

y–axis crossing: $(x, y) = (0, 0)$

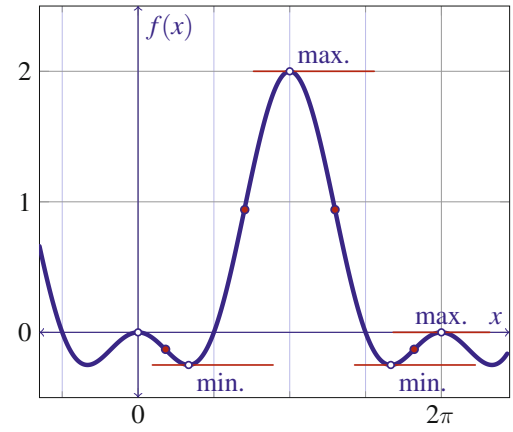


Fig. 3.83 Example P.3.96

zeros: ...

$(x, y) = (\pi/2, 0)$

$(x, y) = (0, 0)$

$(x, y) = (\pi/2, 0)$

$(x, y) = (3\pi/2, 0)$

... (cont.)extremes: ...

$(x, y) = (0, 0)$ (max)

$(x, y) = (\pi/3, -1/4)$ (min)

$(x, y) = (\pi, 2)$ (max)

$(x, y) = (5\pi/3, -1/4)$ (min)

...

inflections:

$$(x, y) = \left(\frac{\pi}{2} + \arcsin\left(\frac{-\sqrt{33}-1}{8}\right), \frac{21-5\sqrt{33}}{32}\right)$$

$$(x, y) = \left(\frac{\pi}{2} + \arcsin\left(\frac{\sqrt{33}-1}{8}\right), \frac{21+5\sqrt{33}}{32}\right)$$

$$(x, y) = \left(\frac{3\pi}{2} - \arcsin\left(\frac{\sqrt{33}-1}{8}\right), \frac{21+5\sqrt{33}}{32}\right)$$

$$(x, y) = \left(\frac{3\pi}{2} - \arcsin\left(\frac{-\sqrt{33}-1}{8}\right), \frac{21-5\sqrt{33}}{32}\right)$$

11. *Graphical representation:* results of the analysis for function in P.3.96 are summarized by $f(x)$ graph (see Fig. 3.83).

3.97. Given function,

$$f(x) = \frac{\sin x}{x}$$

1. *Domain of definition* $\mathbb{D}$: being a sinusoid function, the numerator of $f(x)$ is defined for all real arguments; however, its denominator may cause division by zero; thus, $f(x)$ is defined for $x \in \mathbb{R} | x \neq 0$.
2. *Periodicity:* even though by itself $sin(x)$ is periodic, it is being multiplied by non-periodic inverse function, $1/x$, which causes the amplitude of $f(x)$ to reduce as x varies. Thus, $f(x)$ is not periodic.
3. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\sin(-x)}{-x} = -\frac{\sin x}{-x} = \frac{\sin x}{x} = f(x) \quad \therefore \quad \text{(even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\frac{\sin x}{x} \neq f(x) \quad \therefore \quad \text{(not odd)}$$

   In conclusion, function $\sin x/x$ is even but not odd.
4. *Function's sign:* $(x \neq 0)$

$$\underline{f(x) = 0:} \quad \frac{\sin x}{x} = 0 \quad \therefore \quad \sin x = 0 \quad \therefore \quad x = n\pi, \quad n = \pm 1, \pm 2, \ldots$$

   That is to say,

x	$\cdots$		-2π		$-\pi$		0		π		2π		$\cdots$
$\sin x$	$\cdots$	$-$	0	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$\cdots$
x	$\cdots$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$\cdots$
$f(x)$	$\cdots$	$+$	0	$-$	0	$+$	n.d.	$+$	0	$-$	0	$+$	$\cdots$

   In conclusion, locations of $f(x)$ zeros are periodic with the period of π.
5. *y-axis crossing point* $f(0)$:

$$f(0) = \frac{\sin 0}{0} = \text{not defined}$$

6. *Limits:* limits $\lim_{x\to\pm\infty} \sin x$ are not defined; however, the $f(x)$ *envelope* follows $1/x$ (see Fig. 3.84). On the other hand, $\lim_{x\to 0} f(x)$ exists, as

$$\underline{x \to \pm\infty:} \quad \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\cancel{\sin(x)}^{\leq |1|}}{x} = 0$$

$$\underline{x \to \underset{\leftarrow}{0}:} \quad \lim_{x\to\underset{\leftarrow}{0}} f(x) = \lim_{x\to\underset{\leftarrow}{0}} \frac{\sin x}{x} = \left(\frac{0}{0}\right) \overset{\text{l'H}}{=} \lim_{x\to\underset{\leftarrow}{0}} \frac{\cos x}{1} = 1$$

$$\underline{x \to \underset{\rightarrow}{0}:} \quad \lim_{x\to\underset{\rightarrow}{0}} f(x) = \lim_{x\to\underset{\rightarrow}{0}} \frac{\sin x}{x} = \left(\frac{0}{0}\right) \overset{\text{l'H}}{=} \lim_{x\to\underset{\rightarrow}{0}} \frac{\cos x}{1} = 1$$

Actually, this function is well known as "sinus cardinal," i.e., $f(x) = \sin x/x = \operatorname{sinc}(x)$, and because both left-side and right-side limits are equal, then $\lim_{x\to 0} f(x) = 0$, while $(x \neq 0)$.

7. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\frac{\sin x}{x}}{x} = \lim_{x\to\infty} \frac{\sin x}{x^2} = \left(\frac{\text{not defined}}{\infty}\right)$$

by consequence, there is no oblique asymptote.

8. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2} \quad \therefore \quad x\cos x - \sin x = 0$$

With the help of Newton-Raphson method $f'(x)$ zeros are found (after rounding to three decimal places) at $x_i = [\ldots, -10.904, -7.725, -4.493, 4.493, 7.725, 10.904, \ldots]$, where $f(x_i) = [\ldots, -0.091, 0.128, -0.217, -0.217, 0.128, -0.091, \ldots]$.

Thus, (with truncated decimal numbers)

x	$\cdots$	-7.72		-4.49		0		4.49		7.72	$\cdots$
$x\cos x - \sin x$	$\cdots$	0	$-$	0	$+$	0	$-$	0	$+$	0	$\cdots$
x^2	$\cdots$	$+$	$+$	$+$	$+$	0	$+$	$+$	$+$	$+$	$\cdots$
$f'(x)$	$\cdots$	0	$-$	0	$+$	n.d	$-$	0	$+$	0	$\cdots$
$f(x)$	$\cdots$	0.12	$\searrow$	-0.21	$\nearrow$	n.d.	$\searrow$	-0.21	$\nearrow$	0.12	$\cdots$

Note that $\operatorname{sinc}(x)$ function is not defined at $x = 0$, neither is its first derivative.

Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\frac{x\cos x - \sin x}{x^2}\right)' = \frac{(2 - x^2)\sin x - 2x\cos x}{x^3} = 0$$

$$\therefore \quad N(x) = (2 - x^2)\sin x - 2x\cos x = 0$$

With the help of Newton-Raphson method $f''(x)$ zeros are found (after rounding to three decimal places) at $x_i = [\ldots, -9.206, -5.940, -2.082, 2.082, 5.940, 9.206, \ldots]$, where $f(x_i) = [\ldots, 0.024, -0.057, 0.419, 0.419, -0.057, 0.024, \ldots]$.

Thus, (with truncated decimal numbers)

x	$\cdots$	-5.94		-2.08		0		2.08		5.94	$\cdots$
$N(x)$	$\cdots$	0	$-$	0	$+$	0	$-$	0	$+$	0	$\cdots$
x^3	$\cdots$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$\cdots$
$f''(x)$	$\cdots$	0	$+$	0	$-$	n.d	$-$	0	$+$	0	$\cdots$
$f(x)$	$\cdots$	-0.05	$\cup$	0.41	$\cap$	n.d.	$\cap$	0.41	$\cup$	-0.05	$\cdots$

Note that $\operatorname{sinc}(x)$ function is not defined at $x = 0$, neither is its second derivative.

9. *Function's range:* $f(x)$ is limited to the range $-0.217 \leq f(x) \leq 1$.
10. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{define:}} \quad & x \neq 0 \\
\underline{\text{parity:}} \quad & \text{(even)} \\
\underline{\text{zeros:}} \quad & x = \pm n\pi \\
\underline{\text{y–axis crossing:}} \quad & (x, y) = (0, 1) \quad \text{(not defined)}
\end{aligned}$$

$$\begin{aligned}
\underline{\text{extremes:}} \quad & (x, y) = (\cdots, \cdots) \\
& (x, y) = (-7.725, 0.128) \quad \text{(max)} \\
& (x, y) = (-4.493, -0.217) \quad \text{(min)} \\
& (x, y) = (4.493, -0.217) \quad \text{(min)} \\
& (x, y) = (7.725, 0.128) \quad \text{(max)} \\
& (x, y) = (\cdots, \cdots) \\
\underline{\text{inflections:}} \quad & (x, y) = (\cdots, \cdots) \\
& (x, y) = (-5.940, -0.057) \\
& (x, y) = (-2.082, 0.419) \\
& (x, y) = (2.082, 0.419) \\
& (x, y) = (5.940, -0.057) \\
& (x, y) = (\cdots, \cdots)
\end{aligned}$$

Note that, as oppose to its zeros, extreme and inflection point of sinc (x) are *not* perfectly aligned with the multiples of $\pi/2$.

11. *Graphical representation:* results of the analysis for function in P.3.97 are summarized by $f(x)$ graph (see Fig. 3.84).

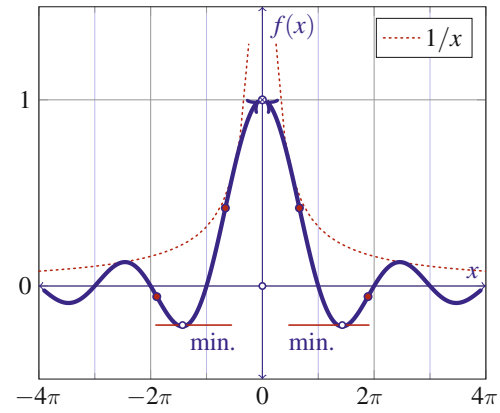


Fig. 3.84 Example P.3.97

3.98. Given function,

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

1. *Domain of definition* $\mathbb{D}$: being a rational function, the numerator of $f(x)$ is defined for all real arguments; however, its denominator may cause division by zero. Thus, $f(x)$ is not defined if $\cos x = 0$, that is to say, vertical asymptotes are located at

$$\sin x = 0 \;\; \therefore \;\; x = \frac{\pi}{2} + n\pi, \;\; n = 0, \pm 1, \pm 2, \ldots$$

and therefore $x \in \mathbb{R} | x \neq \pi/2 + n\pi, \;\; n = 0, \pm 1, \pm 2, \ldots$.

2. *Periodicity:* even though both $sinx$ and $\cos x$ are periodic with $T = 2\pi$, the inverse $1/\cos x$ creates periodic zeros at $T = \pi$; consequently, $f(x)$ is periodic by $T = \pi$, i.e.,

$$\tan(x + \pi) = \tan x$$

3. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \frac{\sin(-x)}{\cos(-x)} = -\frac{\sin x}{\cos x} = -\tan x \neq f(x) \;\; \therefore \;\; \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\left(-\frac{\sin x}{\cos x}\right) = \frac{\sin x}{\cos x} = \tan x = f(x) \;\; \therefore \;\; \text{(odd)}$$

In conclusion, function $\tan x$ is odd but not even.

4. *Function's sign:* $(x \neq \pi/2) + n\pi, n = 0, \pm 1, \pm 2, \ldots$

$$\underline{f(x) = 0:} \quad \frac{\sin x}{\cos x} = 0 \;\; \therefore \;\; \sin x = 0 \;\; \therefore \;\; x = n\pi, \;\; n = \pm 1, \pm 2, \ldots$$

That is to say,

x	$\cdots$		$-\pi/2$		0		$\pi/2$		π		$3\pi/2$		$\cdots$
$\sin x$	$\cdots$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	$\cdots$
$\cos x$	$\cdots$	$-$	0	$+$	1	$+$	0	$-$	$-$	$-$	0	$+$	$\cdots$
$f(x)$	$\cdots$	$+$	n.d.	$-$	0	$+$	n.d.	$-$	0	$+$	n.d.	$-$	$\cdots$

In conclusion, locations of $f(x)$ vertical asymptotes are periodic with the period of π.

5. *y-axis crossing point* $f(0)$:

$$f(0) = \frac{\sin 0}{\cos 0} = 0$$

6. *Limits:* limits $\lim_{x \to \pm\infty} \tan x$ are not defined; however, limits around vertical asymptotes are

$$\underline{x \to -\pi/2 \underset{\to}{:}} \quad \lim_{x \to -\pi/2 \atop \to} f(x) = \lim_{x \to -\pi/2 \atop \to} \frac{\sin x}{\cos x} = \frac{(-1)}{(0) \atop \to} = +\infty$$

$$\{\text{both } \sin x \quad \text{and} \quad \cos x \text{ are negative on the left side of } -\pi/2\}$$

$$\underline{x \to -\pi/2 :} \quad \lim_{\substack{x\to -\pi/2\\ \leftarrow}} f(x) = \lim_{\substack{x\to -\pi/2\\ \leftarrow}} \frac{\sin x}{\cos x} = \frac{(-1)}{(0)_{\leftarrow}} = -\infty$$

$$\{\text{because only } \sin x \text{ is negative on the right side of } -\pi/2\}$$

$$\underline{x \to \pi/2 :} \quad \lim_{\substack{x\to \pi/2\\ \rightarrow}} f(x) = \lim_{\substack{x\to \pi/2\\ \rightarrow}} \frac{\sin x}{\cos x} = \frac{(1)}{(0)_{\rightarrow}} = +\infty$$

$$\{\text{both } \sin x \quad \text{and} \quad \cos x \text{ are positive on the left side of } \pi/2\}$$

$$\underline{x \to \pi/2 :} \quad \lim_{\substack{x\to \pi/2\\ \leftarrow}} f(x) = \lim_{\substack{x\to \pi/2\\ \leftarrow}} \frac{\sin x}{\cos x} = \frac{(1)}{(0)_{\leftarrow}} = -\infty$$

$$\{\text{because only } \cos x \text{ is negative on the right side of } \pi/2\}$$

7. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\tan x}{x} = \left(\frac{\text{not defined}}{\infty}\right)$$

by consequence, there is no oblique asymptote.

8. *Critical points:*

$$\underline{f'(x) = 0 :} \quad f'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

$$\therefore\ 1 + \tan^2 x > 0 \quad \therefore \quad f(x) \quad \text{ascends} \quad \nearrow \quad \text{for all} \quad x$$

because a quadratic function $(\cdot)^2 \geq 0$ is always positive. Then, either inverse or plus "1" is still positive. Therefore, there are no extreme points. Also,

$$\underline{f''(x) = 0 :} \quad f''(x) = \left(\frac{1}{\cos^2 x}\right)' = \frac{2\sin x}{\cos^3 x} = 0$$

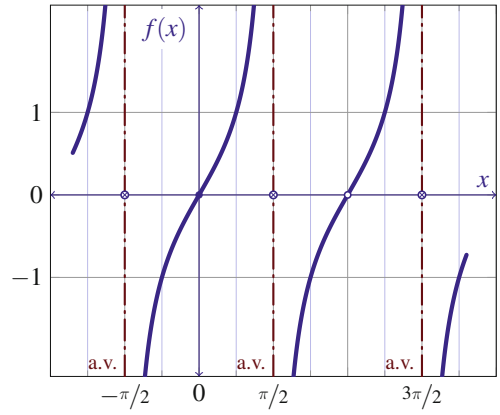
$$\therefore\ \sin x = 0 \quad \therefore \quad x = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Thus,

x	$\cdots$		$-\pi$		$-\pi/2$		0		$\pi/2$		π		$\cdots$
2	$\cdots$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$\sin x$	$\cdots$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$\cdots$
$\cos^3 x$	$\cdots$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	$\cdots$
$f''(x)$	$\cdots$	$-$	0	$+$	n.d.	$-$	0	$+$	n.d.	$-$	0	$+$	$\cdots$
$f(x)$	$\cdots$	$\cap$	0	$\cup$	n.d.	$\cap$	0	$\cup$	n.d.	$\cap$	0	$\cup$	$\cdots$

Inflection points are periodically distributed.

Fig. 3.85 Example P.3.98



9. *Function's range:* $f(x)$ is not limited, i.e., $-\infty \leq f(x) \leq \infty$. Note that for $x = \pm\pi/4$ (and all the other periodic locations) $\sin x = \cos x$; thus $f(x) = \tan x$ takes values ± 1.
10. *Summary of the important points:*

periodicity:	$T = \pi$
parity:	(odd)
v. asymptotes:	$x = \pi/2 \pm n\pi$
zeros:	$x = \pm n\pi$
y–axis crossing:	$(x, y) = (0, 0)$
extremes:	(none, $f(x)$ always ascends $\nearrow$)
inflections:	$(x, y) = (\pm n\pi, 0)$

11. *Graphical representation:* results of the analysis for function in P.3.98 are summarized by $f(x)$ graph (see Fig. 3.85).

3.99. Given that

$$\text{if, } y = \tan x \quad \text{then} \quad x = \arctan y$$

that is to say, geometrical interpretation of the relation between these two function is that, simply put, x-axis and y-axis exchanged their positions. In this case, form of $\arctan(x)$ function is identical to rotated version of $\tan(x)$ (more precisely, its section within the $-\pi/2$ and $\pi/2$ interval.)

1. *Domain of definition* $\mathbb{D}$: given that range of $\tan x$ is $-\infty \leq \tan x \leq +\infty$, then range of $\tan x$ translates into domain of $\arctan x$, i.e., $x \in \mathbb{R}$.
2. *Function's range:* similarly, the $\tan x$ domain translates into range of $\arctan x$, i.e., $-\pi/2 \leq \arctan x \leq \pi/2$. (Trigonometric functions do not have one-to-one mapping; thus this is the principal range, and there are two horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$.)

3. *Periodicity:* $\arctan x$ is not periodic, i.e., $\arctan(x+T) \neq \arctan x$ for all $T \neq 0$.
4. *Function's parity:* one possible method to prove $\arctan x$ function's parity may be as follows.

$$
\begin{aligned}
\underline{\text{odd:}} \quad f(-x) &= \arctan(-x) = \beta \\
&\left\{\tan(\arctan x) = x\right\} \\
&-x = \tan\beta \quad \therefore \quad x = -\tan\beta \\
&\left\{-\tan x = \tan(-x), \quad \text{see P.3.98}\right\} \\
&\therefore \quad x = \tan(-\beta) \\
&\therefore \quad \arctan x = -\beta \\
&\therefore \quad -\arctan x = \beta \\
&\therefore \quad \underbrace{\arctan x}_{f(x)} = -\underbrace{\arctan(-x)}_{f(-x)} \\
&\therefore \quad f(x) = -f(-x) \quad \therefore \quad \text{(odd)} \\
\underline{\text{even:}} \quad f(-x) &= -f(x) \neq f(x) \quad \therefore \quad \text{(not even)}
\end{aligned}
$$

In conclusion, function $\arctan x$ is odd but not even.

5. *Function's sign:*

$$
\underline{f(x) = 0:} \quad \arctan x = 0 \quad \therefore \quad x = \tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0
$$

That is to say, there is one zero at $(0, 0)$. With the help of unity circle, and as it is already found that $\arctan x$ is an odd function, it can be deduced that for $x > 0$, sign of $\arctan x$ is positive, and therefore for $x < 0$, sign of $\arctan x$ is negative.

6. *y-axis crossing point* $f(0)$:

$$
f(0) = \arctan 0 = 0
$$

7. *Limits:* limits $\lim_{x\to\pm\infty} \arctan x$ are deduced with the help of its inverse function $\tan x$, see P.3.98.

$$
\underline{x \to -\pi/2^{+}:} \quad \lim_{x\to -\pi/2^{+}} \tan x = -\infty \quad \therefore \quad \lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}
$$

$$
\underline{x \to \pi/2^{-}:} \quad \lim_{x\to \pi/2^{-}} \tan x = +\infty \quad \therefore \quad \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}
$$

8. *Oblique asymptote:*

$$
a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\arctan x}{x} = \left(\frac{|\pi/2|}{\infty}\right) = 0
$$

by consequence, there is no oblique asymptote.

9. *Critical points:*

$$f'(x) = 0: \quad f'(x) = (\arctan x)' = \frac{1}{x^2+1} > 0 \quad \therefore \quad f(x) \quad \text{ascends} \quad \nearrow \quad \text{for all} \quad x$$

because a quadratic function $(\cdot)^2 \geq 0$ is always positive. Then, either inverse or plus "1" is still positive. Therefore, there are no extreme points.

Also,

$$f''(x) = 0: \quad f''(x) = \left(\frac{1}{x^2+1}\right)' = -2\frac{x}{(x^2+1)^2} = 0$$
$$\therefore \quad x = 0$$

Thus,

x	$-\infty$		0		$+\infty$
-2	$-$	$-$	$-$	$-$	$-$
x	$-\infty$	$-$	0	$+$	∞
$(x^2+1)^2$	∞	$+$	$+$	$+$	∞
$f''(x)$	$0\downarrow$	$+$	0	$-$	$0\downarrow$
$f(x)$	$-\pi/2$	$\cup$	0	$\cap$	$\pi/2$

There is one inflection point at $(x, y) = (0, 0)$.

10. *Summary of the important points:*

parity:	(odd)
y–axis crossing:	$(x, y) = (0, 0)$
zeros:	$(x, y) = (0, 0)$
h. asymptotes:	$y = -\pi/2$
	$y = \pi/2$
extremes:	(none) $f(x)$ always ascends $\nearrow$
inflection:	$(x, y) = (0, 0)$

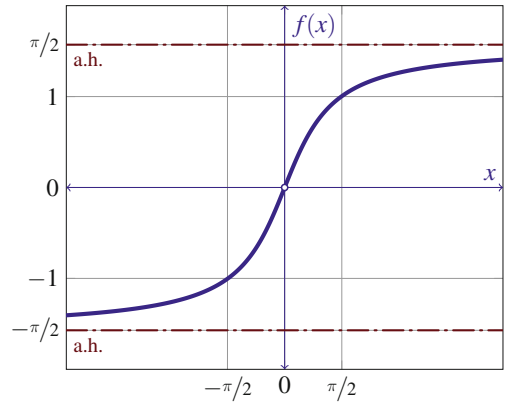
11. *Graphical representation:* results of the analysis for function in P.3.99 are summarized by $f(x)$ graph (see Fig. 3.86).

3.100. Given function,

$$f(x) = \frac{\pi}{2} + \arctan\left(-x^2 + 1\right)$$

1. *Domain of definition* $\mathbb{D}$: a constant plus arctan function is defined for all arguments, i.e., $x \in \mathbb{R}$.
2. *Periodicity:* $\arctan x$ is not periodic, i.e.,

Fig. 3.86 Example P.3.99



$$\frac{\pi}{2} + \arctan\big(-(x+T)^2+1\big) \neq \frac{\pi}{2} + \arctan\big(-x^2+1\big)$$

for all $T \neq 0$.

3. *Function's parity:*

$$\begin{aligned} \underline{\text{even:}} \qquad f(-x) &= \frac{\pi}{2} + \arctan\big(-(-x)^2+1\big) \\ &= \frac{\pi}{2} + \arctan\big(-x^2+1\big) = f(x) \quad \therefore \quad \text{(even)} \\ \underline{\text{odd:}} \quad -f(-x) &= -\frac{\pi}{2} - \arctan\big(-x^2+1\big) \neq f(x) \quad \therefore \quad \text{(not odd)} \end{aligned}$$

In conclusion, function $f(x)$ is even but not odd.

4. *Function's sign:* $f(x)$ is even; thus, it is sufficient to analyze only $x \geq 0$ side. With the help of $\arctan x$ function properties (see P.3.99), it can be deduced that:
   (a) limit $\lim_{x\to-\infty} \arctan(\,\cdot\,) = -\frac{\pi}{2}$ for all real arguments of arctan() function,
   (b) therefore, after adding $\pi/2$ it must be $\lim_{\infty}\big(\pi/2 + \arctan(-x^2+1)\big) = 0$,
   (c) and, $f(0) = \pi/2 + \arctan(0+1) = \pi/2 + \pi/4 = 3\pi/4 > 0$,
   (d) this function is even, therefore, $f(x) > 0$ for all x.
   That is to say, $f(x)$ has no zeros; however, there is one horizontal asymptote at $y = 0$

5. *y-axis crossing point* $f(0)$:

$$f(0) = \frac{\pi}{2} + \arctan\big(-(0)^2+1\big) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

6. *Limits:* limits $\lim_{x\to\pm\infty} \arctan x$ are deduced with the help of its inverse function $\tan x$ (see P.3.98).

$$\underline{x \to \pm\infty:} \quad \lim_{x\to\pm\infty}\left(\frac{\pi}{2} + \arctan\big(-(\pm\infty)^2+1\big)\right) = 0 \qquad \text{(} \arctan\big(-(\pm\infty)^2+1\big) \to -\pi/2 \text{)}$$

that is to say, $y = 0$ is horizontal asymptote.

7. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\arctan(\,\cdot\,)}{x} = \left(\frac{|\pi/2|}{\infty}\right) = 0$$

by consequence, there is no oblique asymptote.

8. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\frac{\pi}{2} + \arctan\left(-x^2+1\right)\right)'$$

$$\left\{\text{composite function: } \left(\arctan g(x)\right)' = \frac{g'(x)}{g^2(x)+1}\right\}$$

$$= \frac{-2x}{(-x^2+1)^2+1} = -2\,\frac{x}{x^4-2x^2+2} = 0 \quad \therefore \quad x = 0$$

where $f(0) = 3\pi/4$. Then,

x		0	
-2	$-$	$-$	$-$
x	$-$	0	$+$
x^4-2x^2+2	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	$3\pi/4$	$\searrow$

There is one maximum point at $(x, y) = (0, 3\pi/4)$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = -2\left(\frac{x}{x^4-2x^2+2}\right)' = -2\,\frac{(x^4-2x^2+2) - x(4x^3-4x)}{(x^4-2x^2+2)^2}$$

$$= 2\,\frac{3x^4-2x^2-2}{(x^4-2x^2+2)^2} = 0$$

$$\therefore \quad 3x^4-2x^2-2 = 0 \quad \therefore \quad \{t = x^2\} \quad \therefore \quad 3t^2-2t-2 = 0 \quad \therefore$$

$$t_{1,2} = \frac{2 \pm \sqrt{4+24}}{6} = \frac{1 \pm \sqrt{7}}{3}, \quad \left(\text{note: } 1-\sqrt{7} < 0\right)$$

$$\therefore \quad x_{1,2} = \pm\sqrt{\frac{1+\sqrt{7}}{3}} \approx \pm 1.102$$

where, $x_2 = -x_1$, and $f(x_{1,2}) \approx 2.405 = y_1$ rounded to three decimal places.

Thus,

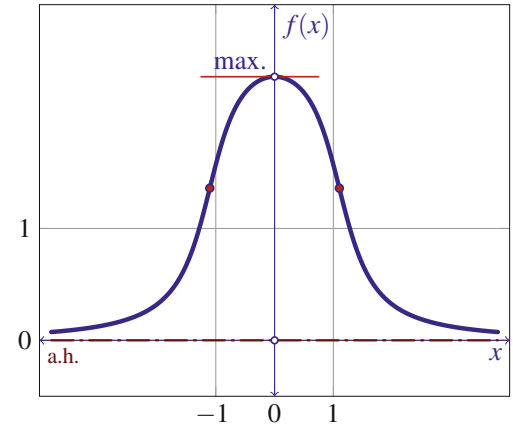


Fig. 3.87 Example P.3.100

x		$-x_1$		x_1	
2	+	+	+	+	+
$x+x_1$	−	0	+	+	+
$x-x_1$	−	−	−	0	+
$(x^4-2x^2+2)^2$	+	+	+	+	+
$f''(x)$	+	0	−	0	+
$f(x)$	∪	y_1	∩	y_1	∪

There are two inflection points, as calculated above.

9. *Function's range:* this functions occupies the interval between its horizontal asymptote, $y=0$, and its maximum at $y=3\pi/4$.
10. *Summary of the important points:*

parity:	(even)
zeros:	(none, $f(x)$ is always positive)
h. asymptote:	$y=0$
y–axis crossing:	$(x,y)=(0,3\pi/4)$
extremes:	$(x,y)=(0,3\pi/4)$ (max)
inflections:	$(x,y)=(-1.102,2.405)$
	$(x,y)=(1.102,2.405)$

11. *Graphical representation:* results of the analysis for function in P.3.100 are summarized by $f(x)$ graph (see Fig. 3.87).

3.101. Given function,

$$f(x)=\frac{1}{2}\left(1-\cos(2x)\right)$$

a simple trigonometric transformation shows that,

$$f(x) = \frac{1}{2}\big(1 - \cos(2x)\big) = \frac{1}{2}\Big(1 - \big(\cos^2 x - \sin^2 x\big)\Big) = \frac{1}{2}\Big(\cancel{\cos^2 x} + \sin^2 x - \cancel{\cos^2 x} + \sin^2 x\Big)$$
$$= \sin^2 x$$

therefore, this function is already analyzed in P.3.91 (see Fig. 3.78).

3.102. Given function,

$$f(x) = 2\sin^2 x + \sin x, \quad x \in [-\pi, \pi]$$

1. *Domain of definition* $\mathbb{D}$: sinusoid functions are defined for all real arguments, and so is their sum, i.e., $x \in \mathbb{R}$.
2. *Periodicity:* $\sin x$ is periodic with $T = 2\pi$ (see P.3.90), while $\sin^2 x$ is periodic with $T = \pi$ (see P.3.91). The lowest common denominator for π and 2π is therefore $T = 2\pi$,

$$2\sin^2(x + 2\pi) + \sin(x + 2\pi) = 2\sin^2 x + \sin x$$

for all $T \neq 0$.

3. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = 2\sin^2(-x) + \sin(-x) = 2\sin(-x)\ \sin(-x) + \sin(-x)$$
$$= 2\sin^2 x - \sin x \neq f(x) \quad \therefore \quad \text{(not even)}$$
$$\underline{\text{odd:}} \quad -f(-x) = -2\sin^2 x + \sin x \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $f(x)$ is neither even nor odd.

4. *Function's sign:* within $x \in [-\pi, \pi]$

$$\underline{f(x) = 0:} \quad 2\sin^2 x + \sin x = 0 \quad \therefore \quad \sin x\,(2\sin x + 1) = 0$$
$$\begin{cases} \sin x = 0 & \therefore \quad x = 0, \pi, -\pi \\ \text{or,} & \\ 2\sin x + 1 = 0 & \therefore \quad \sin x = -\dfrac{1}{2} \quad \therefore \quad x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6} \end{cases}$$

That is to say, over one period of 2π

x	$-\pi$		$-5\pi/6$		$-\pi/6$		0		π
$\sin x$	0	−	−	−	−	−	0	+	0
$2\sin x + 1$	1	+	0	−	0	+	+	+	+
$f(x)$	0	−	0	+	0	−	0	+	0

5. *y-axis crossing point* $f(0)$:

$$f(0) = 2\sin^2 0 + \sin 0 = 0$$

6. *Limits:* limits $\lim_{x\to\pm\infty}\sin x$ and its square are not defined.
7. *Oblique asymptote:*

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{2\sin^2 x+\sin x}{x}=\left(\frac{\text{n.d.}}{\infty}\right)$$

by consequence, there is no oblique asymptote.

8. *Critical points:*

$$\underline{f'(x)=0:}\quad f'(x)=\left(2\sin^2 x+\sin x\right)'$$
$$=4\sin x\cos x+\cos x=\cos x\,(4\sin x+1)=0$$

$$\begin{cases}\cos x=0 & \therefore\ x=\dfrac{\pi}{2},-\dfrac{\pi}{2}\\ \text{or,}\\ 4\sin x+1=0 & \therefore\ \sin x=-\dfrac{1}{4}\ \therefore\ x_1=\arcsin\left(-\dfrac{1}{4}\right)\\ & \therefore\ x_2=\pi+\arcsin\left(\dfrac{1}{4}\right)\\ & \text{(see Fig. 3.88 (left))}\end{cases}$$

where,

$$f(-\pi/2)=2\sin^2(-\pi/2)+\sin(-\pi/2)=2(-1)^2+(-1)=1$$
$$f(\pi/2)=2\sin^2(\pi/2)+\sin(\pi/2)=2(1)^2+(1)=3$$
$$\{\sin(\arcsin x)=x\}$$
$$f(x_{1,2})=2\sin^2\left(\arcsin(-1/4)\right)+\sin\left(\arcsin(-1/4)\right)=2\,(-1/4)^2-1/4=-1/8$$

Then,

x	$-\pi$		x_2		$-\pi/2$		x_1		$\pi/2$		π
$\cos x$	-1	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	-1
$4\sin x+1$	1	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	1
$f'(x)$	-1	$-$	0	$+$	0	$-$	0	$+$	0	$-$	-1
$f(x)$	0	$\searrow$	$-1/8$	$\nearrow$	1	$\searrow$	$-1/8$	$\nearrow$	3	$\searrow$	0

Within the given interval, there are four extreme points, two minimum and two maximum points, as listed above. Also,

$$\underline{f''(x)=0:}\quad f''(x)=(4\sin x\cos x+\cos x)'=4\cos^2 x-4\sin^2 x-\sin x$$
$$=-8\sin^2 x-\sin x+4=0$$
$$\{t=\sin x\}$$

$$\therefore \quad -8t^2 - t + 4 = 0 \quad \therefore \quad t_{1,2} = \frac{1 \pm \sqrt{129}}{-16}$$

$$\therefore \quad x_{1,2} = \arcsin\left(\frac{1 \pm \sqrt{129}}{-16}\right)$$

$$x_{3,4} = \pm\pi \pm \arcsin\left(\frac{1 \pm \sqrt{129}}{-16}\right)$$

where $f''(x)$ zeros x_i are periodic. Within $[-\pi \le x \le \pi]$ period, there are four inflection points.

$$f(x_1) = 2\sin^2\left[-\pi - \arcsin\left(\frac{1+\sqrt{129}}{-16}\right)\right] + \sin\left[-\pi - \arcsin\left(\frac{1+\sqrt{129}}{-16}\right)\right]$$

$$\left\{\begin{aligned} \sin(-x) &= -\sin x; \quad \arcsin(-x) = -\arcsin x \\ \sin(x+y) &= \cos x \sin y + \sin x \cos y \end{aligned}\right\}$$

$$= \frac{61 - \sqrt{129}}{64}$$

$$f(x_2) = 2\sin^2\left[\arcsin\left(\frac{1+\sqrt{129}}{-16}\right)\right] + \sin\left[\arcsin\left(\frac{1+\sqrt{129}}{-16}\right)\right]$$

$$= 2\left(\frac{1+\sqrt{129}}{-16}\right)^2 + \left(\frac{1+\sqrt{129}}{-16}\right) = \frac{61 - 3\sqrt{129}}{64}$$

$$f(x_3) = 2\sin^2\left[\arcsin\left(\frac{1-\sqrt{129}}{-16}\right)\right] + \sin\left[\arcsin\left(\frac{1-\sqrt{129}}{-16}\right)\right]$$

$$= 2\left(\frac{1-\sqrt{129}}{-16}\right)^2 + \left(\frac{1-\sqrt{129}}{-16}\right) = \frac{61 + 3\sqrt{129}}{64}$$

$$f(x_4) = 2\sin^2\left[\pi + \arcsin\left(\frac{1+\sqrt{129}}{-16}\right)\right] + \sin\left[\pi + \arcsin\left(\frac{1+\sqrt{129}}{-16}\right)\right]$$

$$\left\{\begin{aligned} \sin(-x) &= -\sin x; \arcsin(-x) = -\arcsin x \\ \sin(x+y) &= \cos x \sin y + \sin x \cos y \end{aligned}\right\}$$

$$= \frac{61 + \sqrt{129}}{64}$$

Thus,

x	$-\pi$		x_1		x_2		x_3		x_4		π
$x-x_1$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x-x_2$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$
$x-x_3$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$x-x_4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$
$f''(x)$	$+$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$+$
$f(x)$	0	$\cup$	y_1	$\cap$	y_2	$\cup$	y_3	$\cap$	y_4	$\cup$	0

Within the given interval, there are four inflection points as listed above.

9. *Function's range:* this function occupies the interval between its minimum at $y=-1/8$ and its absolute maximum at $y=3$.
10. *Summary of the important points:* within $x\in[-\pi,\pi]$ interval

$$\begin{aligned}
\underline{\text{periodicity:}}\quad & T=2\pi\\
\underline{\text{y–axis crossing:}}\quad & (x,y)=(0,0)\\
\underline{\text{zeros:}}\quad & (x,y)=(-\pi,0)\\
& (x,y)=(-5\pi/6,0)\\
& (x,y)=(-\pi/6,0)\\
& (x,y)=(0,0)\\
& (x,y)=(\pi,0)\\
\underline{\text{extremes:}}\quad & (x,y)=\left(\pi+\arcsin\left(\frac{1}{4}\right),\frac{-1}{8}\right)\ (\text{min})\\
& (x,y)=\left(-\frac{\pi}{2},1\right)\ (\text{max})\\
& (x,y)=\left(\arcsin\left(-\frac{1}{4}\right),\frac{-1}{8},\right)\ (\text{min})\\
& (x,y)=\left(\frac{\pi}{2},3\right)\ (\text{max})
\end{aligned}$$

$$\begin{aligned}
\underline{\text{inflections:}}\quad & (x,y)=\left(-\pi-\arcsin\left(\frac{1+\sqrt{129}}{-16}\right),\frac{61-\sqrt{129}}{64}\right)\\
& (x,y)=\left(\arcsin\left(\frac{1+\sqrt{129}}{-16}\right),\frac{61-3\sqrt{129}}{64}\right)\\
& (x,y)=\left(\arcsin\left(\frac{1-\sqrt{129}}{-16}\right),\frac{61+3\sqrt{129}}{64}\right)\\
& (x,y)=\left(\pi+\arcsin\left(\frac{1+\sqrt{129}}{-16}\right),\frac{61+\sqrt{129}}{64}\right)
\end{aligned}$$

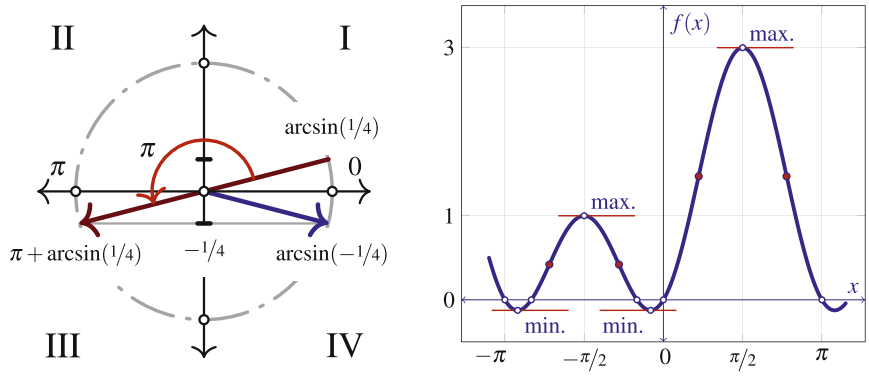


Fig. 3.88 Example P.3.102

11. *Graphical representation:* results of the analysis for function in P.3.102 are summarized by $f(x)$ graph (see Fig. 3.88). Unity circle showing $\arcsin(-1/4)$ angles in **IV** and **III** quadrants where two minimum points are located (left) and complete $f(x)$ graph (right).

3.103. Given function,

$$\begin{aligned} f(x) &= \sin(2x) - \sin\left(x - \frac{\pi}{2}\right) \\ &\quad \left\{\sin(x - \pi/2) = -\cos x; \quad \sin(2x) = 2\sin x \cos x\right\} \\ &= \sin(2x) + \cos x \\ &= 2\sin x \cos x + \cos x \\ &= \cos x(2\sin x + 1), \quad x \in [-\pi, \pi] \end{aligned}$$

1. *Domain of definition* $\mathbb{D}$: sinusoid functions are defined for all real arguments, and so is their sum, i.e., $x \in \mathbb{R}$.
2. *Periodicity:* $\sin 2x$ is periodic with $T = \pi$, while $\sin(x - \pi/2) \equiv -\cos x$ is periodic with $T = 2\pi$. The lowest common denominator for π and 2π is therefore $T = 2\pi$,

$$\sin\left(2(x + \pi)\right) + \cos(x + 2\pi) = \sin(2x) + \cos x$$

for all $T \neq 0$.

3. *Function's parity:*

$$\begin{aligned} &\underline{\text{even:}} \quad f(-x) = \sin\left(2(-x)\right) + \cos(-x) = -\sin(2x) + \cos(x) \neq f(x) \;\therefore \text{ (not even)} \\ &\underline{\text{odd:}} \quad -f(-x) = \sin(2x) - \cos(x) \neq f(x) \quad \therefore \quad \text{(not odd)} \end{aligned}$$

In conclusion, function $f(x)$ is neither even nor odd.

4. *Function's sign:* within $x \in [-\pi, \pi]$

$$\underline{f(x) = 0:} \quad \cos x(2\sin x + 1) = 0$$

$$\begin{cases} \cos x = 0 & \therefore \quad x = \pm\dfrac{\pi}{2} \\ \text{or,} & \\ 2\sin x + 1 = 0 & \therefore \quad \sin x = -\dfrac{1}{2} \quad \therefore \quad x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6} \end{cases}$$

That is to say, over one period $x \in [-\pi, \pi]$

x	$-\pi$		$-5\pi/6$		$-\pi/2$		$-\pi/6$		$\pi/2$		π
$\cos x$	-1	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	-1
$2\sin x + 1$	1	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	1
$f(x)$	-1	$-$	0	$+$	0	$-$	0	$+$	0	$-$	-1

5. *y-axis crossing point* $f(0)$:

$$f(0) = \cos 0\,(2\sin 0 + 1) = 1$$

6. *Limits:* limits $\lim_{x\to\pm\infty} \sin x$ and $\cos x$ and are not defined.
7. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sin(2x) + \cos x}{x} = \left(\frac{\text{n.d.}}{\infty}\right)$$

by consequence, there is no oblique asymptote.
8. *Critical points:*

$$\begin{aligned} \underline{f'(x) = 0:} \quad f'(x) &= (\sin(2x) + \cos x)' = 2\cos(2x) - \sin x \\ &\left\{\cos(2x) = 1 - 2\sin^2 x\right\} \\ &= 2\left(1 - 2\sin^2 x\right) - \sin x = -4\sin^2 x - \sin x + 2 = 0 \\ &\left\{t = \sin x\right\} \\ &\therefore \quad -4t^2 - t + 2 = 0 \quad \therefore \quad t_{1,2} = \frac{\pm\sqrt{33} - 1}{8} \\ &\therefore \quad x_{1,2} = \arcsin\left(\frac{\pm\sqrt{33} - 1}{8}\right) \\ &\qquad x_{3,4} = \pm\pi \pm \arcsin\left(\frac{\pm\sqrt{33} - 1}{8}\right) \end{aligned}$$

$f'(x)$ zeros $x_{\mathtt{i}}$ are periodic. Thus, within $[-\pi \le x \le \pi]$ period, there are four extreme points

$$f(x_{\mathtt{1}}) = \sin\left[2\left(-\pi - \arcsin\left(\frac{-\sqrt{33} - 1}{8}\right)\right)\right] + \cos\left[-\pi - \arcsin\left(\frac{-\sqrt{33} - 1}{8}\right)\right]$$

$$\left\{\begin{array}{l}\sin(-x)=-\sin x;\quad \arcsin(-x)=-\arcsin x;\\ \sin(x+y)=\cos x\sin y+\sin x\cos y;\\ \sin(\arcsin x)=x;\quad \cos(\arcsin x)=\sqrt{1-x^2}\end{array}\right\}$$

$$=\frac{\sqrt{6\left(69-11\sqrt{33}\right)}}{16}$$

$$f(x_2)=\sin\left[2\arcsin\left(\frac{-\sqrt{33}-1}{8}\right)\right]+\cos\left[\arcsin\left(\frac{-\sqrt{33}-1}{8}\right)\right]$$

$$=-\frac{\sqrt{6\left(69-11\sqrt{33}\right)}}{16}$$

$$f(x_3)=\sin\left[2\arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right]+\cos\left[\arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right]$$

$$=\frac{\sqrt{6\left(69+11\sqrt{33}\right)}}{16}$$

$$f(x_4)=\sin\left[2\left(\pi-\arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right)\right]+\cos\left[\pi-\arcsin\left(\frac{\sqrt{33}-1}{8}\right)\right]$$

$$=-\frac{\sqrt{6\left(69+11\sqrt{33}\right)}}{16}$$

Then, within given interval, there are two minimums and two maximums, as listed below.

x	$-\pi$		x_1		x_2		x_3		x_4		π
$x-x_1$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x-x_2$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$
$x-x_3$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$x-x_4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$
$f'(x)$	$+$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$+$
$f(x)$	-1	$\nearrow$	y_1	$\searrow$	y_2	$\nearrow$	y_3	$\searrow$	y_4	$\nearrow$	-1

Also,

$$\underline{f''(x)=0:}\quad f''(x)=(2\cos(2x)-\sin x)'=-4\sin(2x)-\cos x$$
$$=-8\sin x\cos x-\cos x=-\cos x(8\sin x+1)=0$$

therefore,

$$\begin{cases} -\cos x = 0 & \therefore \quad x = \pm\dfrac{\pi}{2} \\ \text{or,} & \\ 8\sin x + 1 = 0 & \therefore \quad \sin x = -\dfrac{1}{8} \quad \therefore \quad x = \pm\pi \pm \arcsin\left(-\dfrac{1}{8}\right) \end{cases}$$

where $f''(x)$ zeros x_i are periodic. Within $[-\pi \le x \le \pi]$ period, there are four inflection points located at

$$f(x_1) = \sin\left[2\left(-\pi - \arcsin\left(-\frac{1}{8}\right)\right)\right] + \cos\left[-\pi - \arcsin\left(-\frac{1}{8}\right)\right] = -\frac{9\sqrt{7}}{32}$$

$$f(x_2) = \sin\left[2\left(-\frac{\pi}{2}\right)\right] + \cos\left(-\frac{\pi}{2}\right) = 0$$

$$f(x_3) = \sin\left[2\left(\arcsin\left(-\frac{1}{8}\right)\right)\right] + \cos\left[\arcsin\left(-\frac{1}{8}\right)\right] = \frac{9\sqrt{7}}{32}$$

$$f(x_4) = \sin\left[2\left(\frac{\pi}{2}\right)\right] + \cos\left(\frac{\pi}{2}\right) = 0$$

Thus, within the given interval, there are four inflection points as listed below.

x	$-\pi$		x_1		$-\pi/2$		x_3		$\pi/2$		π
$x - x_1$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$
$x - x_3$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$x - x_4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$
$f''(x)$	$+$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$+$
$f(x)$	-1	$\cup$	y_1	$\cap$	0	$\cup$	y_3	$\cap$	0	$\cup$	-1

9. *Function's range:* is the interval between its absolute minimum and its absolute maximum.
10. *Summary of the important points:* within $x \in [-\pi, \pi]$ interval

periodicity: $T = 2\pi$

y–axis crossing: $(x, y) = (0, 1)$

zeros: $(x, y) = (-5\pi/6, 0)$

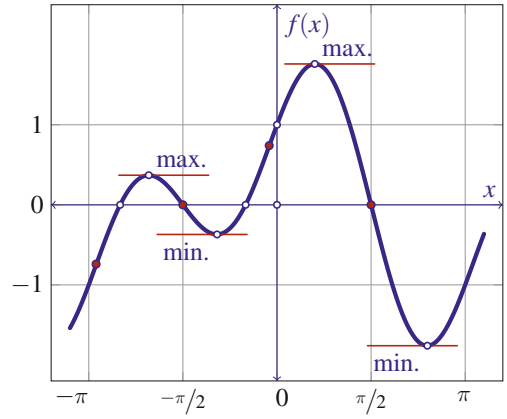
$(x, y) = (-\pi/2, 0)$

$(x, y) = (-\pi/6, 0)$

$(x, y) = (\pi/2, 0)$

extremes: $(x, y) = \left(-\pi - \arcsin\left(\dfrac{-\sqrt{33}-1}{8}\right), \dfrac{\sqrt{6\left(69 - 11\sqrt{33}\right)}}{16}\right)$ (max)

Fig. 3.89 Example
P.3.103



$$(x, y) = \left(\arcsin\left(\frac{-\sqrt{33}-1}{8}\right), \frac{\sqrt{6\left(69-11\sqrt{33}\right)}}{16} \right) \text{ (min)}$$

$$(x, y) = \left(\arcsin\left(\frac{\sqrt{33}-1}{8}\right), \frac{\sqrt{6\left(69+11\sqrt{33}\right)}}{16} \right) \text{ (max)}$$

$$(x, y) = \left(\pi - \arcsin\left(\frac{\sqrt{33}-1}{8}\right), \frac{\sqrt{6\left(69+11\sqrt{33}\right)}}{16} \right) \text{ (min)}$$

inflections: $(x, y) = \left(-\pi - \arcsin(-1/8), -9\sqrt{7}/32\right)$

$(x, y) = (-\pi/2, 0)$

$(x, y) = \left(\arcsin(-1/8), 9\sqrt{7}/32\right)$

$(x, y) = (\pi/2, 0)$

11. *Graphical representation:* see Fig. 3.89.

3.6 Composite Functions

3.104. Given function,

$$f(x) = x^4 - 2x^2 = x^2\left(x^2-2\right) = x^2\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)$$

1. *Domain of definition* $\mathbb{D}$: polynomial functions are defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x) \quad \therefore \quad \text{(even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -x^4 + 2x^2 \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $x^4 - 2x^2$ is even but not odd.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad x^4 - 2x^2 = 0 \quad \therefore \quad x^2\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) = 0$$

$$\therefore \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = -\sqrt{2}, \quad x_4 = \sqrt{2}$$

That is to say, there is one double zero and two single zeros of $f(x)$, as

x		$-\sqrt{2}$		0		$\sqrt{2}$	
x^2	$+$	$+$	$+$	0	$+$	$+$	$+$
$x+\sqrt{2}$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-\sqrt{2}$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$-$	0	$+$

4. *y-axis crossing point* $f(0)$: $f(0) = 0^4 - 2 \cdot 0^2 = 0 = 0$
5. *Limits:* there are two limits to resolve.

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x^4 - 2x^2\right) = +\infty$$

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(x^4 - 2x^2\right) = +\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{x^4 - 2x^2}{x} = \lim_{x\to\infty}\left(x^3 - 2x\right) = \infty$$

by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(x^4 - 2x^2\right)' = 4x^3 - 4x = 4x\left(x^2 - 1\right)$$

$$= 4x\,(x+1)\,(x-1) = 0$$

$$\therefore \quad x_1 = 0, \quad x_2 = -1, \quad x_3 = 1$$

Thus, there are three single zeros of $f'(x)$ as

x		-1		0		1	
4	$+$	$+$	$+$	$+$	$+$	v	v
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	$\searrow$	-1	$\nearrow$	0	$\searrow$	-1	$\nearrow$

There are three extreme points, two minimums at $(x, y) = (-1, -1)$ and $(x, y) = (1, -1)$, and one maximum point at $(x, y) = (0, 0)$. Also,

$$f''(x) = 0: \quad f''(x) = \left(4x^3 - 4x\right)' = 12x^2 - 4 = 4(3x^2 - 1) = 0$$

$$3x^2 - 1 = 0 \quad \therefore \quad x_{1,2} = \pm\sqrt{\frac{1}{3}}$$

where inflection points are located at

$$f\left(\pm\sqrt{\frac{1}{3}}\right) = \left(\pm\sqrt{\frac{1}{3}}\right)^4 - 2\left(\pm\sqrt{\frac{1}{3}}\right)^2 = -\frac{5}{9} = y_{1,2}$$

Thus,

x		x_1		x_2	
4	$+$	$+$	$+$	$+$	$+$
$x - x_1$	$-$	0	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	0	$+$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$	y_1	$\cap$	y_2	$\cup$

There are two inflection points, at $(x, y) = (-\sqrt{1/3}, -5/9)$ and $(x, y) = (\sqrt{1/3}, -5/9)$.

8. *Summary of the important points:*

parity: (even)

zeros: $(x, y) = (0, 0)$

$(x, y) = (0, 0)$

$(x, y) = \left(-\sqrt{2}, 0\right)$

$(x, y) = \left(\sqrt{2}, 0\right)$

extremes: $(x, y) = (-1, -1)$ (min)

$(x, y) = (0, 0)$ (max)

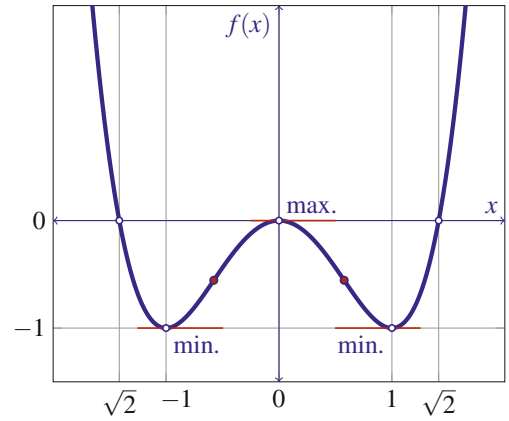


Fig. 3.90 Example
P.3.104

$$(x, y) = (1, -1) \quad \text{(min)}$$

$$\underline{\text{inflections:}} \quad (x, y) = \left(-\sqrt{1/3}, -5/9\right)$$

$$(x, y) = \left(\sqrt{1/3}, -5/9\right)$$

9. *Graphical representation:* the above analysis is summarized in Fig. 3.90.

3.105. Given function,

$$f(x) = e^{1/x}$$

1. *Domain of definition* $\mathbb{D}$: an exponential function is defined for all real arguments, but rational function $1/x$ is not defined for $x = 0$; therefore, $f(x)$ is defined for $x \in \mathbb{R} | x \neq 0$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \exp\left(1/(-x)\right) \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\exp\left(1/(-x)\right) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

   In conclusion, function $\exp\left(1/x\right)$ is neither even nor odd.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad \exp\left(1/(-x)\right) > 0$$

   That is to say, there are no zeros because an exponential function is positive for all arguments.
4. *y-axis crossing point* $f(0)$: $f(0) = \exp\left(1/(0)\right)$ is not defined.
5. *Limits:* there are four limits to resolve.

$$\underline{x \to -\infty:} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \exp\left(1/x\right) = \left(\exp\left(\underline{1/(-\infty)} \to 0\uparrow\right)\right) = \exp(0\uparrow) = 1\uparrow$$

$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \exp\left(1/x\right) = \left(\exp\left(\frac{1}{(+\infty)} \to 0\downarrow\right)\right) = \exp(0\downarrow) = 1\downarrow$$

$$x \to \underset{\to}{0}: \quad \lim_{x\to\underset{\to}{0}} f(x) = \lim_{x\to\underset{\to}{0}} \exp\left(1/x\right) = \exp\left(\lim_{x\to\underset{\to}{0}} \frac{1}{x}\right) = \exp(-\infty) = 0\downarrow$$

$$x \to \underset{\leftarrow}{0}: \quad \lim_{x\to\underset{\leftarrow}{0}} f(x) = \lim_{x\to\underset{\leftarrow}{0}} \exp\left(1/(-x)\right) = \exp\left(\lim_{x\to\underset{\leftarrow}{0}} \frac{1}{x}\right) = \exp(\infty) = +\infty$$

That is to say, $\lim_{x\to 0} f(x)$ does not exist because the two side limits are different (caused by limits of $\lim_{x\to 0} 1/x$). At the same time, there is one horizontal asymptote at $y = 1$ where $\lim_{x\to\pm\infty} f(x)$ tends.

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\exp\left(1/x\right)}{x} = \lim_{x\to\infty} \left(\frac{1}{x}\right) = 0$$

$$\therefore$$

$$b = \lim_{x\to\infty} \left(f(x) - ax\right) \lim_{x\to\infty} f(x) = 1$$

Therefore, there is only horizontal asymptote at $y = 1$.

7. *Critical points:*

$$f'(x) = 0: \quad f'(x) = (\exp(1/x))' = -\frac{\exp(1/x)}{x^2} \neq 0$$

because an exponential function by itself is positive, in this case multiplied by "-1" so that $f(x)$ always descends $\searrow$. Also,

$$f''(x) = 0: \quad f''(x) = \left(-\frac{\exp(1/x)}{x^2}\right)' = \frac{(2x+1)\,\exp(1/x)}{x^4} = 0$$

$$2x + 1 = 0 \quad \therefore \quad x_1 = -\frac{1}{2}$$

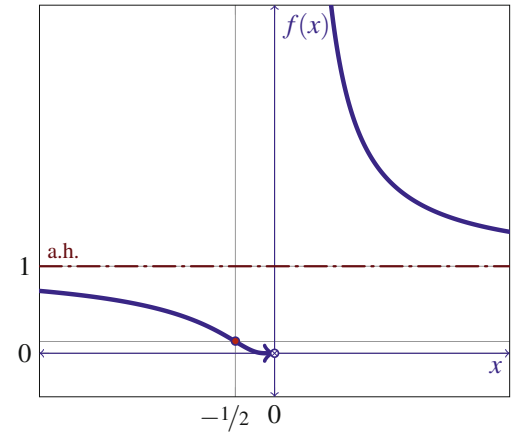
where inflection point is located at

$$f\left(-1/2\right) = \exp\left(1/(-1/2)\right) = \frac{1}{e^2} = y_1$$

Thus,

x		x_1		0	
x^4	+	+	+	+	+
$2x+1$	−	0	+	+	+
$\exp(1/x)$	+	+	+	n.d.	+
$f''(x)$	−	0	+	n.d.	+
$f(x)$	∩	y_1	∪	n.d.	∪

Fig. 3.91 Example
P.3.105



8. *Summary of the important points:*

defined:	$x \neq 0$
h. asymptote:	$y = 1$
v. asymptote:	$x = 0 \;\; (y > 0)$
zeros:	(none, $f(x)$ is always positive)
extremes:	(none), $f(x)$ always descends $\searrow$
inflection:	$(x, y) = \left(-1/2, 1/e^2\right)$

9. *Graphical representation:* the above analysis is summarized in Fig. 3.91.

3.106. Given function,

$$f(x) = x^x$$

1. *Domain of definition* $\mathbb{D}$: the form 0^0 is not defined. In addition,

$$f(x) = x^x = \left\{e^{\ln x} = x\right\} = \left(e^{\ln x}\right)^x = \left\{\left(e^a\right)^b = e^{ab}\right\} = \left(e^x\right)^{\ln x} = e^{x \ln x}$$

that is to say, $x > 0$ due to the domain of this "hidden" $\ln x$ function. Therefore $f(x)$ is defined for $x \in \mathbb{R} | x > 0$.

2. *Function's parity:*

$$\text{even:} \quad f(-x) = (-x)^{(-x)} = \frac{1}{-x^x} \neq f(x) \;\; \therefore \;\; \text{(not even)}$$

$$\text{odd:} \quad -f(-x) = \frac{1}{x^x} \neq f(x) \;\; \therefore \;\; \text{(not odd)}$$

In conclusion, function x^x is neither even nor odd.

3. *Function's sign:*

$$\underline{f(x)=0:}\quad x^x>0$$

That is to say, given that $x>0$, there are no zeros because an exponential function is positive for all arguments.

4. *y-axis crossing point* $f(0)$: $f(0)=0^0$ is not defined.
5. *Limits:* there are two limits to resolve,

$$\begin{aligned}
\underline{x\to+\infty:}\quad \lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} x^x = \left(\infty^\infty\right) = \lim_{x\to+\infty} e^{x\ln x} = e^{\lim_{x\to+\infty} x\ln x}\\
&= e^{+\infty} = +\infty\\
\underline{x\to \underset{\leftarrow}{0}:}\quad \lim_{x\to \underset{\leftarrow}{0}} f(x) &= \lim_{x\to \underset{\leftarrow}{0}} x^x = \left(0^0\right) = \lim_{x\to \underset{\leftarrow}{0}} e^{x\ln x}\\
&= e^{\lim_{x\to \underset{\leftarrow}{0}} x\ln x} \quad \{\text{see P.2.97 for lim xlnx}\}\\
&= e^0 = 1
\end{aligned}$$

6. *Oblique asymptote:*

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^x}{x}=\left(\frac{\infty}{\infty}\right)\overset{\text{l'H}}{=}\lim_{x\to\infty}\frac{x^x\ (\ln x+1)}{1}=\infty$$

Therefore, there is no oblique asymptote.

7. *Critical points:*

$$\begin{aligned}
\underline{f'(x)=0:}\quad f'(x)&=\left(x^x\right)'=\left(e^{x\ln x}\right)'=\underbrace{e^{x\ln x}}_{x^x}\ (x\ln x)'=x^x\left(\ln x+\frac{\cancel{x}}{\cancel{x}}\right)\\
&=x^x\ (\ln x+1)=0,\quad (x\neq 0)\\
&\therefore\ \ln x+1=0\ \ \therefore\ \ln x=-1\ \ \therefore\ \cancel{e^{\ln x}}=e^{-1}\ \ \therefore\ x=\frac{1}{e}
\end{aligned}$$

There is one zero of $f'(x)$ at $x=1/e$ where $f(1/e)=(1/e)^{(1/e)}\approx 0.692$. Also,

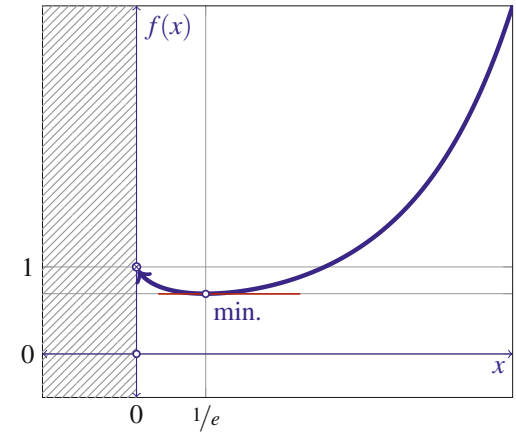
$$\begin{aligned}
\underline{f''(x)=0:}\quad f''(x)&=\left(x^x\ (\ln x+1)\right)'=x^x\ (\ln x+1)\ (\ln x+1)+x^x\ \frac{1}{x}\\
&=x^x\ (\ln x+1)^2+x^{x-1}>0
\end{aligned}$$

because both terms in the sum are positive. Consequently, there are no inflection points, i.e., $f(x)$ is always concave $\cup$.

8. *Summary of the important points:*

$$\begin{aligned}
\underline{\text{defined:}}\quad & x>0\\
\underline{\text{zeros:}}\quad & \text{(none, } f(x) \text{ is always positive)}\\
\underline{\text{extremes:}}\quad & (x,y)=\left(1/e,(1/e)^{1/e}\right)\quad \text{(min)}\\
\underline{\text{inflections:}}\quad & \text{(none, } f(x) \text{ always concave } \cup)
\end{aligned}$$

Fig. 3.92 Example
P.3.106



9. *Graphical representation:* the above analysis is summarized in Fig. 3.92.

3.107. Given function,

$$f(x) = (2 - x^2)\, e^x$$

1. *Domain of definition* $\mathbb{D}$: general form of a polynomial and exponential functions product is defined for all real arguments, because each of the two terms is a continuous function; therefore, $f(x)$ is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = (2 - (-x)^2)\, e^{-x} = \frac{2 - x^2}{e^x} \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\underline{\text{odd:}} \quad -f(-x) = -\frac{2 - x^2}{e^x} \neq f(x) \quad \therefore \quad \text{(not odd)}$$

   In conclusion, function $f(x)$ is neither even nor odd.
3. *Function's sign:*

$$\underline{f(x) = 0:} \quad (2 - x^2)\, e^x = -(x^2 - 2)\, e^x = -(x + \sqrt{2})(x - \sqrt{2})\, e^x = 0$$

$$\therefore \quad x_{1,2} = \pm\sqrt{2}$$

$$\therefore \quad (x, y) = (-\sqrt{2}, 0)$$

$$(x, y) = (\sqrt{2}, 0)$$

   So that,

x		$-\sqrt{2}$		$\sqrt{2}$	
-1	$-$	$-$	$-$	$-$	$-$
$x+\sqrt{2}$	$-$	0	$+$	$+$	$+$
$x-\sqrt{2}$	$-$	$-$	$-$	0	$+$
e^x	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$-$

4. *y-axis crossing point* $f(0)$: $f(0) = (2-0^2)\,e^0 = 2$, i.e., $(x, y) = (0, 2)$.
5. *Limits:* there are two limits to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} (2-x^2)\,e^x = (-\infty)\,(+\infty) = -\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (2-x^2)\,e^x = \lim_{x\to-\infty} 2\,e^{x\,\to 0} - \lim_{x\to-\infty} x^2\,e^x$$

$$= -\lim_{x\to-\infty} x^2\,e^x = (\infty\cdot 0) = -\lim_{x\to-\infty} \frac{x^2}{1/e^x} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} -\lim_{x\to-\infty} \frac{2x}{-1/e^x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} -\lim_{x\to-\infty} \frac{2}{1/e^x} = -\lim_{x\to-\infty} 2\,e^x$$

$$= 0$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{(2-x^2)\,e^x}{x} = \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(-x^2-2x+2)e^x}{1} = -\infty$$

(Note the negative quadratic term.) Therefore, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x)=0:} \quad f'(x) = \left((2-x^2)\,e^x\right)' = -(x^2+2x-2)e^x = 0 \quad \therefore \quad x^2+2x-2=0$$

$$\therefore \quad x_{1,2} = \frac{-2\pm\sqrt{12}}{2} = -1\pm\sqrt{3}$$

There are two zeros of $f'(x)$ where $f(x_{1,2})$ is

$$\underline{f(x_1):} \quad f(x_1) = \left(2-(-1-\sqrt{3})^2\right)\,e^{-1-\sqrt{3}} = -2\left(1+\sqrt{3}\right)\,e^{-(1+\sqrt{3})}$$

$$\underline{f(x_2):} \quad f(x_2) = \left(2-(-1+\sqrt{3})^2\right)\,e^{-1+\sqrt{3}} = -2\left(1-\sqrt{3}\right)\,e^{-(1-\sqrt{3})}$$

$$\therefore \quad (x, y) = \left(-1-\sqrt{3}, -2(1+\sqrt{3})\,e^{-(1+\sqrt{3})}\right)$$

$$(x, y) = \left(-1+\sqrt{3}, -2(1-\sqrt{3})\,e^{-(1-\sqrt{3})}\right)$$

So that,

x		x_1		x_2	
-1	$-$	$-$	$-$	$-$	$-$
$x - x_1$	$-$	0	$+$	$+$	$+$
$x - x_2$	$-$	$-$	$-$	0	$+$
e^x	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\searrow$	y_1	$\nearrow$	y_2	$\searrow$

There are two extreme points, minimum at $(x, y) = (x_1, y_1)$ and maximum at $(x, y) = (x_2, y_2)$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(-(x^2 + 2x - 2)\, e^x\right)' = -(2x + 2)\, e^x - (x^2 + 2x - 2)\, e^x$$
$$= -x(4 + x)\, e^x = 0 \quad \therefore \quad x_1 = -4, \quad x_2 = 0$$

There are two zeros of $f''(x)$ where $f(x_{1,2})$ is

$$\underline{f(x_1):} \quad f(x_1) = \left(2 - (-4)^2\right) e^{-4} = -14\, e^{-4}$$
$$\underline{f(x_2):} \quad f(x_2) = \left(2 - (0)\right) e^0 = 2$$
$$\therefore \quad (x, y) = \left(-4, -14\, e^{-4}\right)$$
$$(x, y) = (0, 2)$$

So that,

x		-4		0	
$-x$	$+$	$+$	$+$	0	$-$
$x + 4$	$-$	0	$+$	$+$	$+$
e^x	$+$	$+$	$+$	$+$	$+$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\cap$	$-14e^4$	$\cup$	2	$\cap$

There are two inflection points, at $(x, y) = (-4, -14e^4)$ and $(x, y) = (0, 2)$.

8. *Summary of the important points:*

$$\underline{\text{horizontal asymptote:}} \quad y = 0, \quad (x < 0)$$
$$\underline{\text{y–axis crossing:}} \quad (x, y) = \left(0, 2\right)$$
$$\underline{\text{zeros:}} \quad (x, y) = (-\sqrt{2}, 0)$$
$$(x, y) = (+\sqrt{2}, 0)$$
$$\underline{\text{extremes:}} \quad (x, y) = \left(-1 - \sqrt{3}, -2(1 + \sqrt{3})\, e^{-(1+\sqrt{3})}\right) \quad \text{(min)}$$

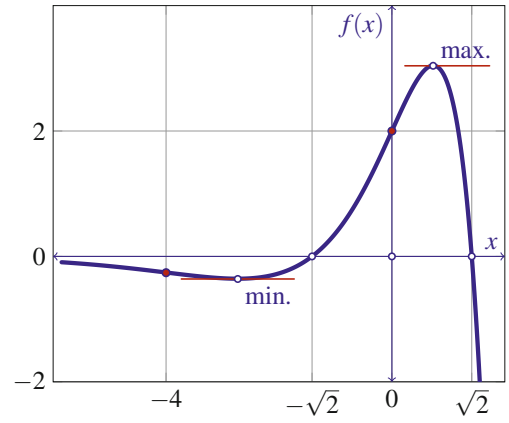


Fig. 3.93 Example
P.3.107

$$(x, y) = \left(-1 + \sqrt{3}, -2(1 - \sqrt{3})\, e^{-(1-\sqrt{3})}\right) \quad \text{(max)}$$

inflections: $(x, y) = \left(-4, -14\, e^{-4}\right)$

$$(x, y) = (0, 2)$$

9. *Graphical representation:* the above analysis is summarized in Fig. 3.93.

3.108. Given function,

$$f(x) = \ln^2\left(\frac{1}{e^x + 1}\right) = \left(\ln\left(\frac{1}{e^x + 1}\right)\right)^2 = \left(\ln(1) - \ln\left(e^x + 1\right)\right)^2 = \ln^2\left(e^x + 1\right)$$

1. *Domain of definition* $\mathbb{D}$: given $e^x + 1 > 0$ for all x, therefore $f(x)$ is also continuous function defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\text{even:} \quad f(-x) = \ln^2\left(e^{(-x)} + 1\right) = \ln^2\left(\frac{1}{e^x} + 1\right) \neq f(x) \quad \therefore \quad \text{(not even)}$$

$$\text{odd:} \quad -f(-x) = -\ln^2\left(\frac{1}{e^x} + 1\right) \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $f(x)$ is neither even nor odd.

3. *Function's sign:*

$$f(x) = 0: \quad \ln^2\left(e^x + 1\right) > 0$$

therefore, there are no zeros of $f(x)$.

4. *y-axis crossing point* $f(0)$: $f(0) = \ln^2\left(e^0 + 1\right) = \ln^2(2)$
5. *Limits:* there are two limits to resolve.

$$x \to +\infty: \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \ln^2\left(e^x + 1\right) = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \ln^2\left(e^x+1\right) = \ln(1) = 0$$

6. *Oblique asymptote:* $\lim_{\pm\infty} f(x)/x$ are different, so each limit separately results in

$$a = \lim_{x\to+\infty} \frac{\ln^2(e^x+1)}{x} = \left(\frac{+\infty}{+\infty}\right) \left\{\lim\big(f(x)g(x)\big) = \lim f(x)\ \lim g(x)\right\}$$

$$\overset{\text{l'H}}{=} 2 \lim_{x\to+\infty} \ln\left(e^x+1\right)^{+\infty} \lim_{x\to+\infty} \frac{e^x}{e^x+1}^{1} = +\infty$$

$$a = \lim_{x\to-\infty} \frac{\ln^2(e^x+1)}{x} = \left(\frac{0}{-\infty}\right) \overset{\text{l'H}}{=} 2 \lim_{x\to-\infty} \ln\left(e^x+1\right)^{0} \lim_{x\to-\infty} \frac{e^x}{e^x+1}^{1} = 0$$

The two limits are different; therefore, the limit does not exist, and by consequence, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(\ln^2\left(e^x+1\right)\right)' = 2e^x \frac{\ln(e^x+1)}{e^x+1} > 0$$

Therefore, there are no extremes, and function always ascends ↗.

Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(2e^x \frac{\ln(e^x+1)}{e^x+1}\right)' = 2\frac{e^x\ln(e^x+1) + e^{2x}}{(e^x+1)^2} > 0$$

Therefore, there are no inflection points, and function is always ∪.

8. *Summary of the important points:*

horizontal asymptote:	$y = 0, \quad (x < 0)$
y–axis crossing:	$(x, y) = \big(0, \ln^2(2)\big)$
zeros:	(none, $f(x)$ is always positive)
extremes:	(none), $f(x)$ is always ascends ↗)
inflections:	(none, $f(x)$ is always concave ∪)

9. *Graphical representation:* the above analysis is summarized in Fig. 3.94.

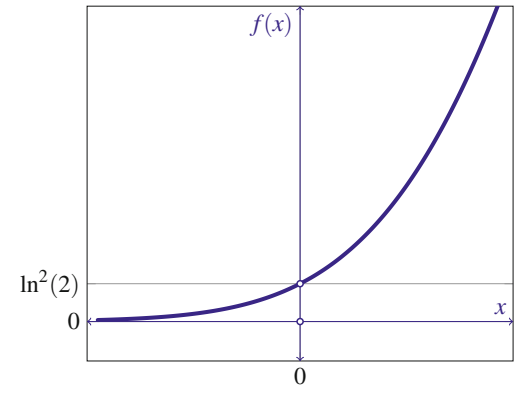
3.109. Given function,

$$f(x) = x\, e^{x^2-x}$$

1. *Domain of definition* $\mathbb{D}$: function is continuous; thus, $f(x)$ is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = (-x)\, e^{(-x)^2-(-x)} = -x\, e^{x^2+x} \neq f(x) \quad \therefore \quad \text{(not even)}$$

Fig. 3.94 Example
P.3.108



$$\underline{\text{odd:}} \quad -f(-x) = x\,e^{x^2+x} \neq f(x) \quad \therefore \quad \text{(not odd)}$$

In conclusion, function $f(x)$ is neither even nor odd.

3. *Function's sign:*

$$\underline{f(x) = 0:} \quad x\,e^{x^2-x} = 0 \quad \therefore \quad x = 0$$

therefore, there is one zero of $f(x)$, where $f(0) = 0\,e^{0^2-0} = 0$; thus

x		0	
x	$-$	0	$+$
$\exp(x^2 - x)$	$+$	$+$	$+$
$f(x)$	$-$	0	$+$

4. *y-axis crossing point* $f(0)$: $f(0) = 0\,e^{0^2-0} = 0$.
5. *Limits:* there are two limits to resolve.

$$\underline{x \to +\infty:} \quad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\,e^{x(x-1)} = (+\infty)(+\infty) = +\infty$$

$$\underline{x \to -\infty:} \quad \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x\,e^{x(x-1)} = (-\infty)(+\infty) = -\infty$$

6. *Oblique asymptote:* there is no oblique asymptote, as

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x\,e^{x(x-1)}}{x} = \infty$$

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \left(x\,e^{x^2-x}\right)' = \left(2x^2 - x + 1\right)e^{x^2-x} > 0$$

Therefore, there are no extreme points, and functions always ascend ↗.

Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\left(2x^2 - x + 1\right) e^{x^2 - x}\right)' = \left(4x^3 - 4x^2 + 7x - 2\right) e^{x^2 - x}$$
$$= 0$$
$$\therefore \quad 4x^3 - 4x^2 + 7x - 2 = 0$$

Third-order polynomial has at least one real zero, here found numerically, at $x = 0.327$ where $f(0.327) = 0.262$. For $x < 0.327$ second derivative is negative (i.e., $f(x)$ is $\cap$), and for $x > 0.327$ second derivative is positive (i.e., $f(x)$ is $\cup$)

8. *Summary of the important points:*

y–axis crossing:	$(x, y) = (0, 0)$
zero:	$(x, y) = (0, 0)$
extremes:	(none), , $f(x)$ is always ascends $\nearrow$)
inflection:	$(x, y) = (0.327, 0.262)$

9. *Graphical representation:* the above analysis is summarized in Fig. 3.95.

3.110. Given function,

$$f(x) = 1 + 2x^3 - x^4$$

1. *Domain of definition* $\mathbb{D}$: function is continuous; thus, $f(x)$ is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = 1 + 2(-x)^3 - (-x)x^4 = 1 - 2x^3 - x^4 \neq f(x)$$
$$\therefore \quad \text{(not even)}$$
$$\underline{\text{odd:}} \quad -f(-x) = -1 + 2x^3 + x^4 \neq f(x) \quad \therefore \quad \text{(not odd)}$$

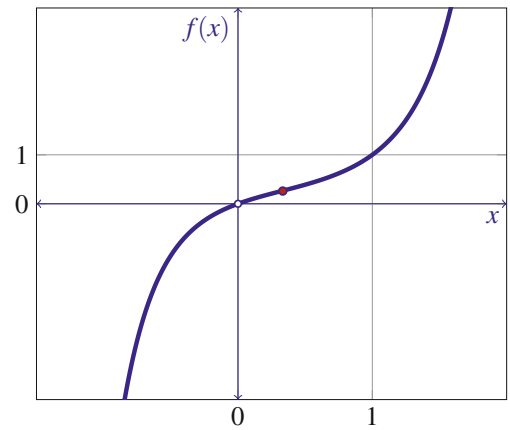


Fig. 3.95 Example P.3.109

In conclusion, function $f(x)$ is neither even nor odd.

3. *Function's sign:*

$$\underline{f(x)=0:}\quad 1+2x^3-x^4=0$$

Two real roots of this fourth-order polynomial, here found numerically, are $x_{1,2} = [-0.717, 2.107]$. That is to say that a general form of $f(x)$ is $(x-x_1)(x-x_2)P_2(x)$, where $P_2(x)$ is a second-order polynomial whose two roots are complex. Note the negative term $-x^4$, which is to say that $\lim_{\infty} f(x)$ tends to $-\infty$; thus, it can be deduced that has to be $P_2(x) < 0$. So,

x		x_1		x_2	
$x-x_1$	$-$	0	$+$	$+$	$+$
$x-x_2$	$-$	$-$	$-$	0	$+$
$P_2(x)$	$-$	$-$	$-$	$-$	$-$
$f(x)$	$-$	0	$+$	0	$-$

4. *y-axis crossing point* $f(0)$: $f(0) = 1+2(0)^3-(0)^4 = 1$.
5. *Limits:* there are two limits to resolve.

$$\underline{x\to\pm\infty:}\quad \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(1+2x^3-x^4\right) = -\infty$$

6. *Oblique asymptote:*

$$a = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{1+2x^3-x^4}{x} = \lim_{x\to\infty}\left(\frac{1}{x}+2x^2-x^3\right) = -\infty$$

Therefore, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x)=0:}\quad f'(x) = \left(1+2x^3-x^4\right)' = -2x^2(2x-3) = 0$$

$$\therefore\ x_1 = 0,\quad x_2 = 0,\quad x_3 = \frac{3}{2}$$

where $f(x_{1,2}) = 1$, and $f(x_3) = 43/16$. Note second (even)-order zero of $f'(x)$ versus the first-order (odd) zero, as

x		0		3/2	
-2	$-$	$-$	$-$	$-$	$-$
x^2	$+$	0	$+$	$+$	$+$
$2x-3$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$+$	$-$	$-$
$f(x)$	$\nearrow$	0	$\nearrow$	$^{43}/_{16}$	$\searrow$

That is to say, an even order zero does not create extreme; consequently, there is only one extreme at $(x, y) = (3/2, 43/16)$, which is an odd zero. Also,

$$f''(x) = 0: \quad f''(x) = \left(-2x^2(2x-3)\right)' = -12x(x-1) = 0$$
$$\therefore \quad x_1 = 0, \quad x_2 = 1$$

where $f(0) = 1$, $f(1) = 2$, so that,

x		0		1	
-12	$-$	$-$	$-$	$-$	$-$
x	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\cap$	1	$\cup$	2	$\cap$

There are two inflection points at $(x, y) = (0, 1)$ and $(x, y) = (1, 2)$.

8. *Summary of the important points:*

horizontal asymptote: (none)

vertical asymptotes: (none)

y–axis crossing: $(x, y) = (0, 1)$

zeros: $(x, y) = (-0.717, 0)$

$(x, y) = (2.107, 0)$

extremes: $(x, y) = (3/2, 43/16)$ (max)

inflections: $(x, y) = (0, 1)$

$(x, y) = (1, 2)$

Note that $(x, y) = (0, 1)$ is not an extreme point, even though $f'(0) = 0$. Recall that $x = 0$ is an even zero of this $f'(x) = 0$ equation.

9. *Graphical representation:* the above analysis is summarized in Fig. 3.96.

3.111. Given function,

$$f(x) = \left|x^2 - |x| - 2\right|$$

1. *Domain of definition* $\mathbb{D}$: an absolute function is continuous; thus, $f(x)$ is defined for $x \in \mathbb{R}$.
2. *Function's parity:* function $f(x)$ is even and not odd, as

$$\text{even:} \quad f(-x) = \left|(-x)^2 - |-x| - 2\right| = \left|x^2 - |x| - 2\right| = f(x) \quad \therefore \quad (\text{even})$$
$$\text{odd:} \quad -f(-x) = -\left|x^2 - |x| - 2\right| \neq f(x) \quad \therefore \quad (\text{not odd})$$

3. *Function's sign:* by definition, an absolute function $|f(x)| \geq 0$ for all real arguments. Composite absolute functions are decomposed as

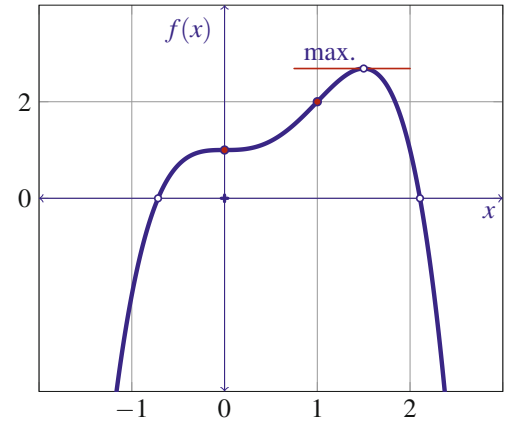


Fig. 3.96 Example
P.3.110

$$\left\{|f(x)| = f(x), (\text{if } f(x) \geq 0) \quad \text{and,} \quad |f(x)| = -f(x), (\text{if } f(x) < 0)\right\}$$

$\underline{f(x) = 0:}\ \left|x^2 - |x| - 2\right| = 0 \quad \therefore$

$$\begin{cases} \underline{x \geq 0:}\ f(x) = \left|x^2 - x - 2\right| \quad \therefore & \begin{cases} x^2 - x - 2 \geq 0: & f_1(x) = x^2 - x - 2 \\ x^2 - x - 2 < 0: & f_2(x) = -x^2 + x + 2 \end{cases} \\ \underline{x < 0:}\ f(x) = \left|x^2 + x - 2\right| \quad \therefore & \begin{cases} x^2 + x - 2 \geq 0: & f_3(x) = x^2 + x - 2 \\ x^2 + x - 2 < 0: & f_4(x) = -x^2 - x + 2 \end{cases} \end{cases}$$

where each of the four versions of $f_i(x)$ is valid within its respective interval (i.e., solutions outside of the given interval are excluded) as

$$\left\{\begin{matrix} ab > 0 \ \therefore \ a > 0 \ \text{ and } \ b > 0, \ \text{ or, } \ a < 0 \ \text{ and } \ b < 0 \\ ab < 0 \ \therefore \ a < 0 \ \text{ and } \ b > 0, \ \text{ or, } \ a > 0 \ \text{ and } \ b < 0 \end{matrix}\right\}$$

$\underline{x \geq 0:}$

$$f_1(x) = x^2 - x - 2 = (x+1)(x-2) \geq 0 \ \therefore \ x \geq -1 \ \text{ and, } \ x \geq 2$$
$$\therefore \ \underline{x \geq 2}$$
$$f_2(x) = -x^2 + x + 2 = -(x+1)(x-2) \geq 0 \ \therefore \ x \geq -1 \ \text{ and, } \ x \leq 2$$
$$\therefore \ \underline{0 \leq x \leq 2}$$

$\underline{x < 0:}$

$$f_3(x) = x^2 + x - 2 = (x-1)(x+2) \geq 0 \ \therefore \ x \leq 1 \ \text{ and, } \ x \leq -2$$
$$\therefore \ \underline{x \leq -2}$$
$$f_4(x) = -x^2 - x + 2 = -(x-1)(x+2) \geq 0 \ \therefore \ x \leq 1 \ \text{ and, } \ x \geq -2$$
$$\therefore \ \underline{-2 \leq x \leq 0}$$

Possible zeros are therefore:

$$f_1(x),\,(x \geq 2):\ x=-1 \quad \text{or,} \quad x=2 \ \ \therefore\ \ x=2$$
$$f_2(x),\,(0 \leq x \leq 2):\ x=-1 \quad \text{or,} \quad x=2 \ \ \therefore\ \ x=2$$
$$f_3(x),\,(x \leq -2):\ x=1 \quad \text{or,} \quad x=-2 \ \ \therefore\ \ x=-2$$
$$f_4(x),\,(-2 \leq x \leq 0):\ x=1 \quad \text{or,} \quad x=-2 \ \ \therefore\ \ x=-2$$

In conclusion, there are two zeros, $x=-2$ and $x=2$. Otherwise, $f(x)>0$.

4. *y-axis crossing point* $f(0)$: $f(0)=\left|0^2-|0|-2\right|=2$.
5. *Limits:* there are two limits to resolve.

$$\underline{x \to \pm\infty:} \quad \lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty} x^2+x-2=+\infty$$

6. *Oblique asymptote:*

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{x^2+x-2}{x}=\lim_{x\to\infty}\left(x+1-\frac{2}{x}\right)=\infty$$

Therefore, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x)=0:}\quad (x\geq 0):\ \left(x^2-x-2\right)'=0,\ \ \therefore\ \ 2x-1=0\ \ \therefore\ \ x_1=1/2$$
$$\left(-x^2+x+2\right)'=0,\ \ \therefore\ \ -2x+1=0\ \ \therefore\ \ x_1=1/2$$
$$\underline{f'(x)=0:}\quad (x<0):\ \left(x^2+x-2\right)'=0,\ \ \therefore\ \ 2x+1=0\ \ \therefore\ \ x_2=-1/2$$
$$\left(-x^2-x+2\right)'=0,\ \ \therefore\ \ -2x-1=0\ \ \therefore\ \ x_2=-1/2$$

where $f(x_{1,2})=9/4$, and

x		-2		$-1/2$		0		$1/2$		2	
$f_1'(x)$										$+$	$+$
$f_2'(x)$						$+$	$+$	0	$-$	$-$	
$f_3'(x)$		$+$	$+$	0	$-$	$-$					
$f_4'(x)$	$-$	$-$									
$f(x)$	$\searrow$	0	$\nearrow$	$9/4$	$\searrow$	2	$\nearrow$	$9/4$	$\searrow$	0	$\nearrow$

There are two maximum extreme points at $(x,y)=(\pm 1/2, 9/4)$, and two minimum extreme points at $(x,y)=(\pm 2, 0)$.

Also,

$$\underline{f''(x)=0:}\quad (x\geq 0):\ f_1''(x)=\left(x^2-x-2\right)''=0,\ \ \therefore\ \ 2>0\ \ \therefore\ \ \cup$$
$$f_2''(x)=\left(-x^2+x+2\right)''=0,\ \ \therefore\ \ -2<0\ \ \therefore\ \ \cap$$
$$\underline{f''(x)=0:}\quad (x<0):\ f_3''(x)=\left(x^2+x-2\right)''=0,\ \ \therefore\ \ 2>0\ \ \therefore\ \ \cup$$
$$f_4''(x)=\left(-x^2-x+2\right)''=0,\ \ \therefore\ \ -2<0\ \ \therefore\ \ \cap$$

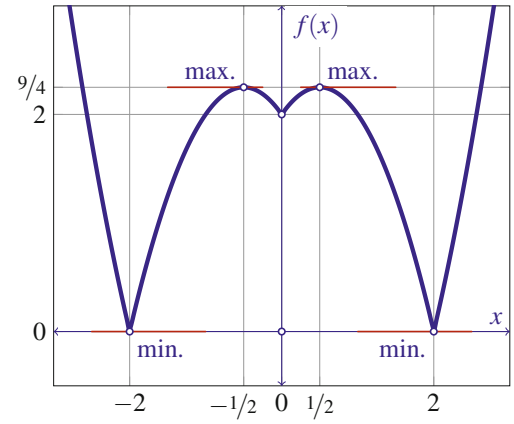


Fig. 3.97 Example
P.3.111

8. *Summary of the important points:*

horizontal asymptote:	(none)
vertical asymptotes:	(none)
y–axis crossing:	$(x, y) = (0, 2)$
zeros:	$(x, y) = (-2, 0)$
	$(x, y) = (2, 0)$
extremes:	$(x, y) = (-2, 0)$ (min)
	$(x, y) = (-1/2, 9/4)$ (max)
	$(x, y) = (2, 0)$ (min)
	$(x, y) = (1/2, 9/4)$ (max)
inflections:	(none)

9. *Graphical representation:* the above analysis is summarized in Fig. 3.97. Note that inflection points do not exist for the reason that (by definition) $f''(x)$ is not defined at $x = 0$ and $x = \pm 2$ points (the left and right side derivative functions are not equal at these points). However, $f(x)$ does change its concavity/convexity at this point.

3.112. Given composite function,

$$f(x) = \ln|x^2 - 2x| = \ln|x(x-2)|$$

1. *Domain of definition* $\mathbb{D}$: absolute function is always positive or equal to zero; thus, logarithm of absolute argument is defined for $x(x-2) \neq 0$, i.e., $x \neq 0$ and $x \neq 2$. That is to say, there are two vertical asymptotes, $x = 0$ and $x = 2$. Thus, domain of definition $\mathbb{D}$ is $x \in \mathbb{R} | x \neq 0, x \neq 2$.
2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \ln|(-x)((-x)-2)| = \ln|x(x+2)| \neq f(x) \quad \text{(not even)}$$
$$\underline{\text{odd:}} \quad -f(-x) = -\ln|x(x+2)| \neq f(x) \quad \text{(not odd)}$$

In conclusion, this function is neither odd nor even.

3. *Function's sign:*

$$\underline{f(x)=0:} \quad \therefore \quad \ln|x^2-2x| = 0 \quad \therefore \quad \begin{cases} x^2-2x=1 & \therefore \quad x^2-2x-1=0 \\ & \therefore \quad x_{1,2} = 1 \pm \sqrt{2} \\ x^2-2x=-1 & \therefore \quad x^2-2x+1=(x-1)^2=0 \\ & \therefore \quad x_{3,4} = 1 \end{cases}$$

That is to say, there are two first-order zeros, $x_1 = 1-\sqrt{2}$ and $x_2 = 1+\sqrt{2}$, and one second-order zero $x_{3,4} = 1$.

x		x_1		0		1		2		x_2	
$(x-x_1)$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-x_2)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x(x-2)$	$+$	1	$+$	0	$-$	-1	$-$	0	$+$	1	$+$
$\lvert x(x-2)\rvert$	>1	1	<1	0	<1	1	<1	0	<1	1	>1
$\ln\lvert x(x-2)\rvert$	$+$	0	$-$	n.d.	$-$	0	$-$	n.d.	$-$	0	$+$
$f(x)$	$+$	0	$-$	n.d.	$-$	0	$-$	n.d.	$-$	0	$+$

4. *y-axis crossing point* $f(0)$: $f(0) = \ln|0(0-2)| = \text{n.d.}$
5. *Limits:* there are six limits to resolve,

$$\underline{x \to \pm\infty:} \quad \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \ln|x^2-2x| = +\infty$$
$$\underline{x \to 0:} \quad \lim_{\substack{x\to 0 \\ \to}} f(x) = \lim_{x\to 0} \ln|x^2-2x| = -\infty$$
$$\underline{x \to 2:} \quad \lim_{x\to 2} f(x) = \lim_{x\to 2} \ln|x^2-2x| = -\infty$$

Note that due to argument of $\ln|\cdot|$ is an absolute function, both left- and right-side limits are equal. Therefore, there are two vertical asymptotes, $x = 0$ and $x = 2$.

6. *Oblique asymptote:* there is no oblique asymptote, as

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\ln|x^2-2x|}{x}$$
$$= \left(\frac{\infty}{\infty}\right) \left\{ |x|' = \frac{x}{|x|} = \frac{|x|}{x} \equiv \text{sign}\,(x) \right\} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(\ln|x^2-2x|)'}{1}$$

$$= \lim_{x\to\infty} \frac{\dfrac{(x^2-2x)(2x-2)}{|x^2-2x|}}{\dfrac{|x^2-2x|}{1}} = \lim_{x\to\infty} \frac{(x^2-2x)(2x-2)}{(x^2-2x)^2} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{(2x-2)'}{(x^2-2x)'} = \lim_{x\to\infty} \frac{2}{2x-2}$$

$$= 0$$

7. *Critical points:*

$$\underline{f'(x)=0:} \quad f'(x) = \left(\ln|x^2-2x|\right)' = \frac{\dfrac{(x^2-2x)(2x-2)}{|x^2-2x|}}{|x^2-2x|} = \frac{2x-2}{x^2-2x} = \frac{2(x-1)}{x(x-2)} = 0$$

$$\therefore \quad 2x-2 \quad \therefore \quad \underline{x=1}$$

$$\therefore \quad f(1) = \ln|1^2-2(1)| = \ln|-1| = 0$$

Therefore, there is one maximum point at $(x, y) = (1, 0)$, as

x		0		1		2	
2	+	+	+	+	+	+	+
x	−	0	+	+	+	+	+
$x-1$	−	−	−	0	+	+	+
$x-2$	−	−	−	−	−	0	+
$f'(x)$	−	n.d.	+	0	−	n.d.	+
$f(x)$	↘	n.d.	↗	0	↘	n.d.	↗

also,

$$\underline{f''(x)=0:} \quad f''(x) = \left(\frac{2x-2}{x^2-2x}\right)' = \frac{2(x^2-2x)-(2x-2)(2x-2)}{(x^2-2x)^2} = \frac{-2(x^2-2x+2)}{(x^2-2x)^2} < 0$$

because denominator is always positive (quadratic function) and numerator is always negative. Note that $x^2-2x+2 > 0$ for all x. Consequently, due to $f''(x) < 0$, there are no inflection points, and function $f(x)$ is always convex $\cap$.

8. *Summary of the important points:*

$$\underline{\text{define:}} \quad x \neq 0, \quad x \neq 2$$

$$\underline{\text{v. asymptote:}} \quad x = 0, \quad x = 2$$

$$\underline{\text{zeros:}} \quad (x, y) = \left(1-\sqrt{2}, 0\right)$$

$$(x, y) = (0, 0)$$

$$(x, y) = (0, 0)$$

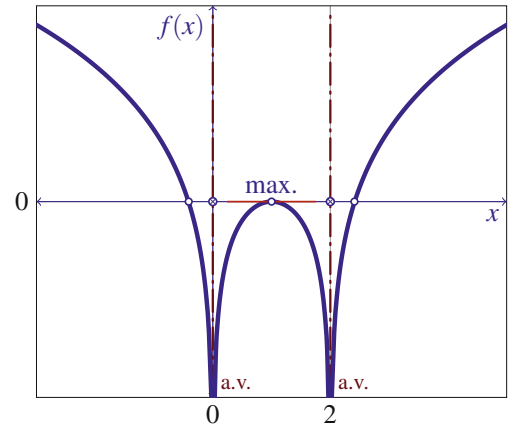


Fig. 3.98 Example
P.3.112

$$(x, y) = \left(1 + \sqrt{2}, 0\right)$$

$$\underline{\text{extremes:}} \quad (x, y) = (1, 0) \quad (\text{max})$$

9. *Graphical representation:* see Fig. 3.98.

3.113. Given function,

$$f(x) = \left(x - \frac{1}{4}\right) \ln\left(1 - \frac{1}{x}\right)$$

1. *Domain of definition* $\mathbb{D}$: inverse function $1/x$ is defined for $x \neq 0$, and logarithmic function is limited to strictly positive arguments; thus, $\left\{\text{recall: } \frac{a}{b} > 0 \;\; \therefore \;\; \frac{+}{+} \;\; \text{or,} \;\; \frac{-}{-}\right\}$

$$1 - \frac{1}{x} = \frac{x-1}{x} > 0 \;\; \therefore \;\; \begin{cases} x - 1 > 0 \;\; \text{and,} \;\; x > 0 \;\; \therefore \;\; x > 1 \;\; \text{and,} \;\; x > 0 \;\; \therefore \;\; \underline{x > 1} \\ x - 1 < 0 \;\; \text{and,} \;\; x < 0 \;\; \therefore \;\; x < 1 \;\; \text{and,} \;\; x < 0 \;\; \therefore \;\; \underline{x < 0} \end{cases}$$

therefore $f(x)$ is defined for $x \in \mathbb{R} | -\infty < x < 0, \quad \text{or,} \quad 1 < x < +\infty$, that is to say, that within the interval $x \in [0, 1]$, this $f(x)$ is not defined. In addition, there are two vertical asymptotes at $y = 0$ and $y = 1$.

2. *Function's parity:*

$$\underline{\text{even:}} \quad f(-x) = \left((-x) - \frac{1}{4}\right) \ln\left(1 - \frac{1}{(-x)}\right) = -\left(x + \frac{1}{4}\right) \ln\left(1 + \frac{1}{x}\right) \neq f(x)$$

$$\therefore \;\; (\text{not even})$$

$$\underline{\text{odd:}} \quad -f(-x) = \left(x + \frac{1}{4}\right) \ln\left(1 + \frac{1}{x}\right) \neq f(x) \;\; \therefore \;\; (\text{not odd})$$

In conclusion, function $f(x)$ is neither even nor odd.

3. *Function's sign:* $\left\{\text{recall: } ab = 0 \Rightarrow a = 0 \quad \text{or,} \quad b = 0\right\}$, as

$$\underline{f(x)=0:}\quad \left(x-\frac{1}{4}\right)\ln\left(1-\frac{1}{x}\right)=0$$

$$\therefore\quad \begin{cases} x-1/4=0 \;\;\therefore\;\; x_1=1/4 \;\;\therefore\;\; x_1\in[0,1] \;\text{ (not in the domain)} \\ 1-1/x=0 \;\;\therefore\;\; x_2=1 \;\;\therefore\;\; x_2\in[0,1] \;\text{ (not in the domain)} \end{cases}$$

therefore, there are no zeros of $f(x)$, thus

x	$-\infty$		0		$1/4$		1		$+\infty$
$x-1/4$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$\ln(1-1/x)$	0	$+$	n.d.	n.d.	n.d.	n.d.	n.d.	$-$	0
$f(x)$	n.d.	$-$	n.d.	n.d.	n.d.	n.d.	n.d.	$-$	n.d.

4. *y-axis crossing point* $f(0)$: $f(0)=(0-1/4)\ln(1-1/0)$ is not defined.
5. *Limits:* there are four limits to resolve.

$$\begin{aligned}
\underline{x\to\pm\infty:}\quad \lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty}\left(x-\frac{1}{4}\right)\ln\left(1-\frac{1}{x}\right)=(\infty\cdot 0)\\
&= \lim_{x\to\pm\infty} x\ln\left(1-\frac{1}{x}\right)-\lim_{x\to\pm\infty}\frac{1}{4}\ln\left(1-1/x^{\to 0}\right)\\
&= \lim_{x\to\pm\infty}\frac{\ln(1-1/x)}{1/x}+0 \overset{\text{l'H}}{=} \lim_{x\to\pm\infty}\frac{1/(x^2-x)}{-1/x^2}\\
&= -\lim_{x\to\pm\infty}\frac{x}{x-1}\overset{\text{l'H}}{=}-\lim_{x\to\pm\infty}\frac{1}{1}\\
&= -1 \;\text{ (horizontal asymptote)}
\end{aligned}$$

$$\underline{x\to 0\atop\to:}\quad \lim_{x\to 0\atop\to} f(x)=\lim_{x\to 0\atop\to}\left(x^{\to 0}-1/4\right)\underbrace{\ln\left(1-1/x^{\to-\infty}\right)}_{\to+\infty}=-\infty$$

$$\underline{x\to 1\atop\leftarrow:}\quad \lim_{x\to 1\atop\leftarrow} f(x)=\lim_{x\to 1\atop\leftarrow}\left(x-1/4^{\to 3/4}\right)\underbrace{\ln\left(1-1/x^{\to 1}\right)}_{\to-\infty}=-\infty$$

Therefore, there is one horizontal asymptote at $y=-1$.

6. *Oblique asymptote:*

$$\begin{aligned}
a &= \lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{(x-1/4)\ln(1-1/x)}{x}\\
&= \underbrace{\lim_{x\to\infty}\frac{x\ln(1-1/x)}{x}}_{\to 0}-\frac{1}{4}\lim_{x\to\infty}\frac{\ln(1-1/x)}{x}=\left(\frac{0}{\infty}\right)\\
&= -\frac{1}{4}\lim_{x\to\infty}\left[\frac{1}{x}\cdot\ln(1-1/x)\right]=(0\cdot 0)=0
\end{aligned}$$

Therefore, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x) = 0:} \quad f'(x) = \big((x - 1/4)\ln(1 - 1/x)\big)' = \ln\left(1 - \frac{1}{x}\right) + \left(x - \frac{1}{4}\right)\frac{1/x^2}{1 - 1/x}$$
$$= \ln\left(1 - \frac{1}{x}\right) + \frac{4x - 1}{4x(x - 1)} = 0$$

This is another transcendental equation form that must be solved numerically. With the help of Newton-Raphson method, the only $f'(x)$ zero is found (after rounding to three decimal places) at $(x, y) = (-0.243, -0.805)$. The same solution can be found by other numerical methods as well: graphical, Taylor expansion polynomial, etc. Sign of $f'(x)$ function is summarized as

x		-0.243		0		1	
$f'(x)$	+	0	−	n.d.	n.d	n.d	+
$f(x)$	↗	-0.805	↘	n.d.	n.d	n.d	↗

There is one maximum point at $(x, y) = (-0.243, -0.805)$. Also,

$$\underline{f''(x) = 0:} \quad f''(x) = \left(\ln\left(1 - \frac{1}{x}\right) + \frac{4x - 1}{4x(x - 1)}\right)' = -\frac{2x + 1}{\big(2x(x - 1)\big)^2} = 0$$
$$\therefore \ 2x + 1 = 0 \ \therefore \ x = -\frac{1}{2}$$

Thus,

x		$-1/2$		0		1	
-1	−	−	−	−	−	−	−
$2x + 1$	−	0	+	+	+	+	+
$\big(2x(x - 1)\big)^2$	+	+	+	0	+	0	+
$f''(x)$	+	0	−	n.d.	n.d.	n.d.	−
$f(x)$	∪	-0.824	∩	n.d.	n.d.	n.d.	∩

There is one inflection point $(x, y) = (-1/2, -0.824)$.

8. *Summary of the important points:*

define:	$x \notin 0 \leq x \leq 1$
horizontal asymptote:	$y = -1$
vertical asymptotes:	$x = 0$
	$x = 1$
y–axis crossing:	(none)
zeros:	(none, $f(x)$ is always negative)

$$\underline{\text{extreme:}} \quad (x, y) = (-0.243, -0.805) \ \ (\text{max})$$

$$\underline{\text{inflection:}} \quad (x, y) = (-1/2, -0.824)$$

9. *Graphical representation:* the above analysis is summarized in Fig. 3.99.

3.114. Given function,

$$f(x) = \sin\left(2\,e^{x/2}\right), \quad x \in [-2\pi, \pi]$$

1. *Domain of definition* $\mathbb{D}$: all terms in this composite function are continuous; therefore, $f(x)$ is defined for $x \in \mathbb{R}$.
2. *Function's parity:*

$$\underline{\text{even:}} \qquad f(-x) = \sin\left(2\,e^{-(x)/2}\right) = \sin\left(\frac{2}{e^{x/2}}\right) \neq f(x) \ \ \therefore \ \ (\text{not even})$$

$$\underline{\text{odd:}} \quad -f(-x) = -\sin\left(\frac{2}{e^{x/2}}\right) \neq f(x) \ \ \therefore \ \ (\text{not odd})$$

   In conclusion, function $f(x)$ is neither even nor odd.
3. *Function's sign:* $(x \in [-2\pi, \pi])$

$$\underline{f(x) = 0:} \quad \sin\left(2\,e^{x/2}\right) = 0 \ \ \therefore \ \ 2\,e^{x/2} = n\pi, \ \ n = 0, \pm 1, \pm 2, \ldots$$

$$\therefore \ \ e^{x/2} = n\,\frac{\pi}{2} \ \ \therefore \ \ x = 2\,\ln\left(n\,\frac{\pi}{2}\right), \ \ n = 1, 2, 3$$

   because a logarithmic function is defined for strictly positive arguments. There are three zeros of $f(x)$ within $x \in [-2\pi, \pi]$ interval. This statement can be verified as

$$\underline{f(x_1):} \quad f(x_1) = \sin\left(2\,e^{\left[\ln(\pi/2)\right]/2}\right) = \sin\left(2\,\frac{\pi}{2}\right) = \sin\pi = 0$$

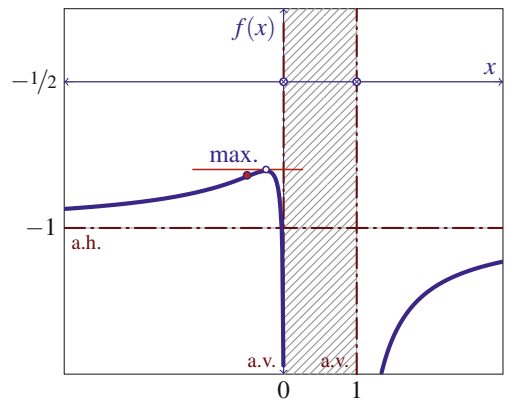


Fig. 3.99 Example P.3.113

$$\underline{f(x_2):}\quad f(x_2)=\sin\left(2\,e^{\left[\frac{1}{2}\ln\left(\frac{1}{2}\pi/\frac{1}{2}\right)\right]/\frac{1}{2}}\right)=\sin 2\pi=0$$

$$\underline{f(x_3):}\quad f(x_3)=\sin\left(2\,e^{\left[\frac{1}{2}\ln(3\pi/2)\right]/\frac{1}{2}}\right)=\sin\left(\cancel{2}\,\frac{3\pi}{\cancel{2}}\right)=\sin 3\pi=0$$

Keep in mind the sign of $\sin(x)$ relative to its argument (see P.3.90 to P.3.94); in this example, the important arguments are π, 2π, 3π, so that intervals are

x	$-\infty$		x_1		x_2		x_3		$+\infty$
$2e^{2\ln(n\pi/2)}$			π		2π		3π		
$\sin(x)$		$+$	0	$-$	0	$+$	0	$-$	
$f(x)$	n.d.	$+$	0	$-$	0	$+$	0	$-$	n.d.

4. *y-axis crossing point* $f(0)$: $f(0)=\sin\left(2\,e^{0/2}\right)=\sin(2)$, i.e. $(x,y)=(0,\sin(2))$ {in radians}.
5. *Limits:* there are two limits to resolve.

$$\underline{x\to+\infty:}\quad \lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\sin\left(2\,e^{x/2}\right)=\sin(\infty)=\text{n.d.}$$

$$\underline{x\to-\infty:}\quad \lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\sin\left((2\,\cancel{e^{x/2}}^{\,0})\right)=\sin(0)=0$$

6. *Oblique asymptote:*

$$a=\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\sin\left(2\,e^{x/2}\right)}{x}=\left(\frac{\text{n.d.}}{\infty}\right)$$

Therefore, there is no oblique asymptote.

7. *Critical points:*

$$\underline{f'(x)=0:}\quad f'(x)=\left(\sin\left(2\,e^{x/2}\right)\right)'=e^{x/2}\cos\left(2\,e^{x/2}\right)=0\quad\therefore\quad\cos\left(2\,e^{x/2}\right)$$

$$\therefore\quad 2\,e^{x/2}=\frac{(2n+1)\pi}{2}\quad\therefore\quad x=2\ln\frac{(2n+1)\pi}{4},\quad n=0,1,2,\ldots$$

There are three zeros of $f'(x)$ in $(x\in[-2\pi,\pi])$ interval where $f(x_{1,2,3})$ is

$$\underline{f(x_1):}\quad f(x_1)=\sin\left(2\,e^{[2\ln(\pi/4)]/2}\right)=\sin\frac{\pi}{2}=1$$

$$\underline{f(x_2):}\quad f(x_2)=\sin\left(2\,e^{[2\ln(3\pi/4)]/2}\right)=\sin\frac{3\pi}{2}=-1$$

$$\underline{f(x_3):}\quad f(x_3)=\sin\left(2\,e^{[2\ln(5\pi/4)]/2}\right)=\sin\frac{5\pi}{2}=1$$

$$\therefore\quad (x,y)=\left(2\,e^{[2\ln(\pi/4)]/2},1\right)$$

$$(x,y)=\left(2\,e^{[2\ln(3\pi/4)]/2},-1\right)$$

$$(x,y)=\left(2\,e^{[2\ln(5\pi/4)]/2},1\right)$$

So that,

x	$-\infty$		x_1		x_2		x_3		$+\infty$
$2e^{2\ln((2n+1)\pi/4)}$			$\pi/2$		$3\pi/2$		$5\pi/2$		
$e^{2\ln((2n+1)\pi/4)}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	
$\cos(x)$		$+$	0	$-$	0	$+$	0	$-$	
$f'(x)$		$+$	0	$-$	0	$+$	0	$-$	
$f(x)$	n.d.	$\nearrow$	1	$\searrow$	-1	$\nearrow$	1	$\searrow$	n.d.

Also,

$$f''(x) = 0: \quad f''(x) = \left(e^{x/2}\cos\left(2\,e^{x/2}\right)\right)' = \frac{1}{2}\,e^{x/2}\cos\left(2\,e^{x/2}\right) - e^{x/2}\,e^{x/2}\sin\left(2\,e^{x/2}\right)$$

$$= \frac{1}{2}\,e^{x/2}\left(\cos\left(2\,e^{x/2}\right) - 2\,e^{x/2}\sin\left(2\,e^{x/2}\right)\right) = 0$$

$$\therefore \quad N(x) = \cos\left(2\,e^{x/2}\right) - 2\,e^{x/2}\sin\left(2\,e^{x/2}\right) = 0$$

With the help of Newton-Raphson method $f''(x)$ zeros are found (after rounding to three decimal places) at $x_{1,2,3} = [-1.687, 1.076, 2.338, 3.122]$, where $f(x_{1,2,3}) = [0.758, -0.280, 0.153, -0.104]$.

Thus,

x	$\cdots$	-1.687		1.076		2.338		3.122		$\cdots$
$N(x)$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$\cdots$
$e^{x/2}$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$f''(x)$	$+$	0	$-$	0	$+$	0	$-$	0	$+$	$\cdots$
$f(x)$	$\cup$	0.758	$\cap$	-0.28	$\cup$	0.153	$\cap$	-0.104	$\cup$	$\cdots$

8. *Summary of the important points:*

horizontal asymptote: (none, must limit on both $\pm\infty$)

y–axis crossing: $(x, y) = \left(0, \sin(2)\right)$

zeros: $(x, y) = (\pi, 0)$

$(x, y) = (2\pi, 0)$

$(x, y) = (3\pi, 0)$

extremes: $(x, y) = \left(2\,e^{[2\ln(\pi/4)]/2}, 1\right)$ (max)

$(x, y) = \left(2\,e^{[2\ln(3\pi/4)]/2}, -1\right)$ (min)

$(x, y) = \left(2\,e^{[2\ln(5\pi/4)]/2}, 1\right)$ (max)

inflections: $(x, y) = (-1.687, 0.758)$

$(x, y) = (1.076, -0.280)$

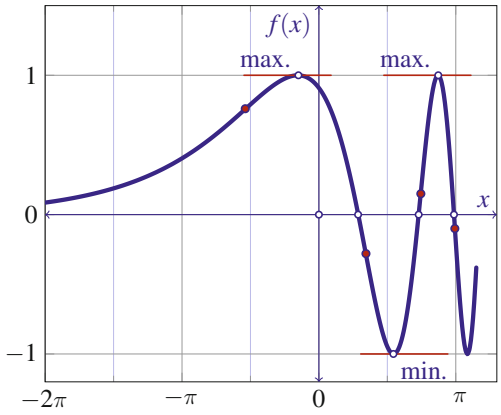


Fig. 3.100 Example
P.3.114

$$(x, y) = (2.338, 0.153)$$
$$(x, y) = (3.122, -0.104)$$

9. *Graphical representation:* the above analysis is summarized in Fig. 3.100.

3.7 Orthogonal Functions

3.115. Given function

$$f(x) = -\frac{x}{2} + 1$$

Given the horizontal coordinate $x_1 = 0$, the vertical coordinate of $f(x_1)$ is

$$f(x_1) = -\frac{(0)}{2} + 1 = 1 = y_1 \quad \therefore \quad \mathscr{A} = (0, 1)$$

By definition, the intersect point is found as

$$f(x) = g(x)$$
$$\therefore$$
$$-\frac{x}{2} + 1 = ax + b$$

The intersect point is at $\mathscr{A} = (0, 1)$, thus

$$y_1 = -\frac{(0)}{2} + b = 1 \quad \therefore \quad b = 1 \quad \therefore \quad g(x) = ax + 1$$

Obviously, one equation is not sufficient to determine two parameters (a and b) at the same time. Therefore, for any given value of a, there is an infinity of linear functions that cross $\mathscr{A} = (0, 1)$ point each line with different slope, as illustrated in Fig. 3.101 (left).

3.116. Given function

$$f(x) = -\frac{x}{2} + 1$$

Given the horizontal coordinate $x_1 = 0$, the vertical coordinate of $f(x_1)$ is

$$f(x_1) = -\frac{(0)}{2} + 1 = 1 = y_1 \quad \therefore \quad \mathscr{A} = (0, 1)$$

By definition, the intersect point is found as

$$f(x) = g(x)$$
$$\therefore$$
$$-\frac{x}{2} + 1 = ax + b$$

The intersect point is at $\mathscr{A} = (0, 1)$, thus

$$y_1 = -\frac{(0)}{2} + b = 1 \quad \therefore \quad b = 1 \quad \therefore \quad g(x) = ax + 1$$

Obviously, one equation is not sufficient to determine two parameters (a and b) at the same time. Therefore, for any given value of a, there is an infinity of linear functions that cross $\mathscr{A} = (0, 1)$ point each line with different slope. The second condition is that $g(x)$ also crosses $\mathscr{B} = (-1, -1)$ point, which is encoded as

$$g(x) = ax + 1$$
$$\therefore$$
$$-1 = a(-1) + 1 \quad \therefore \quad a = 2 \quad \therefore \quad g(x) = 2x + 1$$

as illustrated in Fig. 3.101 (right).

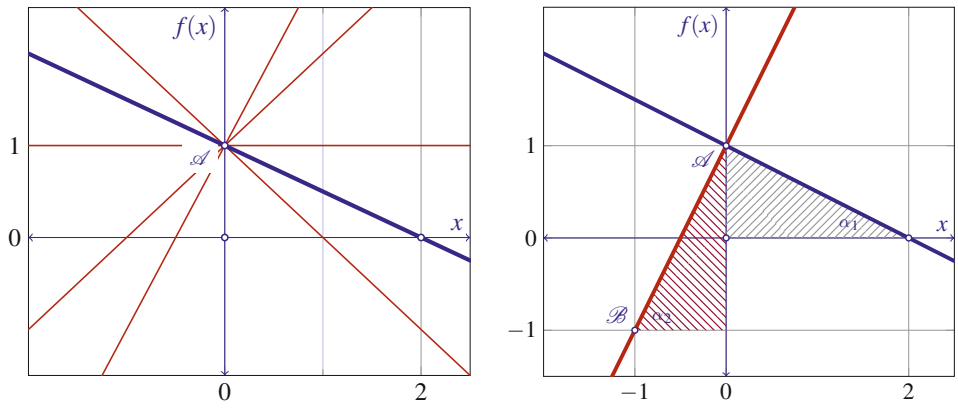


Fig. 3.101 Examples P.3.115, P.3.116

3.117. Given functions

$$f(x) = -\frac{x}{2} + 1$$
$$g(x) = 2x + 1$$

By inspection of graph Fig. 3.101 (right), it should be obvious that two shaded right-angled (orthogonal) triangles) are similar, i.e., rotated versions of each other. Furthermore, clearly they are rotated by $\pi/2$, where their two hypotenuses are aligned with $f(x)$ and $g(x)$. Therefore, $f(x)$ and $g(x)$ are orthogonal.

Formal verification is to calculate angles of the two linear functions. By definition, the ratio of the lengths of the catheti defines tangent of angle α created by hypotenuse and cathetus "facing" α (within a right–angled triangle, Pythagorean theorem), i.e., the two angles are found as

$$\tan\alpha_1 = \frac{1}{2} \quad \therefore \quad \alpha_1 = \arctan\frac{1}{2} = 26.57^\circ$$
$$\tan\alpha_2 = \frac{2}{1} \quad \therefore \quad \alpha_2 = \arctan 2 = 63.43^\circ$$

whose sum is $\alpha = 90^\circ$. Note that $\arctan x$ is calculated in the first quadrant.

3.118. Given function

$$f(x) = 4x + 3$$

1. Given the horizontal coordinate $x_1 = 0$, the vertical coordinate of $f(x)$ is

$$f(x_1) = 4(0) + 3 = 3 = y_1 \quad \therefore \quad \mathscr{A} = (0, 3)$$

Slope $m(x)$ of $f(x)$ for any x is $f'(x) = 4$; therefore, the slope of an orthogonal function must be $m_1 = -1/4$. Linear function $N_1(x)$ passing through $\mathscr{A} = (0, 3)$ must be

$$N_1(x) = -\frac{1}{4}x + b_1 \quad \therefore \quad N_1(0) = -\frac{1}{4}(0) + b_1 = 3 \quad \therefore \quad b_1 = 3$$
$$\therefore \quad N_1(x) = -\frac{1}{4}x + 3$$

2. Similarly, $f(-1) = 4(-1) + 3 = -1$, which completes coordinates for point $\mathscr{B} = (-1, -1)$. In addition, N_2 is normal to $f(x)$, in other words parallel to $N_1(x)$ except that it crosses $\mathscr{B}$ leading to

$$N_2(x) = -\frac{1}{4}x + b_2 \quad \therefore \quad N_2(-1) = -\frac{1}{4}(-1) + b_1 = -1 \quad \therefore \quad b_1 = -\frac{5}{4}$$
$$\therefore \quad N_1(x) = -\frac{1}{4}x - \frac{5}{4}$$

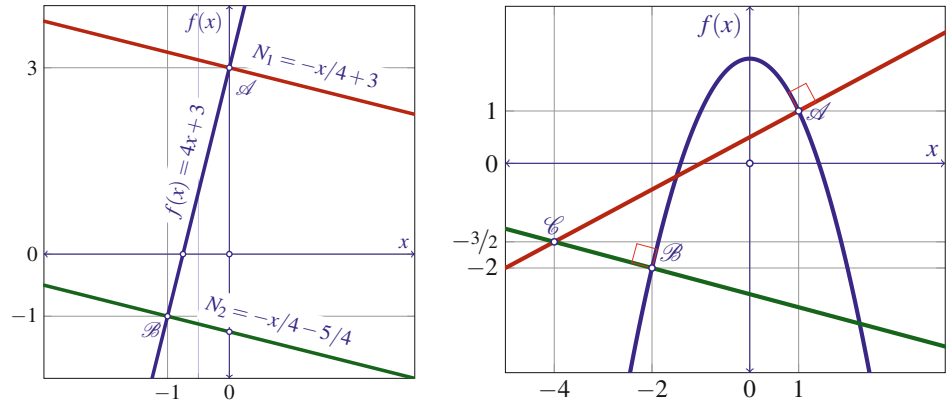


Fig. 3.102 Examples P.3.118, P.3.119

These two vertical functions $N_1(x)$ and $N_2(x)$ are illustrated in Fig. 3.102 (left).

3.119. Given function

$$f(x) = -x^2 + 2$$

By definition, first derivative of function $f(x)$ is a linear function $T(x) = y = mx + b$ that only touches $f(x)$ at point (x, y). Therefore, first, the slope of tangent line m at any point is found as derivative of $f(x)$. Then, among all possible parallel lines (in other words, with the same slope m), there is only one line that crosses specific coordinates (x_1, y_1).

1. Given the horizontal coordinate $x_1 = 1$, the corresponding vertical coordinate of $f(x)$ is

$$f(x_1) = -1^2 + 2 = 1 = y_1 \quad \therefore \quad \mathscr{A} = (1, 1)$$

Slope $m(x)$ of $f(x)$ for any x is $f'(x) = -2x$; therefore, $f'(x_1) = -2$, and the slope of orthogonal function must be $m_1 = 1/2$. Linear function $N_1(x)$ passing through $\mathscr{A} = (1, 1)$ must be

$$N_1(x) = \frac{1}{2}x + b_1 \quad \therefore \quad N_1(1) = \frac{1}{2}(1) + b_1 = 1 \quad \therefore \quad b_1 = \frac{1}{2}$$

$$\therefore \quad N_1(x) = \frac{1}{2}x + \frac{1}{2}$$

2. Similarly, given the horizontal coordinate $x_2 = -2$, the corresponding vertical coordinate of $f(x)$ is

$$f(x_2) = -(-2)^2 + 2 = -2 = y_2 \quad \therefore \quad \mathscr{B} = (-2, -2)$$

Slope $m(x)$ of $f(x)$ for any x is $f'(x) = -2x$; therefore, $f'(x_2) = 4$, and the slope of orthogonal function must be $m_2 = -1/4$. Linear function $N_2(x)$ passing through $\mathscr{B} = (-2, -2)$ must be

$$N_2(x) = -\frac{1}{4}x + b_1 \;\;\therefore\;\; N_2(-2) = -\frac{1}{4}(-2) + b_1 = -2 \;\;\therefore\;\; b_2 = -\frac{5}{2}$$
$$\therefore\;\; N_2(x) = -\frac{1}{4}x - \frac{5}{2}$$

3. Two functions intersect when they share the same point, in other words, when they are equal. In this case,

$$N_1(x) = N_2(x) \;\;\therefore\;\; \frac{1}{2}x + \frac{1}{2} = -\frac{1}{4}x - \frac{5}{2} \;\;\therefore\;\; x = -4 \quad \text{thus,} \quad N_1(-4) = N_2(-4) = -\frac{3}{2}$$

That is to say, the intersect point is $\mathscr{C} = (-4, -3/2)$.

The two orthogonal lines and three relevant points are illustrated in Fig. 3.102 (right).

3.120. Given function $f(x)$ in P.3.45,

$$f(x) = 2 - \frac{2(1-x)}{x^2+1} = \frac{2x(x+1)}{x^2+1}$$

By definition, first derivative of function $f(x)$ is a linear function $T(x) = y = mx + b$ that only touches $f(x)$ at point (x, y). Therefore, first, the slope of tangent line m at any point is found as derivative of $f(x)$. Then, among all possible parallel lines (in other words, with the same slope m), there is only one line that crosses specific coordinates (x_1, y_1).

1. Given the horizontal coordinate $x_1 = 1$, the vertical coordinate of the tangent at point $\mathscr{A}$ is then found as:

$$f(x_1) = \frac{2 \cdot 1(1+1)}{1^2+1} = 2 = y_1 \;\;\therefore\;\; \mathscr{A} = (1, 2)$$

Slope $m(x)$ of tangent $T(x)$ at any point x is derived as

$$m(x) = f'(x) = \left(\frac{2x^2+2x}{x^2+1}\right)' = \frac{(4x+2)(x^2+1) - 2x(2x^2+2x)}{(x^2+1)^2}$$
$$= \frac{\cancel{4x^3} + 2x^2 + 4x + 2 - \cancel{4x^3} - 4x^2}{(x^2+1)^2} = -2\,\frac{x^2-2x-1}{(x^2+1)^2}$$

then, at the specific point $\mathscr{A} = (x_1, y_1) = (1, 2)$, slope m_1 of tangent $T_1(x_1)$ is

$$m_1 = f'(x_1) = \frac{-2(1^2-2-1)}{(1^2+1)^2} = 1 \;\;\therefore\;\; \underline{m_1 = 1}$$

Therefore, equation of tangent at point $\mathscr{A} = (x_1, y_1) = (1, 2)$ is $T_1 = m_1 x + b_1 = 1 \cdot x + b_1$, where constant b_1 is found from the condition that $T_1(x_1) = y_1$. That is to say

$$\underline{x_1 = 1:} \;\;\therefore\;\; \underline{y_1 = 2} \;= T_1(x_1) = m_1 x_1 + b_1 \;\;\therefore\;\; 2 = 1 \cdot 1 + b_1 \;\;\therefore\;\; b_1 = 1 \;\;\therefore\;\; T_1(x) = x + 1$$

This tangent is shown in Fig. 3.103 (left).

2. Given straight line $y = ax + b = -x$ it follows that its slope equals $m = a = -1$ and the constant $b = 0$. In other words, it is a linear function that crosses the origin of the coordinate system $(0, 0)$. As all parallel lines must have the same slope, i.e., $m = a = -1$, then by consequence, they are in the form of

$$T_2 = -x + b_2$$

The question in this example is to find at what value of $x = x_2$ the slope $m = -1$, i.e., the first derivative $f'(x) = -1$?

Thus,

$$f'(x) = \frac{-2(x^2 - 2x - 1)}{(x^2 + 1)^2} = -1 \quad \therefore \quad -2(x^2 - 2x - 1) = -(x^4 + 2x^2 + 1)$$
$$\therefore \quad \underline{x^4 + 4x + 3 = 0}$$

This fourth-order polynomial equation may be resolved by exploiting the factor theorem and factors of number 3, i.e.,

$$3 = (-1)(-3) = (1)(3)$$

First, it is easy to find that $x = -1$ is one of the roots of this polynomial as

$$P(x) = x^4 + 4x + 3 \quad \therefore \quad P(-1) = (-1)^4 + 4(-1) + 3 = 0$$

Therefore, $P(x)$ is divisible by $(x + 1)$ binomial, then long division of $P(x)$ gives

$$(x^4 + 4x + 3) \div (x + 1) = x^3 - x^2 + x + 3$$

Repetition of the factor theorem yields second root $x = -1$, as

$$Q(x) = x^3 - x^2 + x + 3 \quad \therefore \quad Q(-1) = (-1)^3 - (-1)^2 + (-1) + 3 = 0 \quad \therefore$$
$$(x^3 - x^2 + x + 3) \div (x + 1) = x^2 - 2x + 3$$

This quadratic polynomial does not heave real zeros, thus

$$P(x) = x^4 + 4x + 3 = (x + 1)^2(x^2 - 2x + 3)$$

Real zeros of the fourth order equation are

$$(x + 1)^2(x^2 - 2x + 3) = 0 \quad \therefore \quad \underline{x_{1,2} = -1}$$
$$\therefore \quad f'(-1) = 0 \quad \text{as well as} \quad \underline{f(-1) = 0}$$

In conclusion, at the point $\mathscr{B} = (-1, 0)$, the slope of the tangent line to $f(x)$ is $m = -1$. Therefore, the exact form of that tangent T_2 is linear function as

$$T_2 = -x + b_2 \quad \therefore \quad -1 = 0\,x + b_2 \quad \therefore \quad b_2 = -1 \quad \therefore \quad \underline{T_2(x) = -x - 1}$$

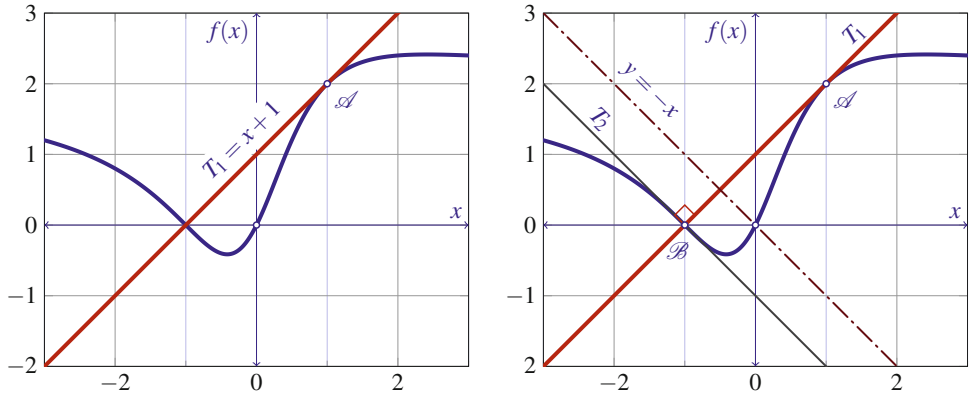


Fig. 3.103 Example P.3.120

3. General orthogonality condition for two lines with slopes m_1 and m_2 is that their slopes (derivatives) at the given point have negative inverse relationship as

$$m_1 = -\frac{1}{m_2}$$

which is exactly the case with T_1 and T_2 (i.e. $1 = -1/(-1)$). Therefore, these two tangents $T_1(x)$ and $T_2(x)$ are normal to each other, as illustrated in Fig. 3.103 (right).

3.121. Given function $f(x)$

$$f(x) = x^3 + 2x^2 - 1$$

By definition, first derivative of function $f(x)$ is a linear function $T(x) = y = m_1 x + b$ that only touches $f(x)$ at point (x, y). Therefore, the tangent's slope m_1 at any point is found as derivative of $f(x)$ at that point. In addition, it is always possible to envision another nonlinear function $g(x)$ whose tangent (i.e., first derivative) at the same point (x, y) is orthogonal to the $f(x)$ tangent, in other words whose first derivative at (x, y) is $m_2 = -1/m_1$.

1. Given the horizontal coordinate $x_1 = -1$, the vertical coordinate of $f(x)$ is

$$f(x_1) = (-1)^3 + 2(-1)^2 - 1 = 0 = y_1 \quad \therefore \quad \mathscr{A} = (-1, 0)$$
$$f'(x) = 3x^2 + 4x \quad \therefore \quad f'(-1) = -1 = m_1$$

therefore slope of the orthogonal function must be $m_2 = -1/m_1 = 1$. The two functions, $f(x)$ and $g(x)$, must satisfy two equations: first, obviously, they must share $\mathscr{A} = (-1, 0)$ point; second, at $\mathscr{A} = (-1, 0)$ point, their derivatives must satisfy the condition of orthogonality, i.e.,

$$\underline{f(-1) = 0 = g(-1)} \quad \therefore \quad ax_1^2 + bx_1 + \frac{1}{2} = a(-1)^2 + b(-1) + \frac{1}{2} = a - b + \frac{1}{2} = 0$$

$$\underline{g'(-1) = 1} \quad \therefore \quad 2ax_1 + b = 2a(-1) + b = -2a + b = 1$$

This system of equations is solved, for example, as

$$a = b - \frac{1}{2} \Rightarrow -2\,(b - 1/2) + b = -2b + 1 + b = 1 \quad \therefore \quad -b = 0 \quad \therefore \quad \underline{b = 0}$$

$$a = 0 - 1/2 \quad \therefore \quad \underline{a = -1/2}$$

In conclusion, quadratic function $g(x) = ax^2 + bx + 1/2$ that is normal to $f(x) = x^3 + 2x^2 - 1$ at point $\mathscr{A} = (-1, 0)$ is in the form (see Fig. 3.104) of

$$g(x) = -\frac{1}{2}x^2 + \frac{1}{2}$$

2. Similarly, given the horizontal coordinate $x_1 = 1$, the vertical coordinate of $f(x)$ is

$$f(x_1) = (1)^3 + 2(1)^2 - 1 = 2 = y_1 \quad \therefore \quad \mathscr{A} = (1, 2)$$

$$f'(x) = 3x^2 + 4x \quad \therefore \quad f'(1) = 7 = m_1$$

therefore, slope of the orthogonal function must be $m_2 = -1/m_1 = -1/7$. The two functions, $f(x)$ and $g(x)$, must satisfy two equations: first, obviously, they must share $\mathscr{B} = (1, 2)$ point; second, at $\mathscr{A} = (1, 2)$ point, their derivatives must satisfy the condition of orthogonality, i.e.,

$$\underline{f(1) = 2 = g(1)} \quad \therefore \quad ax_1^2 + bx_1 + \frac{1}{2} = a(1)^2 + b(1) + \frac{1}{2} = a + b + \frac{1}{2} = 2$$

$$\underline{g'(1) = -1/7} \quad \therefore \quad 2ax_1 + b = 2a(1) + b = 2a + b = -1/7$$

This system of equations is solved, for example, as

$$a = -b + \frac{3}{2} \Rightarrow 2\left(-b + \frac{3}{2}\right) + b = -2b + 3 + b = -1/7 \quad \therefore \quad \underline{b = 22/7}$$

$$a = -22/7 + \frac{3}{2} \quad \therefore \quad \underline{a = -23/14}$$

In conclusion, quadratic function $g(x) = ax^2 + bx + 1/2$ that is normal to $f(x) = x^3 + 2x^2 - 1$ at point $\mathscr{B} = (1, 2)$ is in the form (see Fig. 3.104) of

$$g(x) = -\frac{23}{14}x^2 + \frac{22}{7}x + \frac{1}{2}$$

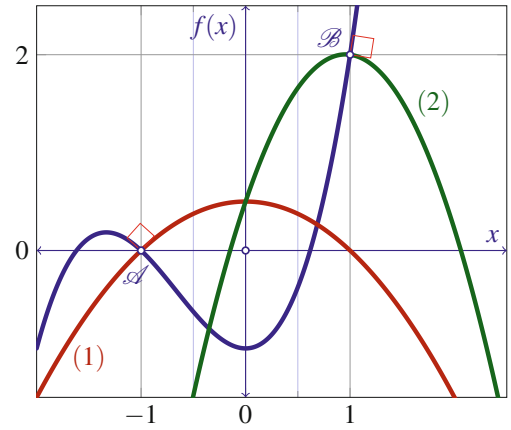


Fig. 3.104 Example
P.3.121

Multivariable Functions

4

Limits: given multivariable functions in the form $f(x, y, z, \dots)$, there are multiple limits to calculate

$$\lim_{\substack{x\to x_0,\\ y\to y_0,\\ (x,z)\to(x_0,z_0),\\ (x,y,z)\to(x_0,y_0,z_0),\\ \dots}} f(x, y, z, \dots)$$

where the function limit exists if all variable limits converge to the same value; otherwise, the limit does not exist. Note that convergence of each individual limit by itself is a necessary but not sufficient condition to prove the existence of the global limit.

Partial derivatives: are calculated for each variable relative to all the others. For example, in the case of a two-variable function $f(x, y)$, the following equivalent notations are used to specifically write the order of second partial derivatives as

$$\left(f'_x\right)_x = f''_{xx} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\left(f'_x\right)_y = f''_{xy} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\left(f'_y\right)_x = f''_{yx} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\left(f'_y\right)_y = f''_{yy} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Integrals: volume of a multidimensional object is calculated as integral along each variable (the integral sum is a commutative operation). For example, in the case of three-dimensional space, we systematically reduce the number of variables as

$$V_{3D} = \iiint_{x,y,z} f(x, y, z)\ dx\ dy\ dz$$

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R. Sobot, *Engineering Mathematics by Example*,
https://doi.org/10.1007/978-3-031-41196-0_4

$$= \iint_{y,z} dy\, dz \underbrace{\int_x f(x, y, z)\, dx}_{f(y,z)}$$

$$= \int_z dz \underbrace{\int_y f(y, z)\, dy}_{f(z)}$$

$$= \int_z f(z)\, dz$$

Problems

4.1 Domains of Typical 3D Surfaces

Sketch plot of surfaces in P.4.2 to P.4.4.

4.1. $f(x, y) = x^2$

4.2. $f(x, y) = x^2 + y^2$

4.3. $z^2 = x^2 + y^2$

4.4. $r^2 = x^2 + y^2 + z^2$

Determine domains of functions in P.4.5 to P.4.8.

4.5. $f(x, y) = x + \sqrt{y}$.

4.6. $f(x, y) = \ln(-x - y)$

4.7. $g(x, y) = \sqrt{9 - x^2 - y^2}$

4.8. $h(x, y) = \sqrt{1 - x^2} + \sqrt{y^2 - 1}$

4.2 Multivariable Limits

Calculate limits of functions in P.4.9 to P.4.12.

4.9. $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$

4.10. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

4.11. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$

4.12. $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2 + y^2}$

4.3 Partial Derivatives

Calculate partial derivatives of functions in P.4.13 to P.4.18. Then, if applicable, calculate derivatives at given points.

4.13. Calculate $f'_x(1, 1)$ and $f'_y(1, 1)$ if $f(x, y) = 4 - x^2 - 2y^2$.

4.14. $f(x, y) = \sin \dfrac{x}{1+y}$.

4.15. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

4.16. Calculate f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy} if $f(x, y) = x^3 + x^2y^3 - 2y^2$.

4.17. Calculate partial derivatives of $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$.

4.18. Given $u = \ln \sqrt{(x-a)^2 + (y-b)^2}$, where a, b are constants, prove the following equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4.4 Multivariable Integrals

Calculate integrals of functions in P.4.19 to P.4.20.

4.19. Calculate the following integral: $\displaystyle\int_0^3 \int_1^2 x^2 y \, dx \, dy$

4.20. Given 3D surface is defined as $\displaystyle\iint_R \sqrt{1 - x^2} \, dA$ where domain R is bound by (x, y) variables. Calculate surface A, given that $|y| \leq 2$.

Answers

4.1 Domains of Typical 3D Surfaces

4.1. Function $f(x, y) = x^2$ does not depend on y; thus, when looking "into" the y-axis at the x-z vertical plane, there is only quadratic function $z = x^2$ visible. However, this quadratic function exists at any position along $y \in [-\infty, \infty]$ axis; see Fig. 4.1.

4.2. While looking into the z-axis, starting with point $(x, y) = (0, 0)$, that is to say, at $z = f(x, y) = 0$, function $f(x, y) = x^2 + y^2$ defines a circle whose radius increases along the z-axis; see Fig. 4.2. At the same time, while looking sideways into either the x- or y-axis direction at the vertical plane, there is a quadratic function: $z = x^2$ if $y = 0$ or $z = y^2$ if $x = 0$.

4.3. While looking into the z-axis, starting with point $(x, y) = (0, 0)$, that is to say, at $z = f(x, y) = 0$, two-sided cone function $z^2 = x^2 + y^2$ defines a circle whose radius increases along the z-axis; see Fig. 4.3. Looking sideways, however, in either x- or y-direction, there is a linear function, $|z| = |x|$ if $y = 0$ and $|z| = |y|$ if $x = 0$.

Fig. 4.1 Example P.4.1

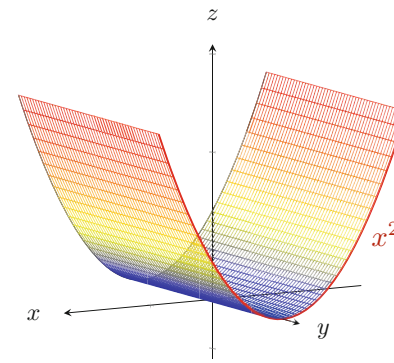


Fig. 4.2 Example P.4.2

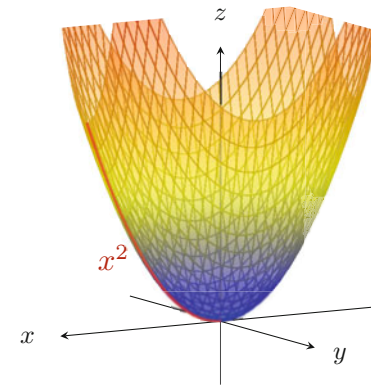


Fig. 4.3 Example P.4.3

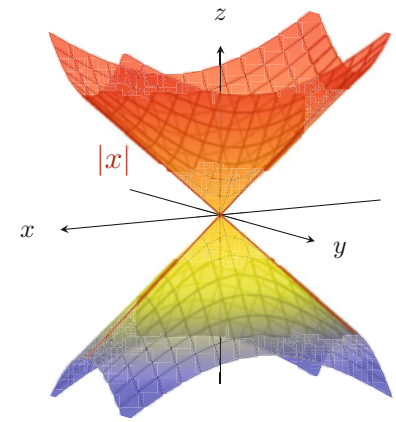
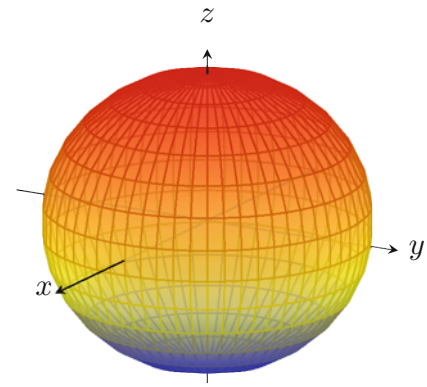


Fig. 4.4 Example P.4.4



4.4. Function $r^2 = x^2 + y^2 + z^2$ defines a sphere whose radius equals r; see Fig. 4.4. Looking from any of the three directions, there is always a circle projection, $r^2 = x^2 + y^2$ if $z = 0$, $r^2 = x^2 + z^2$ if $y = 0$, and $r^2 = y^2 + z^2$ if $x = 0$.

4.5. Function $f(x, y) = x + \sqrt{y}$ is defined for $x \in \mathbb{R}$ and $y \geq 0$.

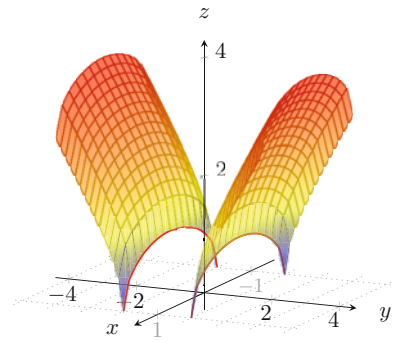
4.6. Function $f(x, y) = \ln(-x - y)$ is limited by ln function; thus, $-x - y > 0 \;\; \therefore \;\; x + y < 0$.

4.7. Given $g(x, y) = \sqrt{9 - x^2 - y^2}$ is limited by the domain of radical function; thus,

$$D = \left\{(x, y) \,\middle|\, 9 - x^2 - y^2 \geq 0\right\} \;\; \therefore \;\; \left\{(x, y) \,\middle|\, x^2 + y^2 \leq 3^2\right\}$$

Thus, in x, y plane, $g(x, y)$ is contained within a circle whose radius $r = 3$, z is limited only to $z \geq 0$, and $\sqrt{9 - x^2 - y^2} \leq 3$ (i.e., when $(x, y) = (0, 0)$; i.e., $0 \leq z \leq 3$, or $g(x, y)$ defines the upper half-sphere).

Fig. 4.5 Example P.4.8



4.8. Function $h(x, y) = \sqrt{1 - x^2} + \sqrt{y^2 - 1}$ is limited by two square roots (see Fig. 4.5); thus,

$$1 - x^2 \geq 0 \quad \therefore \quad x^2 \leq 1 \quad \therefore \quad |x| \leq 1$$
$$y^2 - 1 \geq 0 \quad \therefore \quad y^2 \geq 1 \quad \therefore \quad |y| \geq 1$$

Looking sideways into the y-axis, i.e., when $x \leq 1$, projection of $h(x, y)$ function onto the upper x-z vertical plane is a half-circle function $z^2 + x^2 = 1^2$, i.e., its radius equals one at $y = 1$. At the same time, looking into the x-axis when $y \geq 1$, projection of $h(x, y)$ function onto the upper y-z vertical plane is a hyperbola $y^2 - z^2 = 1$ for $x = 0$.

4.2 Multivariable Limits

4.9. Limits of multivariable functions must be calculated along multiple directions. That is to say, in the first iteration, limits are calculated along each variable. However, in order for a limit to be defined (i.e., given surface is continuous in any direction), *all limits along any arbitrary directions must be equal*. That condition is main difference relative to the single-variable functions where there is only one possible path along the line. In this case, function is continuous along both x and y lines; thus,

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 2^3 - 1^3 2^2 + 3(1) + 2(2) = 11$$

4.10. In the case when a function is not defined at a given point, as

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{0}{0}$$

the existence of such limits must be verified as same as for single-variable functions, for example, by L'Hôpital's rule or some other techniques. As the very first test, limits should be tested along the x- and y-axes separately, as

$$\lim_{(x,y)\to(x,0)} \frac{x^2 - 0^2}{x^2 + 0^2} = \frac{\cancel{x^2}}{\cancel{x^2}} = 1$$

$$\lim_{(x,y)\to(0,y)} \frac{0^2 - y^2}{0^2 + y^2} = \frac{-y^2}{y^2} = -1$$

These two limits are not equal; thus, it is sufficient to conclude that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

4.11. Even if the two limits along the x- and y-axes are equal, that is a necessary but not sufficient condition. As in this example,

$$\lim_{(x,y)\to(x,0)} \frac{xy}{x^2 + y^2} = \frac{x\,(0)}{x^2 + 0} = 0$$

$$\lim_{(x,y)\to(0,y)} \frac{xy}{x^2 + y^2} = \frac{(0)\,y}{0 + y^2} = 0$$

however, limit along $y = x$ is

$$\lim_{(x,y)\to(x,x)} \frac{xy}{x^2 + y^2} = \frac{x\,x}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

Therefore, even though the first two limits are equal, they are different from the third. It is sufficient to find a case that is different to conclude that the limit at a given point does not exist.

4.12. This limit appears to not exist, because

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{however,} \quad \lim_{(x,y)\to(x,0)} \frac{3x^2y}{x^2 + y^2} = \frac{0}{x^2} = 0$$

$$\lim_{(x,y)\to(0,y)} \frac{3x^2y}{x^2 + y^2} = \frac{0}{y^2} = 0$$

$$\lim_{(x,y)\to(x,x)} \frac{3x^2}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{3x}{2} = 0$$

Even though these three limits are equal, it is not certain that there is no some other path in existence on the surface that leads to a different result. The question is how to prove that this limit exists along any arbitrary line? One possible technique is to apply "the squeeze theorem" (see Sec. 1.4).

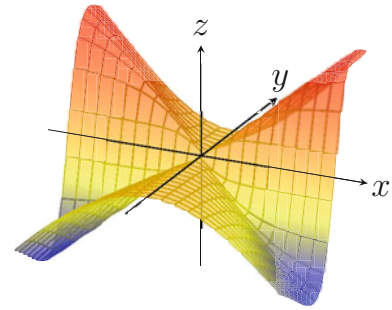
Distance between point $f(x, y) = 0$ and any other arbitrary point on $f(x, y)$ surface may be expressed as

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| = \left| \frac{3x^2y}{x^2 + y^2} \right| = \left| \frac{3x^2}{x^2 + y^2} \right| |y| = \frac{3x^2\,|y|}{x^2 + y^2} \tag{4.1}$$

because "3", x^2, and y^2 are all always positive; thus,

$$\text{as } (3, x^2, y^2) \geq 0 \quad \Rightarrow \quad \left| \frac{3x^2}{x^2 + y^2} \right| = \frac{3x^2}{x^2 + y^2}$$

Fig. 4.6 Example P.4.12



Therefore, the first conclusion is that the lower side limit of (4.1) is zero. On the upper side, because it is always true that $x^2 \leq x^2 + y^2$, then the second conclusion is that

$$x^2 \leq x^2 + y^2 \;\Rightarrow\; \frac{x^2}{x^2+y^2} \leq 1 \;\Rightarrow\; 3 \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} \leq 3$$

Once the two boundaries are established, (4.1) may be written as double inequality

$$\begin{gathered}
\text{lower boundary} \leq \; f(x,y) \; \leq \text{upper boundary} \\
0 \leq \frac{3x^2y}{x^2+y^2} \leq 3\,|y| \\
\therefore \; \{\text{apply the squeeze theorem}\} \\
\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} 3\,|y| \qquad (4.2) \\
0 \leq \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} \leq 0
\end{gathered}$$

As both left- and right-side inequalities in (4.2) tend to the origin, i.e., $(x, y) \to (0, 0)$, then by the virtue of its position between the two extremes, any point at distance $f(x, y)$ is "squeezed" between two limits and must also converge to zero; see Fig. 4.6. In summary, even though $f(0, 0)$ is not defined, this limit exists as in any direction

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

4.3 Partial Derivatives

4.13. Partial derivatives are calculated relative to the indicated variable, while all the other variables are assumed to be constant, i.e.,

$$f'_x = \frac{\partial}{\partial x}(4 - x^2 - 2y^2) = -2x \quad \therefore \quad f'_x(1,1) = -2$$
$$f'_y = \frac{\partial}{\partial y}(4 - x^2 - 2y^2) = -4y \quad \therefore \quad f'_x(1,1) = -4$$

4.14. Partial derivatives of composite multivariable fractions follow the same rules as for single-variable derivatives, i.e.,

$$f'_x = \frac{\partial}{\partial x}\sin\frac{x}{1+y} = \cos\frac{x}{1+y}\,\frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \frac{1}{1+y}\,\cos\frac{x}{1+y}$$
$$f'_y = \frac{\partial}{\partial y}\sin\frac{x}{1+y} = \cos\frac{x}{1+y}\,\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\frac{x}{(1+y)^2}\,\cos\frac{x}{1+y}$$

4.15. Implicit partial derivation is done by following the rules for composite functions as well as for any other ratio/product terms,

$$\frac{\partial z}{\partial x}\Big(\,x^3 + y^3 + z^3 + \underbrace{6y\,xz}_{f(x)f(z)} = 1\,\Big) \quad \therefore \quad 3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$
$$\therefore \quad \frac{\partial z}{\partial x}\left(3z^2 + 6\,xy\right) = -3x^2 - 6\,yz$$
$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y}\Big(\,x^3 + y^3 + z^3 + \underbrace{6x\,yz}_{f(y)f(z)} = 1\,\Big) \quad \therefore \quad 3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$$
$$\therefore \quad \frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

4.16. Second derivatives may be relative to the same variable as the first derivative or mixed.

$$f'_x\left(x^3 + x^2y^3 - 2y^2\right) = 3x^2 + 2xy^3$$
$$f'_y\left(x^3 + x^2y^3 - 2y^2\right) = 3x^2y^2 - 4y$$
$$\therefore$$
$$f''_{xx} = \frac{\partial f'_x}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 + 2xy^3\right) = 6x + 2y^3$$
$$f''_{xy} = \frac{\partial f'_x}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 + 2xy^3\right) = 6xy^2$$
$$f''_{yx} = \frac{\partial f'_y}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y^2 - 4y\right) = 6xy^2$$
$$f''_{yy} = \frac{\partial f'_y}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y^2 - 4y\right) = 6x^2y - 4$$

4.17. Given parametric form of function $z(x, y) = e^x \sin y$, where $x(s, t) = st^2$ and $y(s, t) = s^2 t$ are expressed relative to parameters s and t, its derivatives are calculated by following the composite function rules as

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^x(\sin y)\, t^2 + e^x(\cos y)\, 2st \\
&= t^2\, e^{st^2} \sin(s^2 t) + 2st\, e^{st^2} \cos(s^2 t) \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^x(\sin y)\, 2st + e^x(\cos y)\, s^2 \\
&= 2st\, e^{st^2} \sin(s^2 t) + s^2\, e^{st^2} \cos(s^2 t)
\end{aligned}$$

Note how each fraction relative to one variable may be expanded to include any other variable, e.g.,

$$\frac{\partial y}{\partial x} \equiv \frac{\partial y}{\partial x}\,\frac{\partial t}{\partial t} \equiv \frac{\partial y}{\partial t}\frac{\partial t}{\partial x}$$

4.18. Given

$$u = \ln\sqrt{(x-a)^2 + (y-b)^2}$$

it follows that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}}\;\frac{\partial}{\partial x}\sqrt{(x-a)^2 + (y-b)^2} \\
&= \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}}\;\frac{1}{2\sqrt{(x-a)^2 + (y-b)^2}}\times \\
&\quad \frac{\partial}{\partial x}\left[(x-a)^2 + (y-b)^2\right] \\
&= \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}}\;\frac{1}{2\sqrt{(x-a)^2 + (y-b)^2}}\;\;[2(x-a)] \\
&= \frac{x-a}{(x-a)^2 + (y-b)^2} \\
&\quad \therefore \\
\frac{\partial^2 u}{\partial x^2} &= \frac{-(x-a)^2 + (y-b)^2}{\left[(x-a)^2 + (y-b)^2\right]^2}
\end{aligned}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{y-a}{(x-a)^2 + (y-b)^2} \quad \therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{(x-a)^2 - (y-b)^2}{\left[(x-a)^2 + (y-b)^2\right]^2}$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}=\frac{-(x-a)^2+(y-b)^2}{\left[(x-a)^2+(y-b)^2\right]^2}+\frac{(x-a)^2-(y-b)^2}{\left[(x-a)^2+(y-b)^2\right]^2}=0 \quad \checkmark$$

4.4 Multivariable Integrals

4.19. Fundamentally, integration is an addition; thus, it is a commutative operation, and the order of integration variables can be changed as necessary.

$$\int_0^3\int_1^2 x^2y\,dx\,dy=\int_0^3 x^2\,dx\int_1^2 y\,dy=\int_0^3 x^2\,dx\quad \frac{y^2}{2}\Big|_1^2=\frac{3}{2}\int_0^3 x^2\,dx$$
$$=\frac{3}{2}\,\frac{x^3}{3}\Big|_0^3=\frac{\cancel{3}}{2}\,\frac{27}{\cancel{3}}=\frac{27}{2}$$

4.20. Double integral delivers surface area bound by the integral limits. Thus, given

$$\iint_R \sqrt{1-x^2}\,dA \tag{4.3}$$

and interval $|y|\le 2$, it is necessary to determine the bounds as well as shape of this surface.

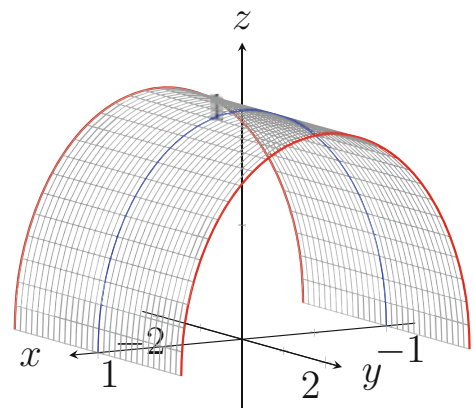
Operation of square root imposes limit $1-x^2\ge 0\ \ \therefore\ \ |x|\le 1$. In addition, $z=\sqrt{1-x^2}$ defines a positive semicircle whose radius $r=1$. After taking into the account the given condition that

$$-2\le y\le 2$$

then (4.3) defines the surface of a half-cylinder whose y-direction length is $l=4$; see Fig. 4.7. The length of half-circumference (see P. 5.101) equals $s=2r\pi/2=\pi$; therefore, the rectangular area that covers this "tunnel" equals

$$A=s\times l=\pi\times 4=4\pi \qquad \therefore$$
$$\iint_R \sqrt{1-x^2}\,dA=4\pi$$

Fig. 4.7 Example P.4.20



Integrals

5

Basic integral solutions are derived from the fundamental theorem of calculus and are traditionally given in the form of "tabular integrals", which are then simply used without additional proofs. A minimal list of tabular integrals and basic properties of integration may be as follows (assuming that constant $a > 0$):

Tabular integrals:

$$\begin{aligned} f(x) &\qquad \int f(x)\,dx \\ x^n &\qquad n\,x^{n-1} + C, \quad (n \neq -1) \\ x^{-1} = \frac{1}{x} &\qquad \ln|x| + C \quad (x \neq 0) \\ a^x &\qquad \frac{a^x}{\ln a} + C \\ \sin x &\qquad -\cos x + C \\ \cos x &\qquad \sin x + C \end{aligned}$$

Basic properties of integration:

$$\int a\,f(x)\,dx = a \int f(x)\,dx \quad \text{(constant } a \text{ is factorized)}$$

$$\int d\big(f(x)\big) = f(x) + C \quad \text{(integral is antiderivative)}$$

$$\int \Big[f(x) \pm g(x)\Big]\,dx = \int f(x)\,dx \pm \int g(x)\,dx \quad \text{(integral of a sum is sum of integrals)}$$

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$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx \quad \text{(partial integration)}$$

Note that the result of integration is *not* unique. That is to say, the integration result is a *family* of functions separated by an arbitrary constant C. That is direct consequence of the fundamental theorem of calculus where, for example, $\int dx = x + C$ because $d(x + C) = dx$.

General strategy to solving integrals:

Tabular forms and basic properties of integration may be used within the following, progressively more involved, steps:

1. Check if the given function is one of the tabular forms (i.e., trivial solution).
2. Decompose the given function's form into tabulated forms.
3. Apply the change of variables method, several times if needed, so that the given function is decomposed into tabulated forms.
4. Apply the method of partial integration, several times if needed.
5. Apply some combination of the above methods.

Problems

5.1 Simple Integrals

Calculate simple integrals in P.5.1 to P.5.8.

5.1. $\int \left(x^3 + 2x - 1\right)\, dx$ **5.2.** $\int \left(x^{-3} + x\right)\, dx$ **5.3.** $\int \frac{x+1}{\sqrt{x}}\, dx$

5.4. $\int (x+1)(2x+1)\, dx$ **5.5.** $\int (3\cos x + \sin x)\, dx$ **5.6.** $\int \sqrt{x\sqrt{x\sqrt{x}}}\, dx$

5.7. $\int \left(e^x + \cos^3 x(1 + \tan^2 x)\right) dx$ **5.8.** $\int \frac{x^2 + x\sqrt[3]{x} + \sqrt{x}}{x\sqrt{x}}\, dx$

5.2 Integration by Change of Variables

Calculate integrals in P.5.11 to P.5.23 by change of variables method. Note that it is not necessary to write the integration constant C during the intermediate steps; it may be summarized and added to the final integration.

5.9. $\int \frac{1}{x \pm a}\, dx$ **5.10.** $\int \frac{1}{(x \pm a)^2}\, dx$ **5.11.** $\int x^2(x^3 + 1)^5\, dx$

5.12. $\int x\sqrt{x^2+1}\,dx$ **5.13.** $\int \frac{x^3}{\sqrt{1-x^4}}\,dx$ **5.14.** $\int \frac{\sin x}{\cos^4 x}\,dx$

5.15. $\int \tan x\,dx$ **5.16.** $\int x\,\cos(x^2+2)\,dx$ **5.17.** $\int \sin(2x)\,dx$

5.18. $\int \sin^2 x\,dx$ **5.19.** $\int \sin^3 x\,dx$ **5.20.** $\int e^{-ax}\,dx$

5.21. $\int x\,e^{ax^2}\,dx$ **5.22.** $\int 5^{x^2}\,x\,dx$ **5.23.** $\int \frac{x^3}{x^2-1}\,dx$

5.24. $\int \cos x\,e^{\sin x}\,dx$

5.3 Integration by Parts

Calculate integrals in P.5.25 to P.5.40 by the partial integration method.

5.25. $\int xe^x\,dx$ **5.26.** $\int xe^{-x}\,dx$ **5.27.** $\int x^2e^x\,dx$

5.28. $\int x^2e^{-x}\,dx$ **5.29.** $\int \ln x\,dx$ **5.30.** $\int x\,\ln\left(1+x^2\right)\,dx$

5.31. $\int x^2\ln x\,dx$ **5.32.** $\int x^3e^{x^2}\,dx$ **5.33.** $\int (x^2-1)\,e^x\,dx$

5.34. $\int x\sin x\,dx$ **5.35.** $\int x\cos(ax)\,dx$ **5.36.** $\int e^x\sin x\,dx$

5.37. $\int \frac{\ln x}{x^2}\,dx$ **5.38.** $\int x^3 5^{x^2}\,dx$ **5.39.** $\int x\,\ln\left(x^2-1\right)\,dx$

5.40. $\int x^2\cos x\,dx$ **5.41.** $\int \sin(2x)\,e^{\sin x}\,dx$ **5.42.** $\int e^{-bx}\,\sin(ax)\,dx$

5.4 Trigonometric Substitutions

Functions in this section are some of classic forms that are solved by applying trigonometric identities and substitutions. Calculate integrals in P.5.43 to P.5.57.

5.43. $\int \sqrt{9-x^2}\,dx$ **5.44.** $\int \frac{x^2}{\sqrt{36-x^2}}\,dx$ **5.45.** $\int \frac{\sqrt{25x^2+4}}{x^4}\,dx$

5.46. $\int \frac{1}{\cos x}\,dx$ **5.47.** $\int \frac{1}{\cos^2 x}\,dx$ **5.48.** $\int \frac{1}{\cos^3 x}\,dx$

5.49. $\int \left(1+\tan^2 x\right)\,dx$ **5.50.** $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$ **5.51.** $\int \sqrt{x^2-a^2}\,dx$

5.52. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ **5.53.** $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$ **5.54.** $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

5.55. $\int \sqrt{a^2+x^2}\,dx$ **5.56.** $\int \frac{1}{1+x^2}\,dx$ **5.57.** $\int \sqrt{e^x-1}\,dx$

5.58. $\int \frac{\sqrt{x^2+2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}\,dx$

5.5 Rational Functions

Calculate integrals in P.5.59 to P.5.66. Note that rational functions are often integrated by applying the partial fraction decomposition technique in Vol. I.

5.59. $\int \frac{x-1}{x^2+x}\,dx$ **5.60.** $\int \frac{x}{(x+1)(x-4)}\,dx$ **5.61.** $\int \frac{1}{x^2-1}\,dx$

5.62. $\int \frac{x+2}{x^3-2x^2}\,dx$ **5.63.** $\int \frac{2x^2}{x^4-1}\,dx$ **5.64.** $\int \frac{1}{x\sqrt{1+x^2}}\,dx$

5.65. $\int \frac{1}{x^3-1}\,dx$ **5.66.** $\int \frac{4}{x^4+1}\,dx$ **5.67.** $\int \frac{x}{1-x^2}\,e^{-2\ln|x+1|}\,dx$

5.6 Definite Integrals

Calculate definite integrals in P.5.68 to P.5.81.

5.68. $\int_2^3 x^3\,dx$ **5.69.** $\int_{-1}^2 x(1+x^3)\,dx$ **5.70.** $\int_{-\pi/4}^{\pi/4} \cos x\,dx$

5.71. $\int_0^{2\pi} \cos x\,dx$ **5.72.** $\int_0^2 \frac{x}{\sqrt{1+3x^2}}\,dx$ **5.73.** $\int_0^{\pi} x^2\cos x\,dx$

5.74. $\int_{-\pi}^{\pi} x^2 \cos(ax)\, dx$

5.75. $\int_{5}^{6} \frac{x}{(x+1)(x-4)}\, dx$

5.76. $\int_{0}^{1} x\, \ln\left(1+x^2\right) dx$

5.77. $\int_{0}^{\ln 2} \sqrt{e^x - 1}\, dx$

5.78. $\int_{-\infty}^{\infty} \frac{a}{a^2+x^2}\, dx$

5.79. $\int_{0}^{\infty} e^{-\omega_0 t\, y}\, dy$

5.80. $\int_{0}^{\infty} e^{-bx}\, \sin(ax)$

5.81. $\int_{-\pi}^{\pi} e^x\, \cos(ax)\, dx$

5.82. $\int_{-\pi}^{\pi} e^x\, \sin(ax)\, dx$

5.7 Area Integral

By using definite integrals, solve problems in P.5.83 to P.5.86.

5.83. Derive the formula for the area of a right angle triangle whose catheti are a and b. To verify the result, use it to calculate the area if the two catheti equal $a = 2 and \;\; b = 1$.

5.84. Derive the formula for the area of a circle whose radius equals r.

5.85. Derive the formula for the volume of a cuboid whose sides equal a, b, c. To verify the result, use it to calculate the volume if the three sides equal $a = 4, b = 3, and c = 2$.

5.86. Calculate $\int_{0}^{1} f(x)^2\, dx$ given that $f(x)$ is a piecewise linear function as in Fig. 5.1.

In P.5.87 to P.5.94, calculate the area between $f(x)$ and $g(x)$ curves. Exclude unbound regions, and, if needed, assume $(x, y \geq 0)$.

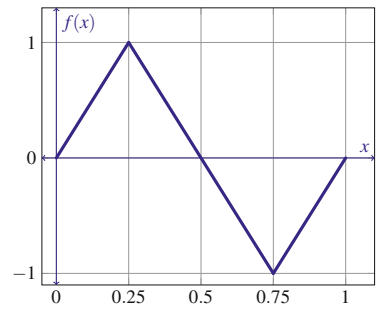
5.87. $f(x) = x^2, \;\; g(x) = 8 - x^2,$

5.88. $7x^2 - 9y + 9 = 0, \;\; 5x^2 - 9y + 27 = 0,$

5.89. $y = x, \;\; x^2 + y - 2 = 0,$

5.90. $y = \sin x, \;\; y = 1/2,$

Fig. 5.1 Example P.5.86



5.91. $x^2 + 4y^2 - 20 = 0, \quad xy - 4 = 0$

5.92. $y = x(x-1)(x-2), \quad y = 0,$

5.93. $x^2 - y^2 = 9, \quad 4x - 5y = 0,$

5.94. $x^2 + y^2 = 8, \quad y^2 = 2x,$

5.8 Volume of a Solid of Revolution

Given a curve and interval (if given; otherwise, calculate the applicable interval), calculate the volume of a solid of revolution in P.5.95 to P.5.100.

5.95. $y = 2, \quad x \in (0, 2)$

5.96. $y = 2x\,, \quad 1 \le x \le 3$

5.97. $y = 4x - x^2$

5.98. $y = 1/x\,, \quad x \in (1, \infty)$

5.99. $y = \ln x\,, \quad y \in (0, 1).$

5.100. $y = \sin x\,, \quad x \in (0, \pi).$

5.9 Line Integral

Given a curve and interval (if given; otherwise, calculate the applicable interval), calculate the line length in P.5.101 to P.5.105. Exclude unbound regions.

Reminder: Arc length s of a continuous function $y = f(x)$ within the interval $x \in (a, b)$ is

$$s = \int_a^b \sqrt{1 + y'^2}\, dx$$

If the function is given in its parametric form, $x = g(t)$ and $y = h(t)$, where the two parametric functions are continuous, then

$$s = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2}\, dt$$

assuming that derivatives are possible.

5.101. $x^2 + y^2 = r^2$

5.102. $y = 4 - x^2$ bound by $y = 0$

5.103. $y = x^2 + 2x$ below $y = 0$

5.104. $\sqrt{x} + \sqrt{y} = 1$

5.105. $x = 1/4\, y^2 - 1/2 \ln y\,, \quad y \in (1, e).$

5.10 Mean of a Function

Reminder: Consider a simple piecewise linear function in Fig. 5.2. Knowing that the geometrical interpretation of a definite integral is the surface of the area under the function over the given interval, it is trivial to conclude that

$$\int_0^4 f(x)\,dx = 4$$

because within [0, 4] interval, there are two non-zero rectangles whose total surface equals four unit squares.

For any arbitrary $f(x)$, it is possible to construct a rectangle whose length equals to the same interval as in the integral (here, four), and at the same time, its surface equals to the value of the integral (here, four). Then, the height (here, one) of that rectangle is referred to as "average value of $f(x)$" and in this example is written as $\langle f(x)\rangle = 1$. In general case, knowing the area and length of a rectangle, its height is calculated as

$$\text{height} = \frac{\text{surface area}}{\text{length}}$$

and equivalently, for any continuous function $f(x)$ within $x \in [a, b]$ interval (length), the formal definition of its average (height) is written as

$$\langle f(x)\rangle = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Calculate average values of given functions $f(x)$ and intervals in P.5.106 to P.5.109.

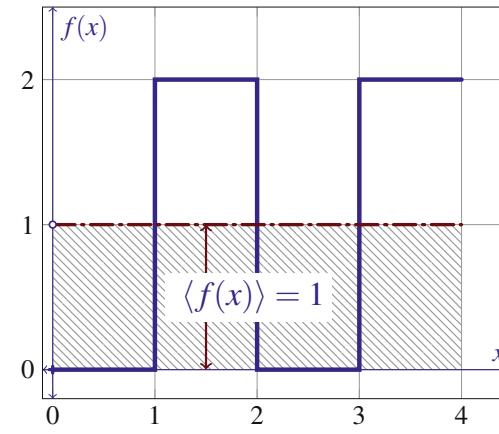


Fig. 5.2 Graphical interpretation of an average value

5.106. $f(x) = \sin x, \quad x \in [0, 2\pi]$

5.107. $f(x) = \cos x, \quad x \in [\pi/2, 3\pi/2]$

5.108. $f(x) = 1 + 0.75 \cos x, \quad x \in [0, 2\pi]$

5.109. $f(x) = \cos^2 x, \quad x \in [0, 2\pi]$

5.11 Improper Integrals

Given definite integrals in P.5.110 to P.5.118, calculate the area under curves.

5.110. $\int_1^\infty \frac{1}{x^2}\, dx$

5.111. $\int_1^\infty \frac{1}{x}\, dx$

5.112. $\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}\, dx$

5.113. $\int_{-\infty}^\infty x e^{-x^2}\, dx$

5.114. $\int_{-\infty}^0 \frac{1}{\sqrt{1-x}}\, dx$

5.115. $\int_0^\infty \frac{1}{1+x^2}\, dx$

5.116. $\int_e^\infty \frac{1}{x \ln^3 x}\, dx$

5.117. $\int_{-1}^1 \frac{1}{x}\, dx$

5.118. $\int_0^{100\pi} \sqrt{1 - \cos 2x}\, dx$

Answers

5.1 Simple Integrals

5.1. $$\int (x^3+2x-1)\,dx = \int x^3\,dx + \int 2x\,dx - \int dx = \left\{\int x^n = \frac{x^{n+1}}{n+1} + C\right\}$$
$$= \frac{x^4}{4} + 2\frac{x^2}{2} - x + C = \frac{x^4}{4} + x^2 - x + C$$

5.2. $$\int (x^{-3}+x)\,dx = \int x^{-3}\,dx + \int x\,dx = \frac{x^{-2}}{-2} + \frac{x^2}{2} + C = -\frac{1}{2x^2} + \frac{x^2}{2} + C$$

5.3. $$\int \frac{x+1}{\sqrt{x}}\,dx = \int \frac{x}{\sqrt{x}}\,dx + \int \frac{1}{\sqrt{x}}\,dx = \int x\,x^{-1/2}\,dx + \int x^{-1/2}\,dx = \int x^{1/2}\,dx + \int x^{-1/2}\,dx$$
$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = \frac{2}{3}\sqrt{x\,x^2} + 2\sqrt{x}$$
$$+ C = \left\{\sqrt{x^2} = |x|\right\} = 2\sqrt{x}\left(\frac{|x|}{3} + 1\right) + C$$

5.4. $$\int (x+1)(2x+1)\,dx = \int (2x^2+3x+1)\,dx = \frac{2x^3}{3} + \frac{3x^2}{2} + x + C$$

5.5. $$\int (3\cos x + \sin x)\,dx = 3\int \cos\,dx + \int \sin x\,dx$$
$$= \left\{\int \cos x\,dx = \sin x, \quad \int \sin x\,dx = -\cos x\right\}$$
$$= 3\sin x - \cos x + C$$

5.6. $$\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int \sqrt{x\sqrt{x\,x^{1/2}}}\,dx = \int \sqrt{x\sqrt{x^{3/2}}}\,dx = \int \sqrt{x\,x^{3/4}}\,dx = \int \sqrt{x^{7/4}}\,dx$$
$$= \int x^{7/8}\,dx = \frac{8}{15}x^{15/8} + C$$

5.7. $$\int \left(e^x + \cos^3 x(1+\tan^2 x)\right)\,dx = \int e^x\,dx + \int \left(\cos^3 x(1+\tan^2 x)\right)\,dx$$
$$= e^x + \int \left[\cos^3 x\left(1 + \frac{\sin^2 x}{\cos^2 x}\right)\right]\,dx$$
$$= e^x + \int \left[\cos^{\not 3} x\left(\frac{\sin^2 x + \cos^2 x}{\not{\cos^2 x}}\right)^{1}\right]\,dx$$
$$= e^x + \int \cos x\,dx = e^x + \sin x + C$$

5.8. $$\int \frac{x^2 + x\sqrt[3]{x} + \sqrt{x}}{x\sqrt{x}}\,dx = \{x \neq 0\} = \int \frac{x^2}{x\sqrt{x}}\,dx + \int \frac{x\sqrt[3]{x}}{x\sqrt{x}}\,dx + \int \frac{\sqrt{x}}{x\sqrt{x}}\,dx$$
$$= \int x^{1/2}\,dx + \int x^{-1/6}\,dx + \ln|x| = \frac{2}{3}x\sqrt{x} + \frac{6}{5}\sqrt[6]{x^5} + \ln|x| + C$$

5.2 Integration by Change of Variables

5.9. $$\int \frac{1}{x \pm a}\,dx = \{x \pm a = t \quad \therefore \quad dx = dt\} = \int \frac{1}{t}\,dt = \ln|t| = \ln|x \pm a| + C$$

5.10. $$\int \frac{1}{(x \pm a)^2}\,dx = \{x \pm a = t \quad \therefore \quad dx = dt\} = \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = -\frac{1}{t} = -\frac{1}{x \pm a} + c$$

5.11. $$\int x^2(x^3+1)^5\,dx = \int (x^3+1)^5\; x^2\,dx$$
$$= \left\{t = x^3 + 1 \quad \therefore \quad \frac{dt}{dx} = \frac{d(x^3+1)}{dx} = 3x^2 \quad \therefore \quad \frac{1}{3}\,dt = x^2\,dx\right\}$$
$$= \frac{1}{3}\int t^5\,dt = \frac{1}{3}\frac{t^6}{6} = \frac{1}{18}(x^3+1)^6 + C$$

5.12. $$\int x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\; x\,dx$$
$$= \left\{t = x^2 + 1 \quad \therefore \quad \frac{dt}{dx} = \frac{d(x^2+1)}{dx} = 2x \quad \therefore \quad x\,dx = \frac{1}{2}\,dt\right\}$$
$$= \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\int t^{1/2}\,dt = \frac{1}{2}\frac{t^{3/2}}{3/2} = \frac{1}{3}\sqrt{(x^2+1)^3} + C$$

5.13. $$\int \frac{x^3}{\sqrt{1-x^4}}\,dx = \left\{t = 1 - x^4 \quad \therefore \quad dt = -4x^3\,dx \quad \therefore \quad x^3\,dx = -\frac{1}{4}\,dt\right\}$$
$$= -\frac{1}{4}\int \frac{dt}{\sqrt{t}} = -\frac{1}{4}\int t^{-1/2}\,dt = -\frac{1}{2}\sqrt{1-x^4} + C$$

5.14. $$\int \frac{\sin x}{\cos^4 x}\,dx = \{\cos x = t \quad \therefore \quad -\sin x\,dx = dt \quad \therefore \quad \sin x\,dx = -\,dt\}$$
$$= -\int \frac{dt}{t^4}\,dt = -\int t^{-4}\,dt = -\frac{t^{-3}}{-3} = \frac{1}{3\cos^3 x} + C$$

5.15. $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \{\cos x = t \quad \therefore \quad -\sin x\,dx = dt\}$$
$$= -\int \frac{dt}{t} = -\ln|t| = -\ln|\cos x| + C$$

5.16. $$\int x\cos(x^2+2)\,dx = \left\{x^2 + 2 = t \quad \therefore \quad 2x\,dx = dt \quad \therefore \quad x\,dx = \frac{1}{2}\,dt\right\}$$

$$= \frac{1}{2}\int \cos t \; dt = \frac{1}{2}\sin(t) = \frac{1}{2}\sin(x^2+2) + C$$

5.17. $$\int \sin(2x)\, dx = \left\{ t = 2x \;\; \therefore \;\; \frac{dt}{dx} = 2 \;\; \therefore \;\; dx = \frac{1}{2}\, dt \right\}$$
$$= \frac{1}{2}\int \sin t \; dt = -\frac{1}{2}\cos t = -\frac{1}{2}\cos(2x) + C$$

5.18. Given trigonometric function is first transformed by the following identities:

$$\sin x \sin x = \frac{1}{2}\big(\cos(x-x) - \cos(x+x)\big)$$
$$\therefore$$
$$\sin^2 x = \frac{1}{2}\big(\cos(0) - \cos(2x)\big) = \frac{1}{2}\big(1 - \cos(2x)\big)$$

Then,

$$\int \sin^2 x \; dx = \int \frac{1}{2}\big(1-\cos(2x)\big)\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\; dx = \frac{x}{2} - \frac{1}{2}\int \cos(2x)\; dx$$
$$= \left\{ t = 2x \;\; \therefore \;\; \frac{1}{2}\, dt = dx \right\} = \frac{x}{2} - \frac{1}{4}\int \cos(t)\; dt = \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

5.19. Given trigonometric function is first transformed by the following identities:

$$\sin^3 x = \sin^2 x \; \sin x = \left(1 - \cos^2 x\right) \sin x$$

Then, the change of variables method is

$$\cos x = t \;\; \therefore \;\; -\sin x \; dx = dt$$

so that

$$\int \sin^3 x \; dx = \int \left(1-\cos^2 x\right) \sin x \; dx = -\int \left(1-t^2\right)\; dt = -\int \; dt + \int t^2 \; dt$$
$$= -t + \frac{1}{3}t^3 = -\cos x + \frac{1}{3}\cos^3 x + C$$

5.20. $$\int e^{-ax}\, dx = \left\{ t = -ax \;\; \therefore \;\; -\frac{1}{a}\, dt = dx \right\} = -\frac{1}{a}\int e^t \; dt = -\frac{1}{a}\frac{e^t}{\cancelto{1}{\ln e}} = -\frac{1}{a}e^{-ax} + C$$

5.21. $$\int x\, e^{ax^2}\, dx = \left\{ t = ax^2 \;\; \therefore \;\; dt = 2ax\, dx \;\; \therefore \;\; \frac{1}{2a}\, dt = dx \right\} = \frac{1}{2a}\int e^t \; dt$$
$$= \frac{1}{2a}\frac{e^t}{\cancelto{1}{\ln e}} = \frac{1}{2a}\, e^{x^2} + C$$

5.22. $$\int 5^{x^2}\, x\, dx = \left\{ t = x^2 \;\; \therefore \;\; dt = 2x\, dx \;\; \therefore \;\; \frac{1}{2}\, dt = x dx \right\} = \frac{1}{2}\int 5^t\, dt = \frac{1}{2}\frac{5^t}{\ln 5} = \frac{1}{2}\frac{5^{x^2}}{\ln 5} + C$$

5.23.
$$\int \frac{x^3}{x^2-1}\, dx = \int \frac{x^2}{x^2-1}\, x\, dx$$
$$= \left\{ t = x^2 - 1 \;\; \therefore \;\; x^2 = t+1 \;\; \therefore \;\; dt = 2x\, dx \;\; \therefore \;\; x\, dx = \frac{1}{2}\, dt \right\}$$
$$= \frac{1}{2}\int \frac{t+1}{t}\, dt = \frac{1}{2}\int dt + \frac{1}{2}\int \frac{1}{t}\, dt = \frac{1}{2}t + \frac{1}{2}\ln|t|$$
$$= \frac{1}{2}(x^2-1) + \frac{1}{2}\ln|x^2-1| + C$$

5.24. $$\int \cos x\, e^{\sin x}\, dx = \left\{ t = \sin x \;\; \therefore \;\; dt = \cos x\, dx \right\} = \int e^t\, dt = e^t = e^{\sin x} + c$$

5.3 Integration by Parts

5.25. $$\int x e^x\, dx = \left\{ \begin{array}{lll} u = x & \therefore & du = dx \\ dv = e^x\, dx & \therefore & v = \int e^x\, dx = e^x \end{array} \right\}$$
$$= x\, e^x - \int e^x\, dx = xe^x - e^x + C = e^x(x-1) + C$$

5.26. $$\int x e^{-x}\, dx = \left\{ \begin{array}{lll} u = x & \therefore & du = dx \\ dv = e^{-x}\, dx & \therefore & v = \int e^{-x}\, dx = \{\text{see P.5.20}\} = -e^{-x} \end{array} \right\}$$
$$= -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1) + C$$

5.27. $$\int x^2 e^x\, dx = \left\{ \begin{array}{lll} u = x^2 & \therefore & du = 2x\, dx \\ dv = e^x\, dx & \therefore & v = \int e^x\, dx = \dfrac{e^x}{\cancel{\ln e}^{\,1}} = e^x \end{array} \right\}$$
$$= x^2\, e^x - 2\int x e^x\, dx = \{\text{see P.5.25}\} = x^2\, e^x - 2\big(e^x(x-1)\big) + C$$
$$= e^x\big(x^2 - 2x + 2\big) + C$$

5.28. $$\int x^2 e^{-x}\, dx = \left\{ \begin{array}{lll} u = x^2 & \therefore & du = 2x\, dx \\ dv = e^{-x}\, dx & \therefore & v = \int e^{-x}\, dx = \{\text{see P.5.20}\} = -e^{-x} \end{array} \right\}$$
$$= -x^2 e^{-x} - \int \left(-2x e^{-x}\right) dx = -x^2 e^{-x} + 2\int x e^{-x}\, dx = \{\text{see P.5.26}\}$$
$$= -x^2 e^{-x} - 2\left(e^{-x}(x+1)\right) = -e^{-x}\big(x^2 + 2x + 2\big) + C$$

5.29. $$\int \ln x\, dx = \left\{\begin{array}{ll} u = \ln x & \therefore\ du = \dfrac{dx}{x} \\ dv = dx & \therefore\ v = x \end{array}\right\}$$
$$= x\ln x - \int \cancel{x}\frac{dx}{\cancel{x}}$$
$$= x\ln x - x + C = x(\ln x - 1) + C$$

5.30. $$\int x\,\ln\left(1+x^2\right) dx = \left\{t = 1 + x^2 \ \ \therefore\ \ dt = 2x\,dx \ \ \therefore\ \ x\,dx = \frac{1}{2}\right\} = \frac{1}{2}\int \ln t\, dt$$
$$= \left\{\text{see A.5.29}\right\} = \frac{t}{2}\,(\ln t - 1) + C$$

5.31. $$\int x^2 \ln x\, dx = \left\{\begin{array}{ll} u = \ln x & \therefore\ du = \dfrac{dx}{x} \\ dv = x^2\,dx & \therefore\ v = \displaystyle\int x^2\,dx = \frac{x^3}{3} \end{array}\right\} = \frac{x^3}{3}\ln x - \frac{1}{3}\int \cancel{x^2}\frac{dx}{\cancel{x}\ 1}$$
$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

5.32. $$\int x^3 e^{x^2}\, dx = \int x^2\, x\, e^{x^2}\, dx$$
$$= \left\{\begin{array}{ll} u = x^2 & \therefore\ du = 2x\,dx \\ dv = x\,e^{x^2}\,dx & \therefore\ v = \displaystyle\int x\,e^{x^2}\,dx = \{\text{see P.5.21}\} = \frac{1}{2}e^{x^2} \end{array}\right\}$$
$$= \frac{1}{2}x^2\,e^{x^2} - \int \frac{1}{\cancel{2}}e^{x^2}\cancel{2}x\,dx = \frac{1}{2}x^2\,e^{x^2} - \int x\,e^{x^2}\,dx = \frac{1}{2}x^2\,e^{x^2} - \frac{1}{2}e^{x^2} + C$$
$$= \frac{e^{x^2}}{2}\left(x^2 - 1\right) + C$$

5.33. $$\int (x^2 - 1)e^x\, dx = \int x^2 e^x\, dx - \int e^x\, dx$$
$$\left\{\int x^2 e^x\, dx = \{\text{see P.5.27}\} = e^x\left(x^2 - 2x + 2\right)\right\}$$
$$= e^x\left(x^2 - 2x + 2\right) - e^x + C = e^x\left(x^2 - 2x + 1\right) + C$$

5.34. $$\int x\sin x\, dx = \left\{\begin{array}{ll} u = x & \therefore\ du = dx \\ dv = \sin x\,dx & \therefore\ v = \displaystyle\int \sin x\,dx = -\cos x \end{array}\right\}$$
$$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$$

5.35. $$\int x\cos(ax)\, dx = \left\{ax = t \quad \therefore\ \ x = \frac{1}{a}t \ \ \therefore\ \ dx = \frac{1}{a}\,dt\right\}$$

$$= \frac{1}{a^2}\int t\cos t\,dt = \left\{\begin{array}{ll} u = t & \therefore\ du = dt \\ dv = \cos t\,dt & \therefore\ v = \int \cos t\,dt = \sin t \end{array}\right\}$$

$$= \frac{1}{a^2}\left(t\sin t - \int \sin t\,dt\right) = \frac{1}{a^2}\big(ax\sin(ax) + \cos(ax)\big) + C$$

5.36. $\underbrace{\int e^x \sin x\,dx}_{\mathtt{I}} = \left\{\begin{array}{ll} u = e^x & \therefore\ du = e^x\,dx \\ dv = \sin x\,dx & \therefore\ v = \int \sin x\,dx = -\cos x \end{array}\right\}$

$$= -e^x\cos x + \int e^x\cos x\,dx$$

$$\left\{\begin{array}{ll} u = e^x & \therefore\ du = e^x\,dx \\ dv = \cos x\,dx & \therefore\ v = \int \cos x\,dx = \sin x \end{array}\right\}$$

$$= -e^x\cos x + e^x\sin x - \underbrace{\int e^x\sin x\,dx}_{\mathtt{I}}$$

This integral is an example of a "circular" form, i.e., after applying the integration by parts method two times, the term in the form of the original integral ("$\mathtt{I}$") is created again. Therefore,

$$\mathtt{I} = -e^x\cos x + e^x\sin x - \mathtt{I} \quad \therefore\ \ 2\,\mathtt{I} = e^x\sin x - e^x\cos x$$

$$\therefore$$

$$\mathtt{I} = \int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

5.37. $\displaystyle\int \frac{\ln x}{x^2}\,dx = \left\{\begin{array}{ll} u = \ln x & \therefore\ du = \dfrac{1}{x}\,dx \\ dv = \dfrac{1}{x^2}\,dx & \therefore\ v = \displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x} \end{array}\right\} = -\frac{\ln x}{x} + \int \frac{1}{x}\frac{1}{x}\,dx$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C = -\frac{1}{x}(\ln x + 1) + C$$

5.38. $\displaystyle\int x^3 5^{x^2}\,dx = \int x^2 5^{x^2}\,x\,dx$

$$= \left\{\begin{array}{ll} u = x^2 & \therefore\ du = 2x\,dx \\ dv = 5^{x^2}\,x\,dx & \therefore\ v = \displaystyle\int 5^{x^2}\,x\,dx = \{\text{see P.5.22}\} = \frac{1}{2\ln 5}\,5^{x^2} \end{array}\right\}$$

$$= x^2\,\frac{1}{2\ln 5}\,5^{x^2} - \frac{1}{\ln 5}\int 5^{x^2}\,x\,dx = x^2\,\frac{1}{2\ln 5}\,5^{x^2} - \frac{1}{\ln 5}\,\frac{1}{2\ln 5}\,5^{x^2} + C$$

$$= \frac{1}{2\ln 5}\,5^{x^2}\left(x^2 - \frac{1}{\ln 5}\right) + C$$

5.39. $\displaystyle\int x\,\ln(x^2-1)\,dx = \left\{\begin{array}{ll} u=\ln(x^2-1) & \therefore\ du=\dfrac{2x}{x^2-1}\,dx \\ dv=x\,dx & \therefore\ v=\dfrac{x^2}{2} \end{array}\right\}$

$$= \frac{x^2}{2}\ln(x^2-1) - \int \frac{x^2}{2}\,\frac{2x}{x^2-1}\,dx = \{\text{see P.5.23}\}$$

$$= \frac{x^2}{2}\ln(x^2-1) - \frac{1}{2}(x^2-1) - \frac{1}{2}\ln|x^2-1| + C$$

5.40. $\displaystyle\int x^2\cos x\,dx = \left\{\int u\,dv = uv - \int v\,du\right\}$

$$= \left\{\begin{array}{l} u=x^2 \Rightarrow \dfrac{du}{dx}=2x \ \ \therefore\ du=2x\,dx \\ dv=\cos x\,dx \Rightarrow v=\displaystyle\int \cos x\,dx = \sin x \end{array}\right\}$$

$$= x^2\sin x - 2\int x\sin x\,dx$$

$$= \left\{\begin{array}{l} u=x \Rightarrow du = dx \\ dv=\sin x\,dx \Rightarrow v=\displaystyle\int \sin x\,dx = -\cos x \\ \therefore \\ \displaystyle\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x \end{array}\right\}$$

$$= \int \left(x^2\sin x - 2(-x\cos x + \sin x)\right) + C$$

$$= \int \left((x^2-2)\sin x + 2x\cos x\right) + C$$

5.41. Given $\displaystyle\int \sin(2x)\,e^{\sin x}\,dx$ can be modified as

$$\int \sin(2x)\,e^{\sin x}\,dx = \{\sin(2x) = 2\sin x\cos x\} = 2\int \sin x\cos x\,e^{\sin x}\,dx$$

so that

$$2\int \sin x\cos x\,e^{\sin x}\,dx$$

$$= \left\{\begin{array}{ll} u=\sin x & \therefore\ du=\cos x\ dx \\ dv=\cos x\,e^{\sin x}\,dx & \therefore\ v=\displaystyle\int \cos x\,e^{\sin x}\,dx = \{\text{see A.5.24}\} = e^{\sin x} \end{array}\right\}$$

$$= 2\sin x\, e^{\sin x} - 2\int \cos x\, e^{\sin x}\, dx = 2\sin x\, e^{\sin x} - 2e^{\sin x}$$

$$= 2e^{\sin x}\big(\sin x - 1\big) + c$$

5.42. $\int e^{-bx}\, \sin(ax)\, dx$ $= \left\{ \begin{array}{ll} \text{partial integration:} & u = e^{-bx} \;\; \therefore \;\; du = -b\, e^{-bx}\, dx \\ & dv = \sin(ax)\, dx \;\; \therefore \;\; v = -\dfrac{1}{a}\cos(ax) \end{array} \right\}$

$$= e^{-bx} \left(-\frac{1}{a}\cos(ax)\right) - \int \left(-\frac{1}{a}\cos(ax)\right) \left(-b\, e^{-bx}\right)\, dx$$

$$= -\frac{1}{a}e^{-bx}\cos(ax) - \frac{b}{a}\underbrace{\int e^{-bx}\cos(ax)\, dx}_{\text{partial integration}}$$

$$\left\{ \begin{array}{ll} \text{partial integration:} & u = e^{-bx} \;\; \therefore \;\; du = -b\, e^{-bx}\, dx \\ & dv = \cos(ax)\, dx \;\; \therefore \;\; v = \dfrac{1}{a}\sin(ax) \end{array} \right\}$$

$$= -\frac{1}{a}e^{-bx}\cos(ax) - \frac{b}{a}\left[\frac{1}{a}\, e^{-bx}\, \sin(ax) + \frac{b}{a}\int e^{-bx}\sin(ax)\, dx\right]$$

Note that after two partial integrations, the last integral on the right side is identical to the original integral on the left side. Thus, subsequent partial integrations do not advance the solution. However, like any other common term in an algebra equation, this integral may be factored out as

$$\left[1 + \left(\frac{b}{a}\right)^2\right] \int e^{-bx}\, \sin(ax)\, dx = -\frac{1}{a}e^{-bx}\cos(ax) - \frac{b}{a}\frac{1}{a}\, e^{-bx}\, \sin(ax)$$

$$\therefore$$

$$\left[\frac{a^2 + b^2}{a^2}\right] \int e^{-bx}\, \sin(ax)\, dx = -\frac{1}{a}e^{-bx}\left[\frac{a\cos(ax) + b\sin(ax)}{a}\right]$$

which gives

$$\int e^{-bx}\, \sin(ax)\, dx = \frac{\cancel{a^2}}{a^2 + b^2}\left(-\frac{1}{\cancel{a}}\right)\frac{1}{\cancel{a}}\, e^{-bx}\, [a\cos(ax) + b\sin(ax)] + C$$

$$= -\frac{1}{a^2 + b^2}\, e^{-bx}\, \big(a\cos(ax) + b\sin(ax)\big) + C$$

5.4 Trigonometric Substitutions

5.43. $\displaystyle\int \sqrt{9-x^2}\,dx = \{x = 3\sin\theta \;\;\therefore\;\; dx = 3\cos\theta\,d\theta \;\;\text{and}\;\; x^2 = 9\sin^2\theta\}$

$$= \int \sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta = \int 3\sqrt{1-\sin^2\theta}\,3\cos\theta\,d\theta$$

$$= \left\{\sin^2\theta+\cos^2\theta = 1 \;\;\therefore\;\; 1-\sin^2\theta = \cos^2\theta \;\;\therefore\;\; \sqrt{\cos^2\theta} = \cos\theta\right\}$$

$$= 9\int \cos\theta\,\cos\theta\,d\theta = 9\int \cos^2\theta\,d\theta = \left\{\cos^2\theta = \frac{1+\cos 2\theta}{2}\right\}$$

$$= \frac{9}{2}\left[\int (1+\cos 2\theta)\,d\theta\right] = \frac{9}{2}\left[\int d\theta + \int \cos 2\theta\,d\theta\right]$$

$$= \left\{t = 2\theta \;\;\therefore\;\; dt = 2\,d\theta \Rightarrow \frac{1}{2}\int \cos t\,dt = \frac{1}{2}\sin 2\theta\right\}$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] = \left\{\sin 2\theta = 2\sin\theta\cos\theta\right\} = \frac{9}{2}\left[\theta + \sin\theta\cos\theta\right]$$

Return to the original variable as

$$\left\{\begin{array}{ll} x = 3\sin\theta & \therefore\;\; \sin\theta = \dfrac{x}{3} \;\;\therefore\;\; \theta = \arcsin\dfrac{x}{3} \\[2ex] 3\cos\theta = \sqrt{9-x^2} & \therefore\;\; \cos\theta = \dfrac{\sqrt{9-x^2}}{3} \end{array}\right\}$$

$$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{\cancel{9}}{2}\frac{x}{\cancel{3}}\frac{\sqrt{9-x^2}}{\cancel{3}} + C = \frac{9}{2}\arcsin\frac{x}{3} + \frac{x}{2}\sqrt{9-x^2} + C$$

5.44. $\displaystyle\int \frac{x^2}{\sqrt{36-x^2}}\,dx = \{x = 6\sin\theta \;\;\therefore\;\; dx = 6\cos\theta\,d\theta \;\;\text{and}\;\; x^2 = 36\sin^2\theta\}$

$$= \int \frac{36\sin^2\theta}{\sqrt{36-36\sin^2\theta}}\,6\,\cos\theta\,d\theta = \int \frac{36\sin^2\theta}{\cancel{6\,\cos\theta}}\cancel{6\,\cos\theta}\,d\theta$$

$$= 36\int \sin^2\theta\,d\theta$$

$$\left\{\sin^2\theta = \frac{1-\cos 2\theta}{2}\right\}$$

$$= 18\int (1-\cos 2\theta)\,d\theta = 18\int d\theta - 18\int \cos 2\theta\,d\theta$$

$$= 18\,\theta - 18\int \cos 2\theta\,d\theta = \left\{y = 2\theta \;\;\therefore\;\; dy = 2\,d\theta\right\}$$

$$= 18\left(\theta - \frac{1}{2}\int \cos y\,dy\right) = 18\left(\theta - \frac{1}{2}\sin 2\theta\right)$$

$$= \left\{\sin 2\theta = 2\sin\theta\cos\theta\right\}$$

$$= 18\,(\theta - \sin\theta\cos\theta) + C$$

Return to the original variable as

$$\left\{\begin{array}{lll} x = 6\sin\theta & \therefore & \sin\theta = \dfrac{x}{6} \quad \therefore \quad \theta = \arcsin\left(\dfrac{x}{6}\right) \\ \sqrt{36 - x^2} = 6\cos\theta & \therefore & \cos\theta = \dfrac{\sqrt{36 - x^2}}{6} \end{array}\right\}$$

$$= 18\arcsin\left(\frac{x}{6}\right) - \cancel{18}\,\frac{x}{\cancel{6}}\,\frac{\sqrt{36 - x^2}}{\cancel{6}\,2} + C = 18\,\arcsin\left(\frac{x}{6}\right) - \frac{x}{2}\sqrt{36 - x^2} + C$$

5.45. $\displaystyle\int \frac{\sqrt{25x^2 + 4}}{x^4}\,dx =$

$$\left\{\begin{array}{l} \text{given form } \sqrt{u^2 + k^2} \quad \text{substitution is} \quad u = k\,\tan\theta \\ \therefore \quad 5x = 2\tan\theta \quad \therefore \quad x = \dfrac{2}{5}\tan\theta \quad \therefore \quad dx = \dfrac{2}{5}\,\dfrac{d\theta}{\cos^2\theta} \\ \therefore \quad \sqrt{25x^2 + 4} = \sqrt{\cancel{25}\,\dfrac{4}{\cancel{25}}\,\tan^2\theta + 4} = \dfrac{2}{\cos\theta} \end{array}\right\}$$

$$= \int \frac{2}{\cos\theta}\,\frac{1}{(2/5\tan\theta)^4}\,\frac{2}{5}\,\frac{d\theta}{\cos^2\theta}$$

$$= \cancel{2}\,\frac{\cancel{625}\,125}{\cancel{16}\,4}\,\frac{\cancel{2}}{\cancel{5}}\int \frac{1}{\cancel{\cos\theta}}\,\frac{\cos^{\cancel{4}}\theta}{\sin^4\theta}\,\frac{d\theta}{\cancel{\cos^2\theta}} = \frac{125}{4}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta$$

$$\left\{t = \sin\theta \quad \therefore \quad dt = \cos\theta\,d\theta\right\}$$

$$= \frac{125}{4}\int \frac{1}{t^4}\,dt = -\frac{125}{12}\,\frac{1}{\sin^3\theta}$$

Return to the original variable as

$$\left\{\begin{array}{lll} 5x = 2\tan\theta & \therefore & \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{5x}{2} \quad \therefore \quad \sin\theta = \dfrac{5x}{2}\cos\theta \\ \sqrt{25x^2 + 4} = \dfrac{2}{\cos\theta} & \therefore & \cos\theta = \dfrac{2}{\sqrt{25x^2 + 4}} \\ & \therefore & \sin\theta = \dfrac{5x}{\cancel{2}}\,\dfrac{\cancel{2}}{\sqrt{25x^2 + 4}} \end{array}\right\}$$

$$= -\frac{\cancel{125}}{12}\left(\frac{\sqrt{25x^2 + 4}}{\cancel{5}x}\right)^3 + C = -\frac{(\sqrt{25x^2 + 4})^3}{12x^3} + C$$

5.46. Given trigonometric function can be converted as follows.

$$\int \frac{1}{\cos x}\,dx\;\frac{\frac{1}{\cos x} + \tan x}{\frac{1}{\cos x} + \tan x} = \int \frac{\left(\frac{1}{\cos^2 x} + \frac{\tan x}{\cos x}\right)\,dx}{\frac{1}{\cos x} + \tan x}$$

$$= \left\{ \begin{array}{ll} t = \dfrac{1}{\cos x} + \tan x \quad \therefore \quad dt & = d\left(\dfrac{1}{\cos x}\right) + d\left(\dfrac{\sin x}{\cos x}\right) \\ & = \left(\dfrac{\sin x}{\cos^2 x} + \dfrac{1}{\cos^2 x}\right) dx = \left(\dfrac{1}{\cos^2 x} + \dfrac{\tan x}{\cos x}\right) dx \end{array} \right\}$$

$$= \int \frac{dt}{t} = \ln|t| = \ln\left|\frac{1}{\cos x} + \tan x\right| + C$$

5.47. Inverse square cosine may be transformed as follows.

$$\int \frac{1}{\cos^2 x}\, dx = \left\{ \begin{array}{l} t = \tan x = \dfrac{\sin x}{\cos x} \quad \Rightarrow \quad \left\{\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}\right\} \\ \dfrac{dt}{dx} = \dfrac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} \\ \quad \therefore \quad dt = \dfrac{1}{\cos^2 t}\, dx \end{array} \right\}$$

$$= \int dt = t = \tan x + C$$

5.48. Inverse cube cosine may be transformed as follows.

$$\int \frac{1}{\cos^3 x}\, dx = \int \frac{1}{\cos^2 x}\, \frac{1}{\cos x}\, dx$$

$$= \left\{ \begin{array}{ll} u = \dfrac{1}{\cos x} & \therefore \quad du = \dfrac{\sin x}{\cos^2 x}\, dx \\ dv = \dfrac{1}{\cos^2 x}\, dx & \therefore \quad v = \{\text{see A.5.47}\} = \tan x = \dfrac{\sin x}{\cos x} \end{array} \right\}$$

$$= \frac{1}{\cos x}\, \frac{\sin x}{\cos x} - \int \frac{\sin x}{\cos x}\, \frac{\sin x}{\cos^2 x}\, dx = \frac{\sin x}{\cos^2 x} - \int \frac{\sin^2 x}{\cos^3 x}\, dt$$

$$= \frac{\sin x}{\cos^2 x} - \int \frac{1 - \cos^2 x}{\cos^3 x}\, dx = \frac{\sin x}{\cos^2 x} - \int \frac{1}{\cos^3 x}\, dx + \int \frac{1}{\cos x}\, dx \quad \{\text{see A.5.46}\}$$

$$= \frac{\sin x}{\cos^2 x} - \int \frac{1}{\cos^3 x}\, dx + \ln\left|\frac{1}{\cos x} + \tan x\right|$$

$\therefore$

$$2 \int \frac{1}{\cos^3 x}\, dx = \frac{\sin x}{\cos^2 x} + \ln\left|\frac{1}{\cos x} + \tan x\right| \quad \Rightarrow \quad \int \frac{1}{\cos^3 x}\, dx$$

$$= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln\left|\frac{1}{\cos x} + \tan x\right| + C$$

5.49. $$\int \left(1 + \tan^2 x\right) dx = \int \left(1 + \frac{\sin^2 x}{\cos^2 x}\right) dx = \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x}\, dx$$

$$= \int \frac{1}{\cos^2 x}\, dx = \{\text{see A.5.47}\} = \int dt = t = \tan x + C$$

5.50. $$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \left\{x = \frac{a}{\cos t} \quad \therefore \quad dx = a\,\frac{\sin t}{\cos^2 t}\,dt\right\} = a\int \frac{1}{\sqrt{\frac{a^2}{\cos^2 t}-a^2}}\,\frac{\sin t}{\cos^2 t}\,dt$$

$$= \int \frac{1}{\sqrt{\frac{1-\cos^2 t}{\cos^2 t}}}\,\frac{\sin t}{\cos^2 t}\,dt = \int \frac{\cancel{\cos t}}{\cancel{\sin t}}\,\frac{\cancel{\sin t}}{\cos^2 t}\,dt = \int \frac{1}{\cos t}\,dt$$

$$= \left\{\text{see A.5.46}\right\} = \ln\left|\frac{1}{\cos t} + \tan t\right| = \left\{x = \frac{a}{\cos t} \quad \therefore \quad t = \arccos\frac{a}{x}\right\}$$

$$= \ln\left|\frac{1}{\cos(\arccos {}^{a}/_{x})} + \tan(\arccos {}^{a}/_{x})\right|$$

$$= \left\{\cos(\arccos z) = z\,; \quad \tan(\arccos z) = \frac{\sqrt{1-z^2}}{z}\right\}$$

$$= \ln\left|\frac{x}{a} + \frac{x}{a}\sqrt{1-(a/x)^2}\right| + C = \ln\left|\frac{x}{a} + \frac{1}{a}\,\frac{x}{|x|}\sqrt{x^2-a^2}\right| + C$$

5.51. $$I = \int \sqrt{x^2-a^2}\,dx = \begin{Bmatrix} u = \sqrt{x^2-a^2} & \therefore \quad du = \dfrac{x}{\sqrt{x^2-a^2}}\,dx \\ dv = dx & \therefore \quad v = x \end{Bmatrix}$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}\,dx - \int \frac{a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \underbrace{\int \sqrt{x^2-a^2}\,dx}_{I} - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \left\{\text{see A.5.50}\right\}$$

$$= x\sqrt{x^2-a^2} - \underbrace{\int \sqrt{x^2-a^2}\,dx}_{I} - a^2\ln\left|\frac{x}{a} + \frac{1}{a}\,\frac{x}{|x|}\sqrt{x^2-a^2}\right|$$

$\therefore$

$$2I = x\sqrt{x^2-a^2} - a^2\ln\left|\frac{x}{a} + \frac{1}{a}\,\frac{x}{|x|}\sqrt{x^2-a^2}\right|$$

$$\therefore \quad \underline{\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|\frac{x}{a} + \frac{1}{a}\,\frac{x}{|x|}\sqrt{x^2-a^2}\right|}$$

5.52. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \left\{x = a\sin t, \quad \therefore \quad dx = a\cos t\,dt \quad \therefore \quad t = \arcsin\frac{x}{a}\right\}$$

$$= \int \frac{a\cos t}{\sqrt{a^2 - a^2\sin^2 t}}\,dt = \int \frac{\cancel{a}\cos t}{\cancel{a}\sqrt{1-\sin^2 t}}\,dt = \int \frac{\cos t}{\sqrt{\cos^2 t}}\,dt$$

$$= \int \frac{\cancel{\cos t}}{\cancel{\cos t}}\,dt = \; t = \arcsin\frac{x}{a} + C$$

5.53. $$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \left\{\begin{array}{ll} x = a\,\tan t & \therefore\ \ t = \arctan\dfrac{x}{a} \quad \therefore\ \ dx = a\,\dfrac{1}{\cos^2 t}\,dt \\ & \therefore\ \ 1+\tan^2 t = \dfrac{1}{\cos^2 t} \quad \{\text{see A.5.49}\} \end{array}\right\}$$

$$= a\int \frac{1}{\sqrt{a^2+(a\,\tan t)^2}}\,\frac{1}{\cos^2 t}\,dt = \int \frac{1}{\sqrt{1+\tan^2 t}}\,\frac{1}{\cos^2 t}\,dt$$

$$= \int \frac{\cancel{\cos t}\ 1}{\cos^{\cancel{2}} t}\,dt = \int \frac{1}{\cos t}\,dt = \left\{\text{see P.5.46}\right\}$$

$$= \ln\left|\frac{1}{\cos t}+\tan t\right| = \ln\left|\frac{1}{\cos\left(\arctan\frac{x}{a}\right)}+\tan\left(\arctan\frac{x}{a}\right)\right|$$

$$= \left\{\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}\ \ ;\ \ \tan(\arctan x) = x\right\}$$

$$= \ln\left|\sqrt{1+\left(\frac{x}{a}\right)^2}+\frac{x}{a}\right| = \ln\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right| + C$$

5.54. The Gaussian integral form

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

is among the most important in science and engineering. There are multiple methods to solve it; one possible method exploits the symmetry of multivariable integral form by declaring the square of the Gaussian integral as follows.

$$I^2 = \underbrace{\int_{-\infty}^{\infty} e^{-y^2}\,dy}_{\mathrm{I}}\underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,dx}_{\mathrm{I}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

Even though there are two variables involved x and y, the two integrals have the same form, therefore the same form of their respective solutions. Furthermore,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Illustration of how Cartesian coordinates (x, y) are transformed into polar coordinates (r, θ) is shown in Fig. 5.3. Observe the position of point $\mathscr{A}$ in the Cartesian system, and note the relations

$$x = r\cos\theta,\quad y = r\sin\theta\ \ \therefore\ \ x^2+y^2 = r^2 \quad \text{and} \quad dx\,dy = r\,dr\,d\theta$$

The geometrical interpretation shows the equivalence of unit differential areas $dx\,dy = r\,dr\,d\theta$. With this change of variables, it follows that the total space in both Cartesian and polar coordinates is covered as

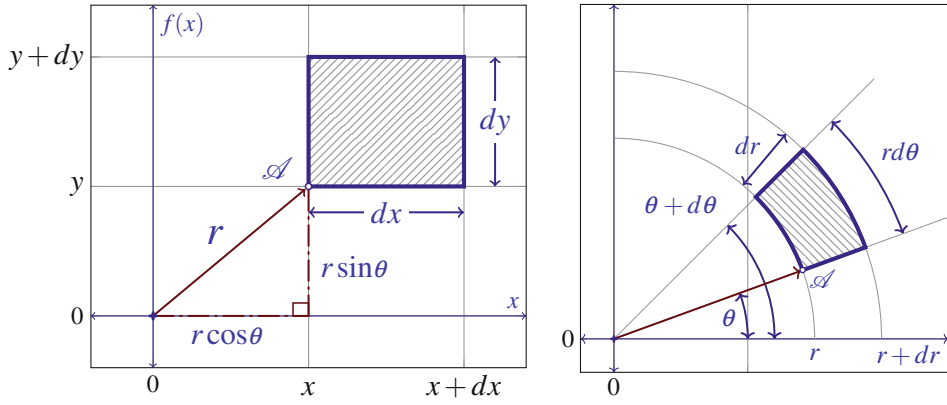


Fig. 5.3 Example P.5.54

$$\left.\begin{array}{l} x \in [-\infty, \infty] \\ y \in [-\infty, \infty] \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r \in [-\infty, \infty] \\ \theta \in [0, 2\pi] \end{array}\right.$$

Thus, the Gaussian integral in polar coordinates is calculated as

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\, dr = 2\pi \int_0^{\infty} e^{-r^2} r\, dr$$

$$\left\{ t = -r^2 \quad \therefore \quad dt = -2r\, dr \quad \therefore \quad \int e^{-r^2} r\, dr = -\frac{1}{2} e^{-r^2} \quad \text{(see A.5.21)} \right\}$$

$$= 2\pi \left(-\frac{1}{2}\right) e^{-r^2}\Big|_0^{\infty} = -\pi \left(\cancel{e^{-\infty}}^{0} - \cancel{e^{0}}^{1} \right) = \pi$$

$\therefore$

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

5.55. $\displaystyle\int \sqrt{a^2+x^2}\, dx = \left\{\begin{array}{ll} u = \sqrt{a^2+x^2} & \therefore \quad du = \dfrac{x}{\sqrt{a^2+x^2}}\, dx \\ dv = dx & \therefore \quad v = x \end{array}\right\}$

$$= x\sqrt{a^2+x^2} - \int \frac{x^2 + a^2 - a^2}{\sqrt{a^2+x^2}}\, dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2}{\sqrt{a^2+x^2}}\, dx + \int \frac{a^2}{\sqrt{a^2+x^2}}\, dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\, dx + \int \frac{a^2}{\sqrt{a^2+x^2}}\, dx$$

That is to say,

$$2\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} + a^2\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \{\text{see A.5.53}\}$$
$$= x\sqrt{a^2+x^2} + a^2 \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|$$

Therefore, in total,

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C$$

5.56. $$\int \frac{1}{1+x^2}\,dx = \left\{x = \tan\theta \;\;\therefore\;\; dx = \frac{1}{\cos^2\theta}\,d\theta = (1+\tan^2\theta)\,d\theta \;\;\therefore\;\; \theta = \arctan x\right\}$$
$$= \int \frac{1}{\cancel{1+\tan^2\theta}}\,\cancel{(1+\tan^2\theta)}\,d\theta = \theta = \arctan x + C$$

5.57. $$\int \sqrt{e^x-1}\,dx = \left\{t^2 = e^x - 1 \;\;\therefore\;\; e^x = t^2+1 \;\;\therefore\;\; 2t\,dt = e^x\,dx \;\;\therefore\;\; dx = \frac{2t}{1+t^2}\,dt\right\}$$
$$= \int \sqrt{t^2}\,\frac{2t}{1+t^2}\,dt = 2\int \frac{t^2+1-1}{1+t^2}\,dt = 2\int dt - 2\int \frac{1}{1+t^2}\,dt$$
$$= \{\text{see A.5.56}\} = 2t - 2\arctan t = 2\sqrt{e^x-1} - 2\arctan\sqrt{e^x-1} + C$$

5.58. $$\int \frac{\sqrt{x^2+2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}\,dx = \int \frac{\sqrt{x^2+2}}{\sqrt{4-x^4}}\,dx - \int \frac{\sqrt{2-x^2}}{\sqrt{4-x^4}}\,dx$$
$$= \int \frac{\sqrt{x^2+2}}{\sqrt{(2-x^2)(2+x^2)}}\,dx - \int \frac{\sqrt{2-x^2}}{\sqrt{(2-x^2)(2+x^2)}}\,dx$$
$$= \int \frac{\cancel{\sqrt{x^2+2}}}{\sqrt{2-x^2}\cancel{\sqrt{2+x^2}}}\,dx - \int \frac{\cancel{\sqrt{2-x^2}}}{\cancel{\sqrt{2-x^2}}\sqrt{2+x^2}}\,dx = I_1 + I_2$$

where

$$I_1 = \int \frac{dx}{\sqrt{2-x^2}} = \left\{x = \sqrt{2}\sin t, \;\;\therefore\;\; dx = \sqrt{2}\cos t\,dt; \;\; \text{see A.5.52}\right\} = \arcsin\frac{x}{\sqrt{2}}$$

and

$$I_2 = \int \frac{dx}{\sqrt{2+x^2}} = \int \frac{dx}{\sqrt{x^2+(\sqrt{2})^2}} = \left\{x = \sqrt{2}\tan t, \;\;\therefore\;\; dx = \frac{\sqrt{2}}{\cos^2 t}\,dt; \;\; \text{see A.5.53}\right\}$$
$$= \ln\left|\sqrt{1+\frac{x^2}{2}} + \frac{x}{\sqrt{2}}\right| + C = \ln\left|\sqrt{\frac{2+x^2}{2}} + \frac{x}{\sqrt{2}}\right| + C$$

Therefore, in total,

$$\int \frac{\sqrt{x^2+2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}\,dx = I_1 + I_2 = \arcsin\frac{x}{\sqrt{2}} + \ln\left|\sqrt{\frac{2+x^2}{2}} + \frac{x}{\sqrt{2}}\right| + C$$

5.5 Rational Functions

5.59. $$\int \frac{x-1}{x^2+x}\,dx = \left\{\text{partial fractions: } -\frac{1}{x}+\frac{2}{x+1}\right\} = -\int \frac{1}{x}\,dx + \int \frac{2}{x+1}\,dx$$
$$= \{x+1=t \quad \therefore \quad dx=dt\} = -\ln|x| + 2\int \frac{dt}{t}$$
$$= -\ln|x| + 2\ln|x+1| + C = \ln\frac{(x+1)^2}{|x|} + C$$

5.60. $$\int \frac{x}{(x+1)(x-4)}\,dx = \left\{\text{partial fractions: } \frac{1}{5(x+1)}+\frac{4}{5(x-4)}\right\}$$
$$= \int \frac{1}{5(x+1)}\,dx + \int \frac{4}{5(x-4)}\,dx = \frac{1}{5}\ln|x+1| + \frac{4}{5}\ln|x-4| + C$$

5.61. Given

$$\int \frac{1}{x^2-1}\,dx$$

partial fraction decomposition of this rational function is (see Vol. I)

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$
$$= \frac{1}{2}\big(\ln|x-1| - \ln|x+1|\big) + C = \frac{1}{2}\ln\frac{|x-1|}{|x+1|} + C$$

5.62. $$\int \frac{x+2}{x^3-2x^2}\,dx = \left\{\text{partial fractions: } -\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x-2} \quad \text{see Vol. I}\right\}$$
$$= \int -\frac{1}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{1}{x-2}\,dx = -\ln|x| + \frac{1}{x} + \ln|x-2| + C$$

5.63. $$\int \frac{2x^2}{x^4-1}\,dx = \left\{\text{partial fraction: } \frac{1}{x^2+1} - \frac{1}{2(x+1)} + \frac{1}{2(x-1)}, \quad \text{see Vol. I}\right\}$$
$$= \arctan x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

5.64. This function may be converted as follows.

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx = \left\{\begin{array}{l} t=\sqrt{1+x^2} \quad \therefore \quad x=\sqrt{t^2-1} \\ dt = \dfrac{x}{\sqrt{1+x^2}}\,dx = \dfrac{\sqrt{t^2-1}}{t}\,dx \quad \therefore \quad dx = \dfrac{t}{\sqrt{t^2-1}}\,dt \end{array}\right\}$$
$$= \int \frac{1}{\sqrt{t^2-1}\,\cancel{t}}\,\frac{\cancel{t}}{\sqrt{t^2-1}}\,dt = \int \frac{1}{t^2-1}\,dt = \{\text{see P.5.61}\}$$
$$= \frac{1}{2}\ln\frac{|t-1|}{|t+1|} = \frac{1}{2}\ln\frac{|\sqrt{1+x^2}-1|}{|\sqrt{1+x^2}+1|} + C$$

5.65. After decomposing given rational function as

$$\begin{aligned}\int \frac{1}{x^3-1}\,dx &= \left\{\text{partial fraction: } \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)},\quad \text{see Vol. I}\right\}\\ &= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx\\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x}{x^2+x+1}\,dx - \frac{2}{3}\int \frac{1}{x^2+x+1}\,dx\end{aligned}$$

The last two integral may be resolved as follows.

1. First,

$$\begin{aligned}\int \frac{x}{x^2+x+1}\,dx &= \left\{x^2+x+1 = x^2+2\,\frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right\}\\ &= \int \frac{x}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx = \left\{t = x+\frac{1}{2}\ \ \therefore\ \ dt = dx\ \ \text{and}\ \ x = \frac{2t-1}{2}\right\}\\ &= \int \frac{2(2t-1)}{4t^2+3}\,dt = 2\int \frac{2t}{4t^2+3}\,dt - 2\int \frac{1}{4t^2+3}\,dt\\ &= \left\{z = 4t^2+3\ \ \therefore\ \ dz = 8\,t\,dt\right\}\\ &= \frac{1}{2}\ln(4t^2+3) - 2\int \frac{1}{4t^2+3}\,dt\end{aligned}$$

$$\left\{\begin{array}{l}\dfrac{1}{4t^2+3} = \dfrac{1}{3}\dfrac{1}{\left(\frac{2t}{\sqrt{3}}\right)^2+1},\quad n = \dfrac{2t}{\sqrt{3}}\ \ \therefore\ \ dn = \dfrac{2}{\sqrt{3}}\,dt\\[2ex] \text{so that}\quad \displaystyle\int \frac{1}{n^2+1}\,dn = \arctan n\end{array}\right\}$$

$$\begin{aligned}&= \frac{1}{2}\ln(4t^2+3) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2t}{\sqrt{3}}\right)\\ &= \frac{1}{2}\ln(4x^2+4x+4) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\end{aligned}$$

2. and, similarly,

$$\int \frac{1}{x^2+x+1}\,dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

Therefore, in total,

$$\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\ln|x-1|$$

$$
\begin{aligned}
&- \frac{1}{6}\ln(4x^2+4x+4) + \frac{1}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) \\
&- \frac{4}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(4x^2+4x+4) - \frac{1}{3\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}
$$

5.66. After decomposing given rational function as

$$
\begin{aligned}
\int \frac{4}{x^4+1}\,dx &= \int \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}\,dx - \int \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}\,dx \\
&= \int \frac{\sqrt{2}x}{x^2+\sqrt{2}x+1}\,dx + \int \frac{2}{x^2+\sqrt{2}x+1}\,dx - \int \frac{\sqrt{2}x}{x^2-\sqrt{2}x+1}\,dx \\
&\quad + \int \frac{2}{x^2-\sqrt{2}x+1}\,dx \\
&= I_1 + I_2 - I_3 + I_4
\end{aligned}
$$

where

1. The first of the four integrals is solved as

$$
\begin{aligned}
I_1 &= \sqrt{2}\int \frac{x}{x^2+\sqrt{2}x+1}\,dx = \sqrt{2}\int \frac{x}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\,dx = \left\{\begin{array}{ll} t &= x+\frac{1}{\sqrt{2}} \quad \therefore \quad dt = dx \\ x &= t-\frac{1}{\sqrt{2}} \end{array}\right\} \\
&= \sqrt{2}\int \frac{t-\frac{1}{\sqrt{2}}}{t^2+\frac{1}{2}}\,dt = \sqrt{2}\sqrt{2}\int \frac{\sqrt{2}t-1}{2t^2+1}\,dt = 2\int \frac{\sqrt{2}t}{2t^2+1}\,dt - 2\int \frac{1}{2t^2+1}\,dt \\
&= 2\sqrt{2}\int \frac{t}{2t^2+1}\,dt - \int \frac{1}{t^2+\left(1/\sqrt{2}\right)^2}\,dt \\
&= \left\{\text{first integral, } 2t^2+1=z \quad \therefore \quad 4t\,dt = dz, \quad \text{etc.;} \quad \text{second integral, see A.5.53}\right\} \\
&= \frac{\sqrt{2}}{2}\ln\left|2t^2+1\right| - \sqrt{2}\arctan\left(\sqrt{2}\,t\right) \\
&= \frac{\sqrt{2}}{2}\ln\left|2\left(x+\frac{1}{\sqrt{2}}\right)^2+1\right| - \sqrt{2}\arctan\left(\sqrt{2}\left(x+\frac{1}{\sqrt{2}}\right)\right) \\
&= \frac{\sqrt{2}}{2}\ln\left|\left(\sqrt{2}x+1\right)^2+1\right| - \sqrt{2}\arctan\left(\sqrt{2}x+1\right)
\end{aligned}
$$

2. Similarly,

$$I_3 = \sqrt{2}\int \frac{x}{x^2-\sqrt{2}x+1}\,dx = \frac{\sqrt{2}}{2}\ln\left|\left(\sqrt{2}x-1\right)^2+1\right| + \sqrt{2}\arctan\left(\sqrt{2}x-1\right)$$

3. The second of the four integrals is solved in the previous step,

$$I_2 = 2\int \frac{1}{x^2+\sqrt{2}x+1}\,dx = 2\sqrt{2}\arctan\left(\sqrt{2}x+1\right)$$

4. as well as the fourth integral

$$I_4 = 2\int \frac{1}{x^2-\sqrt{2}x+1}\,dx = 2\sqrt{2}\arctan\left(\sqrt{2}x-1\right)$$

In total,

$$\begin{aligned}\int \frac{4}{x^4+1}\,dx = {} & \frac{\sqrt{2}}{2}\ln\left|\left(\sqrt{2}x+1\right)^2+1\right| - \sqrt{2}\arctan\left(\sqrt{2}x+1\right) + 2\sqrt{2}\arctan\left(\sqrt{2}x+1\right)\\ & - \frac{\sqrt{2}}{2}\ln\left|\left(\sqrt{2}x-1\right)^2+1\right| + \sqrt{2}\arctan\left(\sqrt{2}x-1\right) + 2\sqrt{2}\arctan\left(\sqrt{2}x-1\right)\\ = {} & \frac{\sqrt{2}}{2}\ln\left|\left(\sqrt{2}x+1\right)^2+1\right| - \frac{\sqrt{2}}{2}\ln\left|\left(\sqrt{2}x-1\right)^2+1\right|\\ & + \sqrt{2}\arctan\left(\sqrt{2}x+1\right) + \sqrt{2}\arctan\left(\sqrt{2}x-1\right)\\ = {} & \left\{\begin{array}{ll}\ln a - \ln b & = \ln\dfrac{a}{b}\\[2ex] \arctan x \pm \arctan y & = \arctan\dfrac{x\pm y}{1\mp y}\end{array}\right\}\end{aligned}$$

which is then simplified as

$$\begin{aligned}\int \frac{4}{x^4+1}\,dx &= \frac{\sqrt{2}}{2}\ln\left|\frac{\left(\sqrt{2}x+1\right)^2+1}{\left(\sqrt{2}x-1\right)^2+1}\right| + \sqrt{2}\arctan\frac{\left(\sqrt{2}x+1\right)+\left(\sqrt{2}x-1\right)}{1-\left(\sqrt{2}x-1\right)}\\ &= \frac{\sqrt{2}}{2}\ln\left|\frac{x^2+\sqrt{2}x+2}{x^2-\sqrt{2}x+2}\right| + \sqrt{2}\arctan\frac{2x}{\sqrt{2}-x}\end{aligned}$$

5.67. Note that

$$-2\ln(x+1) = \left\{a\ln b = \ln b^a\right\} = \ln\frac{1}{(x+1)^2} \quad \therefore \quad \left\{e^{\ln x} = x\right\}$$

$$\therefore$$

$$e^{-2\ln|x+1|} = e^{\ln(1/(x+1)^2)} = \frac{1}{(x+1)^2}$$

because a square function is always positive so it can replace the absolute function. Which is to say,

$$\frac{x}{1-x^2}\,e^{-2\ln|x+1|} = \frac{x}{(1-x)(1+x)}\,e^{-2\ln|x+1|} = \frac{x}{(1-x)(1+x)(1+x)^2} = \frac{x}{(1-x)(1+x)^3}$$

Partial fraction decomposition of the last rational function (see Vol. I) is

$$\frac{x}{(1-x)(x+1)^3} = -\frac{1}{8(x-1)} + \frac{1}{8(x+1)} + \frac{1}{4(x+1)^2} - \frac{1}{2(x+1)^3}$$

And by consequence (see P.5.9 and P.5.10),

$$\begin{aligned}\int \frac{x}{1-x^2}\,e^{-2\ln|x+1|}\,dx &= -\frac{1}{8}\int\frac{dx}{(x-1)} + \frac{1}{8}\int\frac{dx}{(x+1)} + \frac{1}{4}\int\frac{dx}{(x+1)^2} - \frac{1}{2}\int\frac{dx}{(x+1)^3}\\ &= -\frac{1}{8}\ln|x-1| + \frac{1}{8}\ln|x+1| - \frac{1}{4}\frac{1}{x+1} + \frac{1}{4}\frac{1}{(x+1)^2} + c\\ &= \frac{1}{8}\ln\frac{|x+1|}{|x-1|} + \frac{1}{4}\left(\frac{1}{(x+1)^2} - \frac{1}{x+1}\right) + c \quad \{a\ln b = \ln b^a\}\\ &= \frac{1}{16}\ln\frac{(x+1)^2}{(x-1)^2} - \frac{1}{4}\frac{x}{(x+1)^2} + c\end{aligned}$$

5.6 Definite Integrals

5.68. Geometrical interpretation of definite integral is area A under curve $f(x)$; see Fig. 5.4. Note that when calculating finite integrals, the integration constants C are cancelled. Thus,

$$A = \int_2^3 x^3\,dx = \left\{\int x^n dx = \frac{x^{n+1}}{n}\right\} = \frac{x^4}{4}\Big|_2^3 = \frac{1}{4}\left(3^4 - 2^4\right) = \frac{1}{4}\,(81-16) = \frac{65}{4}$$

5.69. Note that the result of definite integrals are signed areas; see Fig. 5.5 (left).

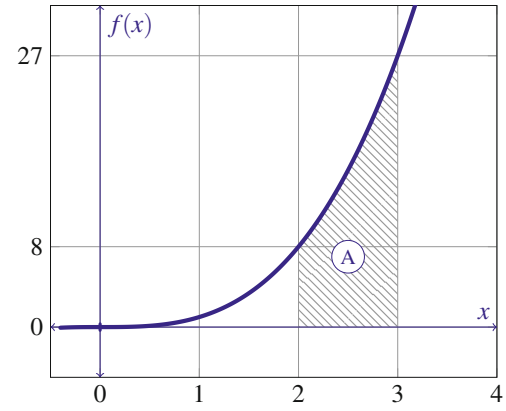


Fig. 5.4 Example P.5.68

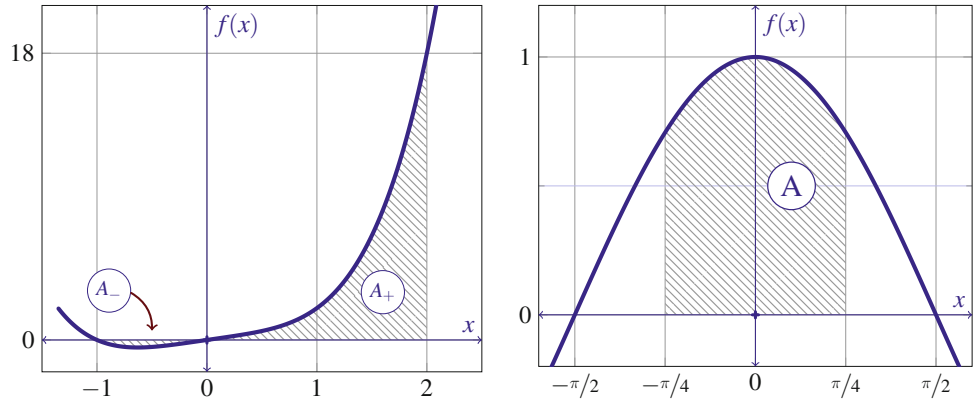


Fig. 5.5 Examples P.5.69 and P.5.70

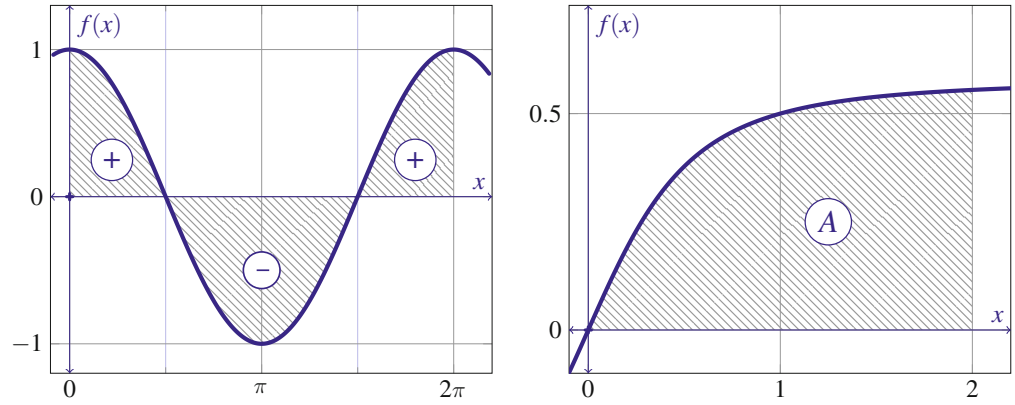


Fig. 5.6 Examples P.5.71 and P.5.72

$$A = \int_{-1}^{2} x(1+x^3)\,dx = \int_{-1}^{2} x\,dx + \int_{-1}^{2} x^4 dx = \frac{x^2}{2}\Big|_{-1}^{2} + \frac{x^5}{5}\Big|_{-1}^{2}$$
$$= \frac{1}{2}\big(2^2 - (-1)^2\big) + \frac{1}{5}\big(2^5 - (-1)^5\big)$$
$$= \frac{3}{2} + \frac{33}{5} = \frac{81}{10}$$

5.70. Trigonometric functions and irrational numbers are interleaved; see Fig. 5.5 (right).

$$\int_{-\pi/4}^{\pi/4} \cos x\,dx = \sin x\Big|_{-\pi/4}^{\pi/4} = \sin\frac{\pi}{4} - \sin\frac{-\pi}{4} = \Big\{\sin(-x) = -\sin(x)\Big\} = 2\sin\frac{\pi}{4} = 2\frac{\sqrt{2}}{2} = \sqrt{2}$$

5.71. Within one period, the areas' sum equals zero; see Fig. 5.6 (left). That is, the *average* of the sinusoidal function over the integer number of full periods is zero.

$$\int_0^{2\pi} \cos x \, dx = \sin x \Big|_0^{2\pi} = \sin(2\pi) - \sin(0) = 0$$

5.72. As a matter of style and practicality, when using the change of variables method, the interval boundaries may either be recalculated relative to the new variable or kept and applied once the original variable is reintroduced. The area under curve is illustrated in Fig. 5.6 (right), as

$$\int_0^2 \frac{x}{\sqrt{1+3x^2}}\, dx = \left\{ \begin{array}{l} t = 1 + 3x^2 \;\; \therefore \;\; dt = 6x\, dx \;\; \therefore \;\; x\, dx = \dfrac{dt}{6} \\ \text{if } x = 0 \;\; \Rightarrow \;\; t = 1, \quad \text{and if} \quad x = 2 \;\; \Rightarrow \;\; t = 13 \end{array} \right\}$$

$$= \int_1^{13} \frac{1}{6} \frac{dt}{\sqrt{t}} = \left\{ \sqrt[n]{x^m} = x^{m/n}, \;\; \frac{1}{x^n} = x^{-n} \right\}$$

$$= \frac{1}{6} \int_1^{13} t^{-1/2}\, dt = \frac{1}{\cancel{6}\;3} \frac{t^{1/2}}{\cancel{1/2}} \Big|_1^{13} = \frac{1}{3} \left(\sqrt{13} - \sqrt{1}\right)$$

5.73. While using the integration by parts method, it is not mandatory to write the interval limits; however, they must be accounted for at the end of calculations. Being signed area, in this case, the total calculated area is found to be negative; see Fig. 5.7.

$$\int_0^{\pi} x^2 \cos x \, dx = \{\text{see A.5.40}\} = \left((x^2 - 2)\sin x + 2x \cos x\right) \Big|_0^{\pi}$$

$$= (\pi^2 - 2) \cancelto{0}{\sin \pi} - (0 - 2) \cancelto{0}{\sin 0} + 2\pi \cancelto{-1}{\cos \pi} - 2\,(0) \cancelto{0}{\cos 0}$$

$$= -2\pi$$

5.74. Square function is even,

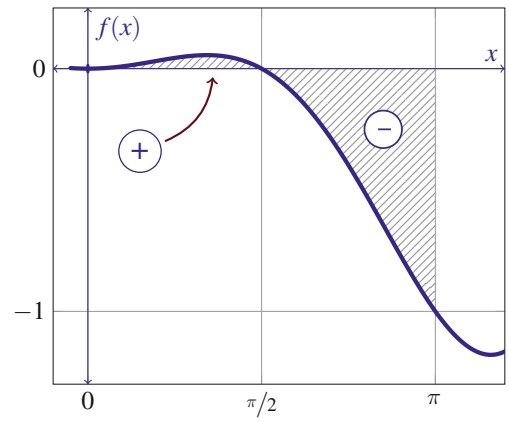


Fig. 5.7 Example P.5.73

$$\int_{-\pi}^{\pi} x^2 \cos(ax)\, dx$$

$$= \left\{ \int u\, dv = u\, v - \int v\, du \right\} = \left\{ \begin{array}{l} u = x^2 \Rightarrow \dfrac{du}{dx} = 2x \quad \therefore \quad du = 2x\, dx \\ \\ dv = \cos(ax)\, dx \Rightarrow v = \displaystyle\int \cos(ax)\, dx = \frac{1}{a} \sin(ax) \end{array} \right\}$$

$$= \frac{x^2}{a} \sin(ax) - \frac{2}{a} \int x\, \sin(ax)\, dx$$

$$= \left\{ \begin{array}{l} u = x \Rightarrow du = dx \\ \\ dv = \sin(ax)\, dx \Rightarrow v = \displaystyle\int \sin(ax)\, dx = -\frac{1}{a} \cos(ax) \\ \\ \therefore \\ \\ \displaystyle\int x \sin(ax)\, dx = -\frac{x}{a} \cos(ax) + \frac{1}{a} \int \cos(ax)\, dx = -\frac{x}{a} \cos(ax) + \frac{1}{a^2} \sin(ax) \end{array} \right\}$$

$$= \left(\frac{1}{a} x^2\, \sin(ax) + \frac{2}{a^2}\, x\, \cos(ax) - \frac{2}{a^3} \sin(ax) \right) \Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{a} \pi^2\, \sin(a\pi) + \frac{2}{a^2}\, \pi\, \cos(a\pi) - \frac{2}{a^3} \sin(a\pi)$$

$$- \left(\frac{1}{a} (-\pi)^2\, \sin(-a\pi) + \frac{2}{a^2}\, (-\pi)\, \cos(-a\pi) - \frac{2}{a^3} \sin(-a\pi) \right)$$

$$= \frac{1}{a} \pi^2\, \sin(a\pi) + \frac{2}{a^2}\, \pi\, \cos(a\pi) - \frac{2}{a^3} \sin(a\pi)$$

$$+ \frac{1}{a} \pi^2\, \sin(a\pi) + \frac{2}{a^2}\, \pi\, \cos(a\pi) - \frac{2}{a^3} \sin(a\pi)$$

$$= \frac{2}{a} \pi^2\, \sin(a\pi) - \frac{4}{a^3} \sin(a\pi) + \frac{4}{a^2}\, \pi\, \cos(a\pi)$$

Similarly,

$$\int_{-\pi}^{\pi} x^2 \sin(ax)\, dx = \left(\frac{2}{a^3} \cos(ax) - \frac{x^2}{a}\, \cos(ax) + \frac{2}{a^2} x \sin(ax) \right) \Bigg|_{-\pi}^{\pi}$$

$$= \left(\frac{2}{a^3} \cos(a\pi) - \frac{\pi^2}{a}\, \cos(a\pi) + \frac{2}{a^2}\, \pi \sin(a\pi) \right)$$

$$- \left(\frac{2}{a^3} \cos(-a\pi) - \frac{\pi^2}{a}\, \cos(-a\pi) - \frac{2}{a^2} \pi \sin(-a\pi) \right) = 0$$

5.75. Given rational function is decomposed by the partial fraction decomposition technique (see Vol. I), so that

$$\int_5^6 \frac{x}{(x+1)(x-4)}\, dx = \frac{1}{5} \int_5^6 \frac{dx}{x+1} + \frac{4}{5} \int_5^6 \frac{dx}{x-4}$$

$$= \left\{ \begin{array}{l} x+1=t \;\; \therefore \;\; x \to 5 \Rightarrow t \to 6 \quad \text{and} \quad x \to 6 \Rightarrow t \to 7 \\ x-4=r \;\; \therefore \;\; x \to 5 \Rightarrow r \to 1 \quad \text{and} \quad x \to 6 \Rightarrow r \to 2 \end{array} \right\}$$

$$= \{\text{see A.5.9}\}$$

$$= \frac{1}{5}\int_6^7 \frac{dt}{t} + \frac{4}{5}\int_1^2 \frac{dr}{r} = \frac{1}{5}\,\ln(t)\,\Big|_6^7 + \frac{4}{5}\,\ln(r)\,\Big|_1^2$$

$$= \frac{1}{5}(\ln 7 - \ln 6) + \frac{4}{5}(\ln 2 - \cancelto{0}{\ln 1})$$

$\therefore$

(the answer may be in any of the following forms)

$$= \frac{1}{5}(\ln 7 - \ln 6 + 4\ln 2) = \frac{1}{5}\ln\left(\frac{7\times 2^4}{6}\right) = \frac{1}{5}\ln\frac{56}{3} = \ln\sqrt[5]{\frac{56}{3}}$$

$$= 0.58534...$$

as illustrated in Fig. 5.8. Note that by visual inspection of the graph, the calculated area A may be estimated to be indeed approximately 0.6 (a little more than half a 1×1 square).

5.76. $\displaystyle\int_0^1 x\,\ln\left(1+x^2\right)dx = \{\text{see A.5.30}\} = \frac{1}{2}\,t\,(\ln t - 1)\,\Big|_1^2 = \frac{1}{2}\,2(\ln 2 - 1) - \frac{1}{2}\,(\cancelto{0}{\ln 1} - 1)$

$$= \ln 2 - 1 + \frac{1}{2} = \ln 2 - \frac{1}{2}$$

5.77. $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\,dx = \{\text{see A.5.57}\} = 2\,\sqrt{e^x-1}\,\Big|_0^{\ln 2} - 2\,\arctan\sqrt{e^x-1}\,\Big|_0^{\ln 2}$

$$= 2 - 2\,\arctan 1 - 0 = 2 - \frac{\pi}{2}$$

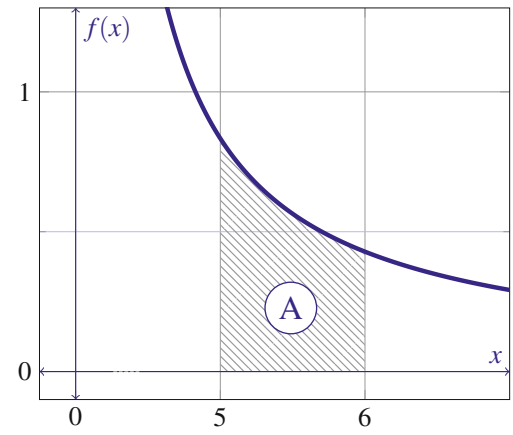


Fig. 5.8 Example P.5.75

5.78. $$\int_{-\infty}^{\infty} \frac{a}{a^2+x^2}\,dx = \frac{a}{a^2}\int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx = \left\{\frac{x}{a}=t \quad \therefore \quad dx = |a|\,dt\right\}$$
$$= \frac{|a|}{a}\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt = \frac{|a|}{a}\ \arctan(t)\ \Big|_{-\infty}^{\infty}$$
$$= \frac{|a|}{a}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$
$$= \pi\ \text{sign}\,(a)$$

5.79. $$\int_0^{\infty} e^{-\omega_0\, t\, y}\,dy = \left\{\begin{array}{l}\text{change variable:} \quad -\omega_0\, t\, y = x \quad \therefore \quad dy = -\dfrac{1}{\omega_0\, t}\,dx \\ \qquad\qquad\qquad\qquad y = 0 \Rightarrow x = 0, \quad \text{and} \quad y = \infty \Rightarrow x = \infty\end{array}\right\}$$
$$= -\frac{1}{\omega_0\, t}\int_0^{\infty} e^x\,dx = -\frac{1}{\omega_0\, t}\ e^x\ \Big|_0^{\infty} = -\frac{1}{\omega_0\, t}\left(\cancel{e^{-\infty}}^{\,0} - \cancel{e^0}^{\,1}\right)$$
$$= \frac{1}{\omega_0\, t}$$

5.80. See P.5.42; then

$$\int_0^{\infty} e^{-bx}\,\sin(ax) = -\frac{1}{a^2+b^2}\ e^{-bx}\left(a\cos(ax) + b\sin(ax)\right)\ \Big|_0^{\infty}$$
$$= -\frac{1}{a^2+b^2}\left\{\underbrace{\cancel{e^{-b\infty}}^{\,0}\left[a\ \underbrace{\cos(a\,\infty)}_{\le 1} + b\ \underbrace{\sin(a\,\infty)}_{\le 1}\right]}_{=0}\right.$$
$$\left. - \cancel{e^{-b\cdot 0}}^{\,1}\left[a\ \underbrace{\cos(a\cdot 0)}_{=1} + b\ \underbrace{\sin(a\cdot 0)}_{=0}\right]\right\}$$
$$= \frac{a}{a^2+b^2}$$

Therefore,

$$\int_0^{\infty} e^{-bx}\,\sin(ax) = \frac{a}{a^2+b^2}$$

By repeating the same idea,

$$\int_0^{\infty} e^{-bx}\,\cos(ax) = \frac{b}{a^2+b^2}$$

5.81. $$\int e^x\,\cos(ax)\,dx = \left\{\begin{array}{l}\text{partial integration:} \quad u = e^x \quad \therefore \quad du = e^x\,dx \\ \qquad\qquad\qquad\qquad dv = \cos(ax)\,dx \quad \therefore \quad v = \dfrac{1}{a}\sin(ax)\end{array}\right\}$$

$$= e^x \ \frac{1}{a}\sin(ax) - \frac{1}{a}\underbrace{\int e^x \sin(ax)\,dx}_{\text{partial integration}} = \{\text{see A.5.82}\}$$

$$= e^x \ \frac{1}{a}\sin(ax) + \frac{1}{a}\frac{1}{a^2+1}\,e^x\left(a\cos(ax) - \sin(ax)\right)$$

$$= e^x \ \frac{1}{a}\sin(ax) - \frac{1}{a}\frac{1}{a^2+1}\,e^x \ \sin(ax) + \frac{1}{\cancel{a}}\frac{1}{a^2+1}\,e^x\,\cancel{a}\cos(ax)$$

$$= \frac{a}{a^2+1}\,e^x \ \sin(ax) + \frac{1}{a^2+1}\,e^x \ \cos(ax) + C$$

which gives

$$\int_{-\pi}^{\pi} e^x \cos(ax)\,dx\,dx = \left(\frac{a}{a^2+1}\,e^x \ \sin(ax) + \frac{1}{a^2+1}\,e^x \ \cos(ax)\right)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{a}{a^2+1}\,e^{\pi} \ \sin(a\pi) + \frac{1}{a^2+1}\,e^{\pi} \ \cos(a\pi)$$

$$- \frac{a}{a^2+1}\,e^{-\pi} \ \sin(-a\pi) - \frac{1}{a^2+1}\,e^{-\pi} \ \cos(-a\pi)$$

$$= \frac{a}{a^2+1} \ \sin(a\pi)\left(e^{\pi} + e^{-\pi}\right) + \frac{1}{a^2+1} \ \cos(a\pi)\left(e^{\pi} - e^{-\pi}\right)$$

5.82. $\int e^x \sin(ax)\,dx =$

$$\left\{\begin{array}{ll}\text{partial integration:} & u = e^x \ \ \therefore \ \ du = e^x\,dx \\ & dv = \sin(ax)\,dx \ \ \therefore \ \ v = -\frac{1}{a}\cos(ax)\end{array}\right\}$$

$$= e^x \ \left(-\frac{1}{a}\cos(ax)\right) - \int\left(-\frac{1}{a}\cos(ax)\right)\,e^x\,dx$$

$$= -\frac{1}{a}e^x\cos(ax) + \frac{1}{a}\underbrace{\int e^x\cos(ax)\,dx}_{\text{partial integration}}$$

$$\left\{\begin{array}{ll}\text{partial integration:} & u = e^x \ \ \therefore \ \ du = e^x\,dx \\ & dv = \cos(ax)\,dx \ \ \therefore \ \ v = \frac{1}{a}\sin(ax)\end{array}\right\}$$

$$= -\frac{1}{a}e^x\cos(ax) + \frac{1}{a}\left[\frac{1}{a}\,e^x \ \sin(ax) - \frac{1}{a}\int e^x\sin(ax)\,dx\right]$$

The last integral may be factored out and moved to the right side as

$$\left(1 + \frac{1}{a^2}\right)\int e^x \ \sin(ax)\,dx = -\frac{1}{a}e^x\cos(ax) + \frac{1}{a^2}\,e^x \ \sin(ax)$$

$$\therefore$$

$$\int e^x \ \sin(ax)\,dx = -\frac{\cancel{a^2}}{a^2+1}\,\frac{1}{\cancel{a^2}}\,e^x\,(a\cos(ax) - \sin(ax))$$

which gives

$$\begin{aligned}\int_{-\pi}^{\pi} e^x \sin(ax)\,dx &= -\frac{1}{a^2+1}\; e^x\left(a\cos(ax)-\sin(ax)\right)\Big|_{-\pi}^{\pi}\\ &= -\frac{1}{a^2+1}\left(e^{\pi}\left(a\cos(a\pi)-\sin(a\pi)\right)-e^{-\pi}\left(a\cos(-a\pi)-\sin(-a\pi)\right)\right)\\ &= -\frac{1}{a^2+1}\left(e^{\pi}\left(a\cos(a\pi)-\sin(a\pi)\right)-e^{-\pi}\left(a\cos(a\pi)+\sin(a\pi)\right)\right)\\ &= -\frac{a}{a^2+1}\cos(a\pi)\left(e^{\pi}-e^{-\pi}\right)+\frac{1}{a^2+1}\sin(a\pi)\left(e^{\pi}+e^{-\pi}\right)\end{aligned}$$

5.7 Area Integral

5.83. A right-angled triangle may be mapped into a linear function so that the two catheti a and b are the horizontal and vertical coordinates, Fig. 5.9, of the hypotenuse.

The angle of a linear function is, by definition of tangent, the ratio of the two catheti; therefore, the analytical form of this affine function is

$$f(x)=\frac{b}{a}x$$

Area A under a function is therefore

$$A=\int_0^a f(x)\,dx=\int_0^a \frac{b}{a}x\,dx=\frac{b}{a}\frac{1}{2}\;\frac{x^2}{2}\Big|_0^a=\frac{1}{2}\,\frac{b}{\cancel{a}}\,(a^{\cancel{2}}-0)=\frac{ab}{2}$$

which is a well-known formula given in the elementary school that results in $A=ab/2=2\times 1/2{=}1$.

5.84. The area of a circle is a classic problem that presented challenge over thousands of years until the development of calculus in the late seventeenth century by Newton and Leibniz. One possible

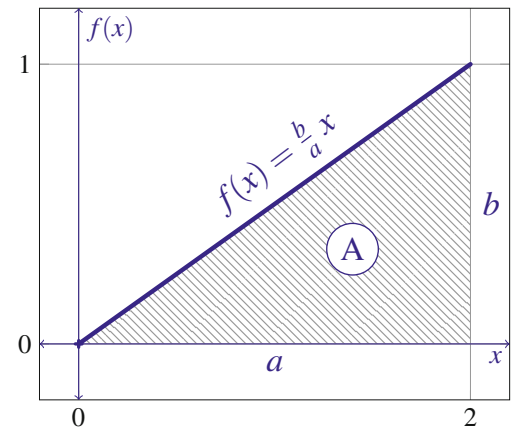


Fig. 5.9 Example P.5.83

technique to calculate the circle area is as follows.

From the algebraic equation for circle, given $r \geq 0$, it follows that

$$x^2 + y^2 = r^2 \quad \therefore \quad y = \sqrt{r^2 - x^2} = f(x)$$

where the function's boundaries may be $x \in [0, r]$, $y \in [0, r]$. This is possible for the reason of symmetry; it is sufficient to calculate the area of $1/4$ of the circle, specifically the one in the first (positive) quadrant.

That being the case, trigonometric substitution may be used to solve the following definite integral, as

$$\begin{aligned}
A_4 &= \int_0^1 f(x)\,dx = \int_0^1 \sqrt{r^2 - x^2}\,dx = \{\text{see A.5.43}\} \\
&= \left\{x = r\,\cos\theta \quad \therefore \quad \frac{dx}{d\theta} = -r\,\sin\theta \quad \therefore \quad dx = -r\,\sin\theta\,\,d\theta\right\} \\
&= \int \sqrt{r^2 - r^2\,\cos^2\theta}\,(-r\,\sin\theta\,\,d\theta) = -r^2 \int \sqrt{1 - 1\cos^2\theta}\,\sin\theta\,\,d\theta \\
&= \left\{\text{where, if } x = 0 \Rightarrow \theta = \arctan(0) = \frac{\pi}{2}, \quad \text{if} \quad x = 1 \Rightarrow \theta = \arctan(1) = 0\right\} \\
&= -r^2 \int_{\pi/2}^0 \sqrt{1 - \cos^2\theta}\,\sin\theta\,d\theta = \left\{\sin^2 x + \cos^2 x = 1 \quad \therefore \quad \sin x = \sqrt{1 - \cos^2 x}\right\} \\
&= r^2 \int_0^{\pi/2} \sin^2\theta\,d\theta = \left\{\sin^2 x = \frac{1}{2}\,(1 - \cos(2x))\right\}\left\{\text{(note negative sign and order of } 0 \leftrightarrow \pi/2)\right\} \\
&= \frac{r^2}{2} \int_0^{\pi/2} \big(1 - \cos(2\theta)\big)\,d\theta = \frac{r^2}{2} \int_0^{\pi/2} d\theta - \frac{r^2}{2} \int_0^{\pi/2} \cos(2\theta)\,d\theta
\end{aligned}$$

The second change of variables results in

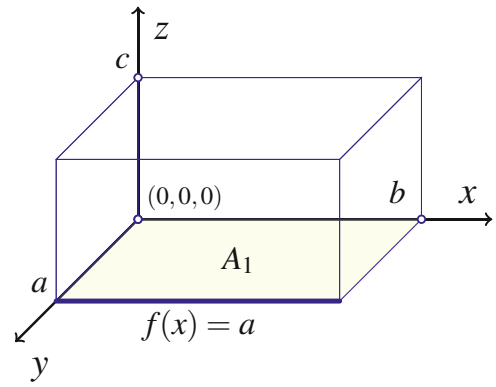
$$\begin{aligned}
A_4 &= \left\{\begin{array}{l} t = 2\theta \quad \therefore \quad dt = 2d\theta \quad \therefore \quad d\theta = \dfrac{dt}{2} \\ \\ \theta = 0 \quad \therefore \quad t = 0, \quad \theta = \dfrac{\pi}{2} \quad \therefore \quad t = \pi \end{array}\right\} = \frac{r^2}{2}\,\theta\,\Big|_0^{\pi/2} - \frac{r^2}{2} \int_0^{\pi} \cos t\,\frac{dt}{2} \\
&= \frac{r^2}{2}\left[\frac{\pi}{2} - 0\right] - \frac{r^2}{4}\,\sin t\,\Big|_0^{\pi} = \frac{r^2\pi}{4} - \frac{r^2}{4}(\cancelto{0}{\sin\pi} - \cancelto{0}{\sin 0}) = \frac{r^2\pi}{4}
\end{aligned}$$

which equals to $1/4$ of the circle area; thus, the full circle area is four times greater, i.e.,

$$A = 4\,A_4 = \cancel{4}\,\frac{r^2\pi}{\cancel{4}} = r^2\pi$$

5.85. Calculation of volumes is done as "integral of an integral", that is to say, a multiple integrals. More precisely, there is one integration for each variable, while all the remaining variables are considered as constants. Visually, the volume of a rectangular cuboid is similar to a book, where

Fig. 5.10 Example P.5.85



the area of each rectangular (horizontal) page is calculated as a definite integral of a constant (first integral). Then, all pages are piled on top of each other along the vertical dimension until the full height is reached (second integral).

For example, two sides of the bottom horizontal rectangular area (see Fig. 5.10) are $x \in [0, b]$ interval on the one side and $y \in [0, a]$ on the other. That is, area A_1 is found as

$$A_1 = \int_0^b f(x)\,dx = \int_0^b a\,dx = a\,x\,\Big|_0^b = a(b-0) = ab$$

which is an already known area of a rectangle formula. Next, all "pages" are added in the vertical $z \in [0, c]$ direction as

$$V = \int_0^c A_1\,dz = \{A_1 = ab = \text{const.}\} = ab\,z\,\Big|_0^c = ab(c-0) = abc$$

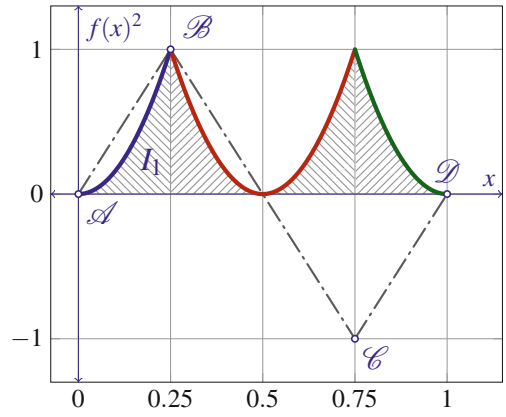
which is known as the cuboid volume formula, e.g., $V = abc = 4 \times 3 \times 2 = 24$. This process of double integration is formally written in one line as

$$V = \int_0^b \int_0^c f(x)\,dx\,dz = \int_0^b \int_0^c a\,dx\,dz = a \int_0^c dz \int_0^b dx$$
$$= a \int_0^c dz\,x\,\Big|_0^b = ab \int_0^c dz = ab\,z\,\Big|_0^c = abc$$

Note that the order of addition is not relevant; thus, the most convenient one may be chosen. In addition, this trivial example illustrates the general principle for volume calculations in 3D space. What is more, the same reasoning and techniques apply to higher-order spaces and non-trivial objects (i.e., non-constant functions).

5.86. Piecewise linear function is one of the most often used approximations in engineering. Given its graphical representation, it is necessary to derive analytical forms for *each* of the linear sections

Fig. 5.11 Example P.5.86



separately. Therefore, knowing a priori the analytical form of each linear piece, the overall integral is reduced to the sum of simple integrals of each section.

Linear section $y = ax + b$ is determined by two points in the plane, where the two constants (a, b) are determined with a simple algebra. Linear sections are outlined in by points $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})$; see Fig. 5.11. In addition, $f(x)^2$ of each section is shown – note the symmetry of the four intervals, each resolved as follows.

Method 1:

1. $\overline{\mathscr{A}\mathscr{B}}$: at two end points (x, y) of $y = ax + b$, it follows that:

$$(0, 0) \quad \therefore \quad 0 = a \times 0 + b \Rightarrow b = 0$$

$$(0.25, 1) \quad \therefore \quad 1 = a \times \frac{1}{4} + b \Rightarrow a = 4$$

Therefore, the analytical form of $\overline{\mathscr{A}\mathscr{B}}$ linear segment is $f(x) = 4x$, which is valid within the interval $(0, 0.25)$. Therefore,

$$I_1 = \int_0^{1/4} f(x)^2 \, dx = \int_0^{1/4} (4x)^2 \, dx = 16 \left. \frac{x^3}{3} \right|_0^{1/4} = \frac{16}{3} \left(\frac{1}{4^3} - 0 \right) = \frac{1}{12}$$

2. $\overline{\mathscr{B}\mathscr{C}}$: at two end points (x, y) of $y = ax + b$, it follows that:

$$(0.25, 1) \quad \therefore \quad 1 = a \times \frac{1}{4} + b$$

$$(0.75, -1) \quad \therefore \quad -1 = a \times \frac{3}{4} + b$$

The solution of this system of two linear equations is $(a, b) = (-4, 2)$. Therefore, the analytical form of $\overline{\mathscr{C}\mathscr{D}}$ linear segment is $f(x) = -4x + 2$, which is valid in the interval $(0.25, 0.75)$. Then,

$$I_2 = \int_{1/4}^{3/4} f(x)^2 \, dx = \int_{1/4}^{3/4} (-4x + 2)^2 \, dx = \int_{1/4}^{3/4} (16x^2 - 16x + 4) \, dx$$

$$= 16\ \frac{x^3}{3}\Big|_{1/4}^{3/4} - \cancel{16}\ 8\ \frac{x^2}{\cancel{2}}\Big|_{1/4}^{3/4} + 4\,x\Big|_{1/4}^{3/4} = \frac{\cancel{16}}{3}\frac{13}{\cancel{32}\ 2} - \cancel{8}\ 4\ \frac{1}{\cancel{2}} + \cancel{4}\ \frac{2}{\cancel{4}} = \left(\frac{13}{6} - 2\right) = \frac{1}{6}$$

3. $\overline{\mathscr{CD}}$: at two end points (x, y) of $y = ax + b$, it follows that:

$$(0.75, -1) \quad \therefore \quad -1 = a \times \frac{3}{4} + b$$
$$(1, 0) \quad \therefore \quad 0 = a \times 1 + b$$

The solution of this system of two linear equations is $(a, b) = (4, -4)$. Therefore, the analytical form of $\overline{\mathscr{CD}}$ linear segment is $f(x) = 4\,x - 4$, which is valid in the interval $(0.75, 1)$. Then,

$$I_3 = \int_{3/4}^{1} f(x)^2\,dx = \int_{3/4}^{1} (4x-4)^2\,dx = \int_{3/4}^{1} (16x^2 - 32x + 16)\,dx$$
$$= \frac{16}{3}\,x^3\Big|_{3/4}^{1} - 16\,x^2\Big|_{3/4}^{1} + 16\,x\Big|_{3/4}^{1} = \frac{\cancel{16}}{3}\frac{37}{\cancel{64}\ 4} - \cancel{16}\ \frac{7}{\cancel{16}} + \cancel{16}\ 4\ \frac{1}{\cancel{4}} = \frac{37}{12} - 3 = \frac{1}{12}$$

The total area under $f(x)^2$ curve (see Fig. 5.11) is

$$I = \int_0^1 f(x)^2\,dx = I_1 + I_2 + I_3 = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$$

As an additional exercise, a simple integral

$$I = \int_0^1 f(x) = \int_0^{\frac{1}{4}} 4x\,dx + \int_{\frac{1}{4}}^{\frac{3}{4}} (-4x+2)\,dx + \int_{\frac{3}{4}}^{1} (4x-4)\,dx = 0$$

which is evident by inspection of Fig. 5.11; the area of a triangle above zero (i.e., positive) equals to the area of a triangle below zero (i.e., negative); therefore, their sum equals zero. In other words, the average of this piecewise linear function by itself equals zero.

Method 2:
Note that due to symmetry, all four interval areas under $f(x)^2$ are equal; thus, it is sufficient to calculate I_1 and multiply by four, as

$$I = 4\,I_1 = 4\int_0^{1/4} f(x)^2\,dx = \cancel{4}\ \frac{1}{\cancel{12}\ 3} = \frac{1}{3}$$

5.87. Given $f(x)$ and $g(x)$, first, it is necessary to determine the applicable interval. In this case, it can be visualized as if the total area under $g(x)$ (see Fig. 5.12 (left, top)) is created first and then the area under $f(x)$ (see Fig. 5.12 (left, bottom)) is "cut out" and removed. That action leaves only area A between the two functions; see Fig. 5.12 (right).

The applicable interval therefore is determined by two points $\mathscr{A}$ and $\mathscr{B}$ that belong to both functions, in other words when $f(x) = g(x)$, i.e.,

$$f(x) = g(x) \quad \therefore \quad x^2 = 8 - x^2$$

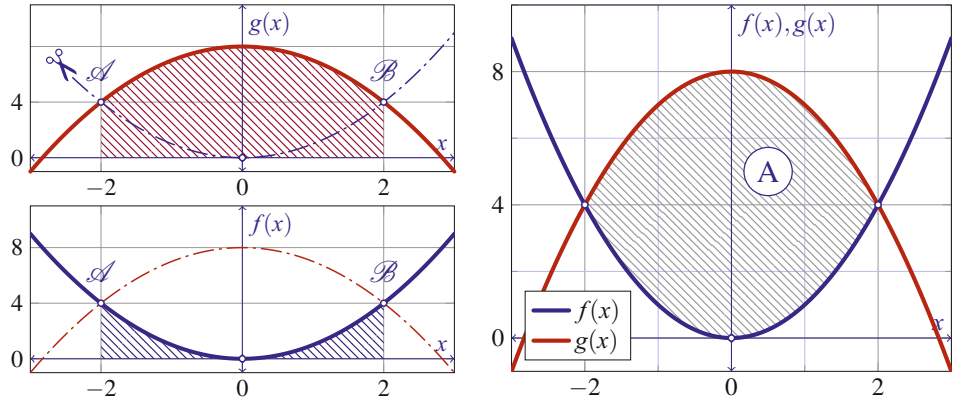


Fig. 5.12 Example P.5.87

$$2x^2 = 8$$
$$x^2 = 4 \;\; \therefore \;\; \underline{x = \pm 2}$$

Evidently, the "cutting out" operation is then a simple difference of

$$A = \int_{-2}^{2} \big(g(x) - f(x)\big)\, dx = \int_{-2}^{2} \big(8 - x^2 - x^2\big)\, dx = 8\int_{-2}^{2} dx - 2\int_{-2}^{2} x^2\, dx$$
$$= 8\, x \Big|_{-2}^{2} - 2\,\frac{x^3}{3}\Big|_{-2}^{2} = 8(2 - (-2)) - \frac{2}{3}\big(2^3 - (-2)^3\big) = \frac{64}{3}$$

5.88. The two functions may be written explicitly as functions of x, as

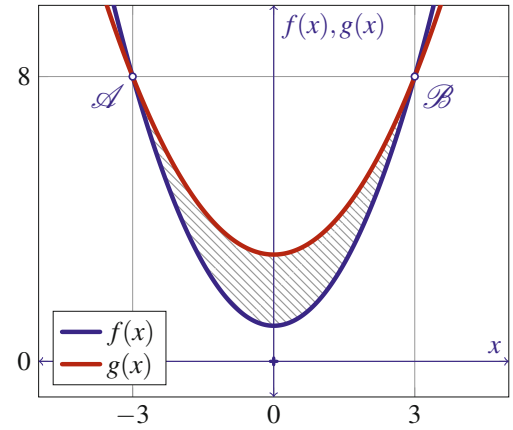
$$7x^2 - 9y + 9 = 0 \Rightarrow y = f(x) = \frac{7x^2 + 9}{9}$$
$$5x^2 - 9y + 27 = 0 \Rightarrow y = g(x) = \frac{5x^2 + 27}{9}$$

The area between the two functions is defined by an interval defined by intersect points $\mathscr{A}$ and $\mathscr{B}$ calculated as

$$f(x) = g(x)$$
$$\frac{7x^2 + 9}{\cancel{9}} = \frac{5x^2 + 27}{\cancel{9}}$$
$$7x^2 + 9 = 5x^2 + 27 \Rightarrow 2x^2 = 18 \;\; \therefore \;\; \underline{x_{1,2} = \pm 3}$$

so that coordinates of the two points are

Fig. 5.13 Example P.5.88



$$f(-3) = g(-3) = 8 \;\; \therefore \;\; \mathscr{A} = (-3, 8)$$
$$f(3) = g(3) = 8 \;\; \therefore \;\; \mathscr{B} = (3, 8)$$

(see Fig. 5.13). The area between the two parabolas is therefore

$$\int_{-3}^{3} \big(g(x) - f(x)\big)\, dx = \int_{-3}^{3} \left(\frac{5x^2+27}{9} - \frac{7x^2+9}{9}\right) dx = \int_{-3}^{3} \frac{5x^2+27-7x^2-9}{9}\, dx$$
$$= \frac{1}{9}\int_{-3}^{3} \big(18 - 2x^2\big)\, dx = \frac{\cancel{18}^{\,2}}{\cancel{9}}\; x \,\Big|_{-3}^{3} - \frac{2}{27}\; x^3 \,\Big|_{-3}^{3} = 2\cdot 6 - \frac{2\cdot \cancel{54}^{\,2}}{\cancel{27}} = 8$$

5.89. Two equations may be written explicitly as functions of x, as

$$x^2 + y - 2 = 0 \Rightarrow y = f(x) = 2 - x^2$$
$$y = g(x) = x$$

The area between the two functions is defined by an interval defined by intersect points $\mathscr{A}$ and $\mathscr{B}$ calculated as

$$f(x) = g(x)$$
$$2 - x^2 = x \Rightarrow x^2 + x - 2 = 0 \;\; \therefore \;\; x^2 + 2x - x - 2 = x(x+2) - (x+2) = (x+2)(x-1) = 0$$
$$\therefore \;\; x_1 = -2, \;\; x_2 = 1$$

so that coordinates of the two intersect points are

$$f(-2) = g(-2) = -2 \;\; \therefore \;\; \mathscr{A} = (-2, -2)$$
$$f(1) = g(1) = 1 \;\; \therefore \;\; \mathscr{B} = (1, 1)$$

Note that there are both positive and negative signed areas bound by these two functions; see Fig. 5.14 (left). Therefore, if a single function integral is calculated, the resulting area is signed where the sign changes if the upper and lower integral limits are interchanged.

Method 1: Direct application of the Newton-Leibniz formula to calculate the integral between the two curves (see Fig. 5.14 (left)) results in

$$\int_{-2}^{1} \big(g(x) - f(x)\big)\, dx = \int_{-2}^{1} \big(2 - x^2 - x\big)\, dx = 2\,x \,\Big|_{-2}^{1} - \frac{1}{3}\,x^3 \,\Big|_{-2}^{1} - \frac{1}{2}\,x^2 \,\Big|_{-2}^{1} = 6 - 3 + \frac{3}{2} = \frac{9}{2}$$

Note that the *difference* between the two polynomial functions is continuous (see Sect. 5.11) and already accounted for at the beginning; by consequence, the total area being calculated is correct.

Method 2: There are multiple ways to cut the total area and then to add the individual areas. For example,

1. Quadratic $f(x)$ function crosses the horizontal axis at $x^2 = 2 \;\;\therefore\;\; x_{1,2} = \pm\sqrt{2}$. That being said, total area A_1 above the horizontal axis and below $f(x)$ (see Fig. 5.14 (center)) is

$$A_1 = \int_{-\sqrt{2}}^{\sqrt{2}} f(x)\, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \big(2 - x^2\big)\, dx = 2\,x \,\Big|_{-\sqrt{2}}^{\sqrt{2}} - \frac{1}{3}\,x^3 \,\Big|_{-\sqrt{2}}^{\sqrt{2}} = 4\,\sqrt{2} - \frac{4\sqrt{2}}{3} = \frac{8}{3}\sqrt{2}$$

2. Triangular area A_2 above the horizontal axis and below $g(x)$ (see Fig. 5.14 (right)) is

$$A_2 = \int_0^1 g(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}\,x^2 \,\Big|_0^1 = \frac{1}{2}$$

3. Arc area A_3 below $f(x)$ from $x = 1$ to $x = \sqrt{2}$ (see Fig. 5.14 (right)) is

$$\begin{aligned} A_3 = \int_1^{\sqrt{2}} f(x)\, dx = \int_1^{\sqrt{2}} \big(2 - x^2\big)\, dx = 2\int_1^{\sqrt{2}} dx - \int_1^{\sqrt{2}} x^2\, dx = 2\,x \,\Big|_1^{\sqrt{2}} - \frac{1}{3}\,x^3 \,\Big|_1^{\sqrt{2}} \\ = 2\big(\sqrt{2} - 1\big) - \frac{1}{3}\left((\sqrt{2})^3 - 1\right) = 2\sqrt{2} - 2 - \frac{2\sqrt{2}}{3} + \frac{1}{3} = \frac{4\sqrt{2}}{3} - \frac{5}{3} \end{aligned}$$

4. Triangular area A_4 *above* $g(x)$ and *below* the horizontal axis from $x = 0$ to $x = -2$ (see Fig. 5.14 (center)) is

$$A_4 = \int_0^{-2} g(x)\, dx = \int_0^{-2} x\, dx = \frac{1}{2}\,x^2 \,\Big|_0^{-2} = 2$$

   (Note that the area sign is positive due to the order of integral limits.)
5. Arc area A_5 *above* $f(x)$ and *below* the horizontal axis from $x = -\sqrt{2}$ to $x = -2$ (see Fig. 5.14 (right)) is

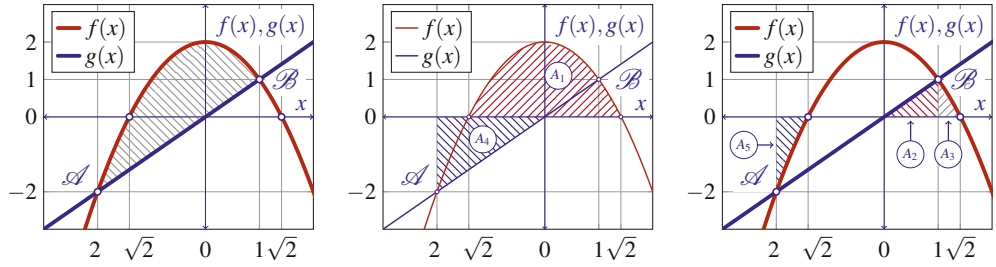


Fig. 5.14 Example P.5.89

$$A_5 = \int_{-\sqrt{2}}^{-2} f(x)\,dx = \int_{-\sqrt{2}}^{-2} (2 - x^2)\,dx = 2\,x\,\Big|_{-\sqrt{2}}^{-2} - \frac{1}{3}\,x^3\,\Big|_{-\sqrt{2}}^{-2} = -4 + 2\sqrt{2} + \frac{8}{3} - \frac{2\sqrt{2}}{3}$$
$$= \frac{4\sqrt{2}}{3} - \frac{4}{3}$$

In total, the area between $f(x)$ and $g(x)$ equals the sum of area above and below the horizontal axis, i.e.,

$$A = (A_1 - A_2 - A_3) + (A_4 - A_5) = \frac{8}{3}\sqrt{2} - \frac{1}{2} - \frac{4\sqrt{2}}{3} + \frac{5}{3} + 2 - \frac{4\sqrt{2}}{3} + \frac{4}{3} = \frac{9}{2}$$

5.90. Within the positive interval of the first period, two intercept points $\mathscr{A}$ and $\mathscr{B}$ are

$$f(x) = g(x)$$
$$\sin x = \frac{1}{2} \Rightarrow x_1 = \frac{\pi}{6}, \quad x_2 = \frac{5\pi}{6} \quad \{\text{see unit circle, Vol. I}\}$$

Direct calculation of the integral between the two curves results in

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin x - \frac{1}{2}\right) dx = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin x\,dx - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} dx = -\cos x\,\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{1}{2}\,x\,\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$
$$= -\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$$

5.91. The two functions may be written explicitly as functions of x, as

$$x^2 + 4y^2 - 20 = 0 \quad \therefore \quad y = f(x) = \sqrt{\frac{20 - x^2}{4}}$$
$$xy - 4 = 0 \quad \therefore \quad y = g(x) = \frac{4}{x}$$

The area between the two functions is within the interval defined by intersect points $\mathscr{A}$ and $\mathscr{B}$, as

$$f(x) = g(x) \quad \therefore \quad \sqrt{\frac{20 - x^2}{4}} = \frac{4}{x} \quad \therefore \quad \frac{20 - x^2}{4} = \frac{16}{x^2} \quad \therefore \quad x^4 - 20x^2 + 64 = 0$$

where roots of this biquadratic polynomial are found as

$$\begin{aligned} x^4 - 20x^2 + 64 = 0 \quad & \{x^2 = t\} \\ t^2 - 20t + 64 = 0 \quad & \\ t^2 - 4t - 16t + 64 = 0 \quad & \\ t(t-4) - 16(t-4) = 0 \quad & \\ (t-16)(t-4) = 0 \quad & \Rightarrow t_1 = 16, \quad t_2 = 4 \quad \therefore \quad x_{1,2} = \pm 4, \quad x_{3,4} = \pm 2 \end{aligned}$$

Coordinates of points $\mathscr{A}$ and $\mathscr{B}$ in $x > 0$ intervals (see Fig. 5.15 (left)) are

$$\begin{aligned} f(2) = g(2) = 2 \quad & \therefore \quad \mathscr{A} = (2, 2) \\ f(4) = g(4) = 1 \quad & \therefore \quad \mathscr{B} = (4, 1) \end{aligned}$$

Direct calculation of the integral between the two curves in the first quadrant results in

$$\int_2^4 \big(f(x) - g(x)\big)\, dx = \int_2^4 \left(\sqrt{\frac{20 - x^2}{4}} - \frac{4}{x}\right) dx = \frac{1}{2}\int_2^4 \sqrt{20 - x^2}\, dx - 4\int_2^4 \frac{1}{x}\, dx$$

$$\left\{x = \sqrt{20}\sin t \quad \therefore \quad dx = \sqrt{20}\cos t\, dt\right\}$$

$$= \frac{20}{2}\int \sqrt{1 - \sin^2 t}\, \cos t\, dt - 4\, \ln|x|\,\Big|_2^4 \quad \left\{1 - \sin^2 t = \cos^2 t\right\}$$

$$= 10\int \cos^2 t\, dt - 4\big(\ln 4 - \ln 2\big) \quad \left\{\begin{array}{l} \cos^2\theta = \dfrac{1 + \cos(2\theta)}{2} \\ \ln a - \ln b = \ln\dfrac{a}{b} \end{array}\right\}$$

As same as in A.5.18, the change of variables results in

$$\frac{1}{2}\int_2^4 \sqrt{20 - x^2}\, dx - 4\int_2^4 \frac{1}{x}\, dx = 10\,\frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right) - 4\ln 2 \quad \left\{t = \arcsin\left(\frac{x}{\sqrt{20}}\right)\right\}$$

$$= 5\left(\arcsin\left(\frac{x}{\sqrt{20}}\right)\Big|_2^4 + \frac{1}{2}\,\sin\left(2\,\arcsin\left(\frac{x}{\sqrt{20}}\right)\right)\Big|_2^4\right) - 4\ln 2$$

$$= \left\{\begin{array}{l} \arcsin\alpha \pm \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right) \\ \sin(2\alpha) = (\sin\alpha + \cos\alpha)^2 - 1 \\ \sin^2\alpha + \cos^2\alpha = 1 \end{array}\right\}$$

where the above trigonometric identities may be used as follows:

$$\begin{aligned}\arcsin\left(\frac{x}{\sqrt{20}}\right)\Big|_2^4 &= \arcsin\left(\frac{2}{\sqrt{5}}\right) - \arcsin\left(\frac{1}{\sqrt{5}}\right)\\ &= \arcsin\left(\frac{2}{\sqrt{5}}\sqrt{1-\left(\frac{1}{\sqrt{5}}\right)^2} - \frac{1}{\sqrt{5}}\sqrt{1-\left(\frac{2}{\sqrt{5}}\right)^2}\right)\\ &= \arcsin\frac{3}{5}\end{aligned}$$

and

$$\begin{aligned}\sin\left(2\arcsin\left(\frac{x}{\sqrt{20}}\right)\right)\Big|_2^4 &= \left(\left(\sin\left(\arcsin\left(\frac{x}{\sqrt{20}}\right)\right) + \cos\left(\arcsin\left(\frac{x}{\sqrt{20}}\right)\right)\right)^2 - 1\right)\Big|_2^4\\ &= \left\{\begin{array}{l}\sin(\arcsin\alpha) = \alpha\\ \cos(\arcsin\alpha) = \sqrt{1-\alpha^2}\end{array}\right\} = \left(\frac{x}{\sqrt{20}} + \sqrt{1-\left(\frac{x}{\sqrt{20}}\right)^2} - 1\right)\Big|_2^4\\ &= \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} - 1 - (-1) = 0\end{aligned}$$

Therefore, in total,

$$\begin{aligned}\int_2^4 \big(f(x) - g(x)\big)\,dx &= \int_2^4 \left(\sqrt{\frac{20-x^2}{4}} - \frac{4}{x}\right)dx = 5\left(\arcsin\frac{3}{5} + \frac{1}{2}(0)\right) - 4\ln 2\\ &= 5\arcsin\frac{3}{5} - 4\ln 2\end{aligned}$$

5.92. Given two functions

$$y = f(x) = x(x-1)(x-2) \text{ and } \quad y = g(x) = 0$$

form two symmetrical areas, one positive and one negative; see Fig. 5.15 (right). Direct application of the Newton-Leibniz formula to calculate the integral between the two curves results in

$$\begin{aligned}\int_0^2 \big(f(x) - g(x)\big)\,dx &= \int_0^2 f(x)\,dx = \int_0^2 \big(x(x-1)(x-2)\big)\,dx = \int_0^2 \big(x^3 - 3x^2 + 2x\big)\,dx\\ &= \frac{1}{4}x^4\Big|_0^2 - x^3\Big|_0^2 + x^2\Big|_0^2 = 4 - 8 + 4 = 0\end{aligned}$$

which is a consequence of symmetrical function with the initial (i.e., $f(0)$) and final (i.e., $f(2)$) value equal zero. Instead, it is necessary to calculate the absolute value of both sections and add them thereafter, as

$$\begin{aligned}\int_0^1 \big(f(x) - g(x)\big)\,dx &= \int_0^1 f(x)\,dx = \int_0^1 \big(x(x-1)(x-2)\big)\,dx = \int_0^1 \big(x^3 - 3x^2 + 2x\big)\,dx\\ &= \frac{1}{4}x^4\Big|_0^1 - x^3\Big|_0^1 + x^2\Big|_0^1 = \frac{1}{4} - 1 + 1 = \frac{1}{4}\end{aligned}$$

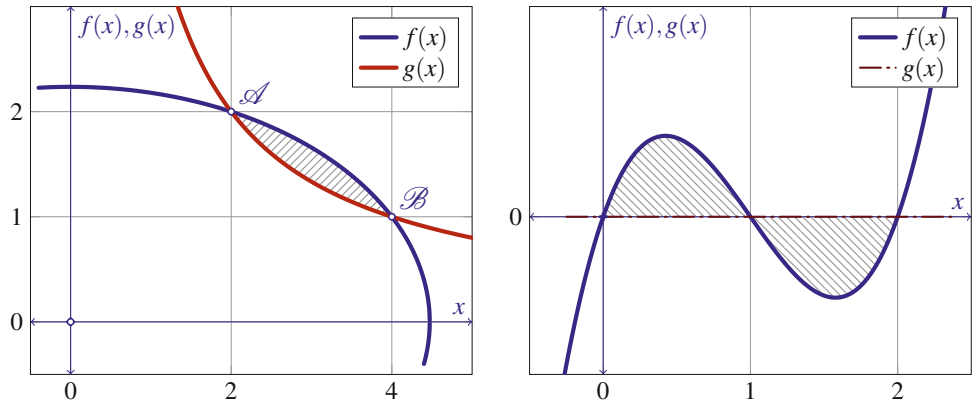


Fig. 5.15 Examples P.5.91 and P.5.92

and

$$\int_1^2 \big(f(x)-g(x)\big)\,dx = \int_1^2 f(x)\,dx = \int_1^2 \big(x(x-1)(x-2)\big)\,dx = \int_1^2 \big(x^3-3x^2+2x\big)\,dx$$
$$= \frac{1}{4}x^4\Big|_1^2 - x^3\Big|_1^2 + x^2\Big|_1^2 = \frac{1}{4}(16-1)-(8-1)+(4-1) = -\frac{1}{4}$$

which is to say that either the absolute value or reversal of the integral limits results in the results. In total, the area between these two functions equals $1/4 + |-1/4| = 1/2$.

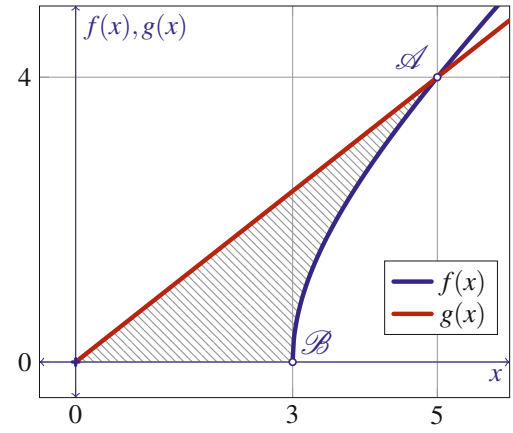
5.93. Given functions may be written in the explicit form, given that $(x, y \geq 0)$, as

$$x^2 - y^2 = 9 \quad \Rightarrow y = f(x) = \sqrt{x^2-9}$$
$$4x - 5y = 0 \quad \Rightarrow y = g(x) = \frac{4}{5}x$$

where $f(x)$ is defined for $x \geq 3$. Coordinates of the intercept point between $f(x)$ and $g(x)$ are found as

$$f(x) = g(x)$$
$$\sqrt{x^2-9} = \frac{4}{5}x$$
$$x^2 - \frac{16}{25}x^2 - 9 = 0 \quad \times 25$$
$$25x^2 - 16x^2 - 225 = 0$$
$$9x^2 - 225 = 0 \quad \div 9$$
$$x^2 - 25 = 0 \quad \Rightarrow \underline{x = 5} \quad \therefore \quad f(5) = 4 = g(5)$$

Fig. 5.16 Example P.5.93



Thus, the intercept point is $\mathscr{A} = (5, 4)$; see Fig. 5.16. Area A between these two functions may be found as the difference between their respective area as

$$A_1 = \int_0^5 g(x)\, dx$$

$$A_2 = \int_3^5 f(x)\, dx$$

$$\therefore$$

$$A = A_1 - A_2$$

where

1. The triangular area under $g(x)$ where $x \in (0, 5)$ is

$$A_1 = \int_0^5 g(x)\, dx = \frac{4}{5}\int_0^5 x\, dx = \frac{4}{10}\, x^2 \Big|_0^5 = 10$$

2. The arc area under $f(x)$ where $x \in (3, 5)$ is

$$I = \int f(x)\, dx = \int \sqrt{x^2 - 9}\, dx = \{\text{see A.5.51}\} = \frac{x}{2}\sqrt{x^2 - 9} - \frac{9}{2}\ln|x + \sqrt{x^2 - 9}|$$

so that the arc area under $f(x)$ in $x \in (3, 5)$ interval is

$$\begin{aligned} A_2 &= \left(\frac{x}{2}\sqrt{x^2 - 9} - \frac{9}{2}\ln|x + \sqrt{x^2 - 9}|\right)\Big|_3^5 \\ &= \left(\frac{5}{2}\sqrt{5^2 - 9} - \frac{9}{2}\ln|5 + \sqrt{5^2 - 9}|\right) - \left(\frac{3}{2}\sqrt{3^2 - 9} - \frac{9}{2}\ln|3 + \sqrt{3^2 - 9}|\right) \\ &= 10 - \frac{9}{2}\ln|9| + \frac{9}{2}\ln|3| = 10 - \frac{9}{2}\ln|3| \end{aligned}$$

In summary, area A above the horizontal axis bound between $g(x)$ and $f(x)$ equals

$$A = A_1 - A_2 = 10 - \left(10 - \frac{9}{2}\ln|3|\right) = \frac{9}{2}\ln|3|$$

5.94. Given functions may be written in the explicit form, as

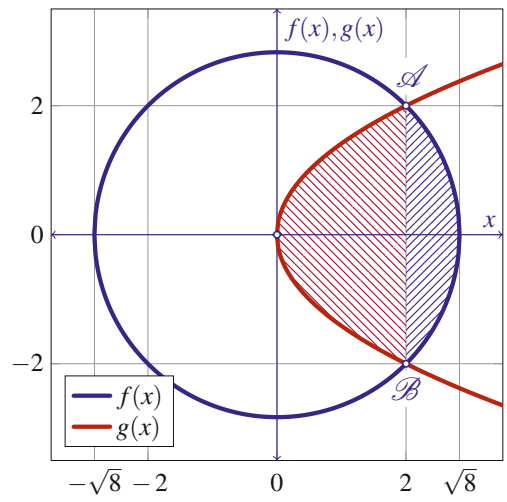
$$x^2 + y^2 = 8 \quad \Rightarrow y = f(x) = \sqrt{8 - x^2}$$
$$y^2 = 2x \quad \Rightarrow y = g(x) = \sqrt{2x}$$

where $f(x)$ is defined for $x \le \sqrt{8}$. Coordinates of the intercept point between $f(x)$ and $g(x)$ are found as

$$\begin{aligned}
f(x) &= g(x) \\
\sqrt{8 - x^2} &= \sqrt{2x} \quad \{\text{note that there are } \pm\surd \text{ values}\} \\
x^2 + 2x - 8 &= 0 \\
x^2 - 2x + 4x - 8 &= 0 \\
x(x-2) + 4(x-2) &= 0 \\
(x-2)(x+4) = 0 \quad &\Rightarrow \underline{x_{1,2} = \pm 2} \quad \therefore \quad f(\pm 2) = \pm 2 = g(\pm 2) \quad \left\{|\pm 4| > |\sqrt{8}|\right\}
\end{aligned}$$

Thus, for $x \ge 0$, intercept points are $\mathscr{A} = (2, 2)$ and $\mathscr{B} = (2, -2)$; see Fig. 5.17. Area A between these two functions may be found as the sum of their respective areas. Due to symmetry, it is sufficient to calculate the area for $y \ge 0$ and then multiply by two. That is to say,

Fig. 5.17 Example P.5.94



$$A_1 = \int_0^2 g(x)\,dx$$

$$A_2 = \int_2^{2\sqrt{2}} f(x)\,dx$$

$$\therefore$$

$$A = 2A_1 + 2A_2$$

where

1. The arc area under $g(x)$ is

$$A_1 = \int_0^2 g(x)\,dx = \sqrt{2}\int_0^2 \sqrt{x}\,dx = \sqrt{2}\,\frac{2}{3}\,x\sqrt{x}\,\Big|_0^2 = \frac{2\sqrt{2}}{3}\left(2\sqrt{2}\right) = \frac{8}{3}$$

2. The arc area under $f(x)$ is

$$A_2 = \int_2^{2\sqrt{2}} f(x)\,dx = \int_2^{2\sqrt{2}} \sqrt{8-x^2}\,dx = \{\text{see A.5.43}\}$$

$$= \left(4\arcsin\frac{x}{2\sqrt{2}} + \frac{x}{2}\sqrt{8-x^2}\right)\Big|_2^{2\sqrt{2}}$$

$$= \left(4\arcsin\frac{\cancel{2\sqrt{2}}\,1}{\cancel{2\sqrt{2}}} + \frac{\cancel{2}\sqrt{2}}{\cancel{2}}\cancel{\sqrt{8-(2\sqrt{2})^2}}^{\,0}\right) - \left(4\arcsin\frac{\cancel{2}\,1}{\cancel{2}\sqrt{2}} + \frac{\cancel{2}}{\cancel{2}}\sqrt{8-2^2}\right)$$

$$= 0 - (\pi - 2) \quad\Rightarrow\quad A_2 = \pi - 2 \quad \{\text{both positive and negative side}\}$$

In summary, the total enclosed area is

$$A = 2A_1 + 2A_2 = 2\,\frac{8}{3} + 2\pi - 4 = \frac{4}{3} + 2\pi$$

5.8 Volume of a Solid of Revolution

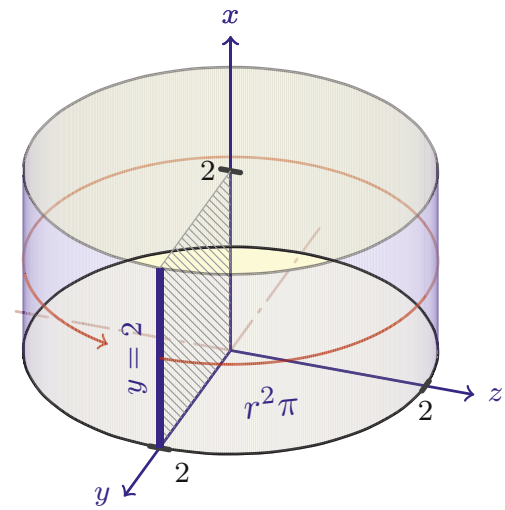
5.95. The circle area whose radius equals r is $A = \pi r^2$; see A.5.84. If each circle is imagined as a paper sheet whose thickness equals zero, then adding an infinite number of "pages" creates a solid cylindrical body, where radius may not be constant change along the vertical direction.

In the simplest case of a horizontal line $y = 2$ (i.e., radius of the future solid) within interval $x \in (0, 2)$, thus parallel to the x-axis (see Fig. 5.18), the volume of a solid cylindrical body is

$$V_x = \pi\int_0^2 y^2\,dx = \pi\int_0^2 2^2\,dx = 4\pi\,x\,\Big|_0^2 = 8\pi$$

which is a simple cylinder, radius $r = 2$, and height $h = 2 - 0 = 2$.

Fig. 5.18 Example P.5.95



5.96. Rotation of a linear function $y = 2x$ creates a cone, whose volume is then

$$V_x = \pi \int_1^3 y^2\,dx = \pi \int_1^3 (2x)^2\,dx = 4\pi\ x^3\Big|_1^3 = 4\frac{\pi}{3}(27-1) = \frac{104\pi}{3}$$

5.97. Rotation of a function $y = 4x - x^2$ creates a solid body, whose volume is then bound by an interval defined by points at the x-axis,

$$4x - x^2 = x(4-x) = 0 \quad \therefore \quad x_1 = 0, \quad x_2 = 4$$

$$V_x = \pi \int_0^4 y^2\,dx = \pi \int_0^4 (4x - x^2)^2\,dx = \pi \int_0^4 (x^4 - 8x^3 + 16x^2)\,dx$$
$$= \pi \left(\frac{1}{5}\,x^5\Big|_0^4 - 2\,x^4\Big|_0^4 + \frac{16}{3}\,x^3\Big|_0^4 \right) = \pi \left(\frac{1024}{5} - 512 + \frac{1024}{3} \right) = \frac{512\pi}{15}$$

5.98. Rotation of a function $y = 1/x$ creates a solid body. However, its volume must be calculated by an improper integral as

$$V_x = \pi \int_1^\infty y^2\,dx = \pi \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx = \pi \lim_{a\to\infty} \left(-\frac{1}{x}\right)\Big|_1^a = \pi \lim_{a\to\infty} \left(\cancelto{0}{-\frac{1}{a}} - (-1) \right) = \pi$$

5.99. Solid body may be created by rotation around any axis, here around y-axis. Rotation of a function $y = \ln x$ around the y-axis creates a solid body, whose volume is bound by

$$y = \ln x \quad \therefore \quad x = e^y \quad \therefore \quad V_y = \pi \int_0^1 x^2\,dy = \pi \int_0^1 \left(e^x\right)^2\,dy = \frac{\pi}{2}\,e^{2x}\Big|_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$$

5.100. Rotation of function $y = \sin x$ creates a solid, whose volume is then

$$V_x = \pi \int_0^{\pi} y^2\, dx = \pi \int_0^{\pi} \sin^2 x\, dx \quad \{\text{see A.5.18}\} = \frac{\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi}$$
$$= \frac{\pi}{2}\left(\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right) = \frac{\pi^2}{2}$$

5.9 Line Integral

5.101. Given circle equation $x^2 + y^2 = r^2$ may be written in its parametric form as

$$x(\theta) = r\,\cos\theta \quad \therefore \quad x'(\theta) = \frac{dx(\theta)}{d\theta} = -r\,\sin\theta$$
$$y(\theta) = r\,\sin\theta \quad \therefore \quad y'(\theta) = \frac{dy(\theta)}{d\theta} = r\,\cos\theta$$

Therefore, by definition of line length integral in parametric form, $\big(r \geq 0,\ \theta \in (0, 2\pi)\big)$

$$s = \int_0^{2\pi} \sqrt{x'^2 + y'^2}\, d\theta = \int_0^{2\pi} \sqrt{\big(-r\,\sin\theta\big)^2 + \big(r\,\cos\theta\big)^2}\, d\theta = r\int_0^{2\pi} \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$
$$= r\,\theta\,\Big|_0^{2\pi} = 2r\pi$$

5.102. Given function $y = f(x) = 4 - x^2$, it is bound by $x \in (-2, 2)$ interval. This function is even, so it is sufficient to calculate the half-length (e.g., $x \in (0, 2)$) and then multiply by two. Thus, by definition of line length integral, it follows that

$$\frac{s}{2} = \int_0^2 \sqrt{1 + y'^2}\, dx = \int_0^2 \sqrt{1 + \big(-2x\big)^2}\, dx = \int_0^2 \sqrt{1 + 4x^2}\, dx = 2\int_0^2 \sqrt{(1/2)^2 + x^2}\, dx$$
$$= \{\text{see A.5.55}\}$$
$$= \left(\frac{x}{2}\sqrt{(1/2)^2 + x^2} + \frac{(1/2)^2}{2}\ln\left|\frac{\sqrt{(1/2)^2 + x^2}}{1/2} + \frac{x}{1/2}\right|\right)\Bigg|_0^2$$
$$= \left(\frac{x}{2}\,\frac{\sqrt{1 + 4x^2}}{2} + \frac{1}{8}\ln\left|\sqrt{1 + 4x^2} + 2x\right|\right)\Bigg|_0^2 = \frac{\sqrt{17}}{2} + \frac{1}{8}\ln\left|\sqrt{17} + 4\right| - 0$$
$$\therefore \quad \underline{s = \sqrt{17} + \frac{1}{4}\ln\left|\sqrt{17} + 4\right|}$$

5.103. Given function may be written as

$$f(x) = y = x^2 + 2x = x(x + 2) \quad \therefore \quad x \in (-2, 0) \quad \text{and} \quad y' = 2x + 2$$

so that

$$s = \int_{-2}^{0} \sqrt{1+y'^2}\,dx = \int_{-2}^{0} \sqrt{1+(2x+2)^2}\,dx = \left\{ \begin{array}{l} t = 2x+2 \;\;\therefore\;\; dt = 2\,dx \;\;\therefore\;\; dx = \frac{1}{2}\,dt \\ x \to -2 \Rightarrow t \to -2; \quad \text{and} \quad x \to 0 \Rightarrow t \to 2 \end{array} \right\}$$

$$= \frac{1}{2}\int_{-2}^{2} \sqrt{1+t^2}\,dt = \int_{0}^{2} \sqrt{1+t^2}\,dt = \{\text{see A.5.55}\} = \left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left|\sqrt{1+t^2}+t\right|\right)\Big|_0^2$$

$$= \sqrt{5} + \frac{1}{2}\ln\left|\sqrt{5}+2\right|$$

5.104. Given function may be written as

$$\sqrt{x}+\sqrt{y}=1 \;\;\therefore\;\; f(x) = y = \sqrt{1-\sqrt{x}} \;\;\therefore\;\; x \in (0,1) \;\;\therefore\;\; y' = \frac{\sqrt{x}-1}{\sqrt{x}}$$

so that

$$s = \int_0^1 \sqrt{1+y'^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{\sqrt{x}-1}{\sqrt{x}}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{x-2\sqrt{x}+1}{x}\right)}\,dx$$

$$= \int_0^1 \sqrt{\frac{2\left(x-\sqrt{x}+\frac{1}{2}\right)}{x}}\,dx = \sqrt{2}\int_0^1 \sqrt{\frac{\left(x-\sqrt{x}+\frac{1}{4}\right)-\frac{1}{4}+\frac{1}{2}}{x}}\,dx$$

$$= \sqrt{2}\int_0^1 \frac{\sqrt{\left(\sqrt{x}-\frac{1}{2}\right)^2+\frac{1}{4}}}{\sqrt{x}}\,dx$$

Variable substitution may be as

$$\sqrt{x}-\frac{1}{2} = \frac{t}{2} \;\;\therefore\;\; \frac{dx}{\sqrt{x}} = dt \;\;\therefore\;\; x \to 0 \Rightarrow t \to -1 \quad \text{and} \quad x \to 1 \Rightarrow t \to 1$$

which further leads to

$$s = \sqrt{2}\int_0^1 \sqrt{\left(\sqrt{x}-\frac{1}{2}\right)^2+\frac{1}{4}}\,\frac{dx}{\sqrt{x}} = \sqrt{2}\int_{-1}^1 \sqrt{\left(\frac{t}{2}\right)^2+\frac{1}{4}}\,dt = \frac{\sqrt{2}}{2}\int_{-1}^1 \sqrt{t^2+1}\,dt$$

$$= \left\{\text{see A.5.55}\right\} = \frac{\sqrt{2}}{2}\left(t\sqrt{1+t^2} + \ln\left|\sqrt{1+t^2}+t\right|\right)\Big|_0^1$$

$$= \frac{\sqrt{2}}{2}\left(\sqrt{2}+\ln\left|\sqrt{2}+1\right|\right)$$

5.105. Given function

$$f(y) = x = \frac{1}{4}y^2 - \frac{1}{2}\ln y \;\;\therefore\;\; x' = \frac{y^2-1}{2y}$$

Line integral can be calculated for y variable as well, as

$$s = \int_1^e \sqrt{1 + x'^2}\, dy = \int_1^e \sqrt{1 + \left(\frac{y^2-1}{2y}\right)^2}\, dy = \int_1^e \sqrt{\frac{(y^2+1)^2}{4y^2}}\, dy = \int_1^e \frac{y^2+1}{2y}\, dy$$
$$= \frac{1}{2}\int_1^e y\, dy + \frac{1}{2}\int_1^e \frac{1}{y}\, dy = \frac{1}{4}\, y^2 \Big|_1^e + \frac{1}{2}\, \ln y \Big|_1^e = \frac{1}{4}(e^2+1)$$

5.10 Mean of a Function

5.106. The average of a periodic function as calculated over one period is

$$\langle \sin x \rangle = \frac{1}{2\pi}\int_0^{2\pi} \sin x\, dx = \frac{1}{2\pi}\, (-\cos x) \Big|_0^{2\pi} = \frac{1}{2\pi}(-(-1)-1) = 0$$

To interpret the result that $\langle \sin x \rangle = 0, \quad x \in [0, 2\pi]$, as well as $\langle \cos x \rangle = 0, \quad x \in [0, 2\pi]$ (see Fig. 5.6), note that within one period of sin/cos functions, positive and negative areas are perfectly matched, thus cancelled to zero.

5.107. The average value of sin/cos functions over an interval *that is equal to a non-integer number of function's periods*, as in this example, is

$$\langle \cos x \rangle = \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} \cos x\, dx = \frac{1}{\pi}\, \sin x \Big|_{\pi/2}^{3\pi/2} = \frac{1}{\pi}(-1-1) = -\frac{2}{\pi}$$

that is to say, not necessarily equal to zero.

5.108. The average value over one period of a sinusoid function ("AC") that is added to a constant ("DC") is found as follows.

$$\langle f(x) \rangle = \frac{1}{2\pi}\int_0^{2\pi} (1 + 0.75\, \cos x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} dx + \frac{1}{2\pi}\int_0^{2\pi} 0.75\, \cos x\, dx \;\to\; 0$$
$$= \frac{1}{2\pi}\, x \Big|_0^{2\pi} = \frac{1}{2\pi}(2\pi - 0) = 1$$

This is an important general property to notice, and graphical interpretation is shown in Fig. 5.19.

5.109. Integral that involves the square of a sinusoid function is solved by the change of variables technique in addition to trigonometric identity transformation as follows.

$$\langle f(x) \rangle = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 x\, dx = \left\{\cos^2 x = \frac{1}{2}(1 + \cos(2x))\right\}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}(1 + \cos(2x))\, dx = \frac{1}{4\pi}\, x \Big|_0^{2\pi} + \frac{1}{4\pi}\int_0^{2\pi} \cos(2x)\, dx$$

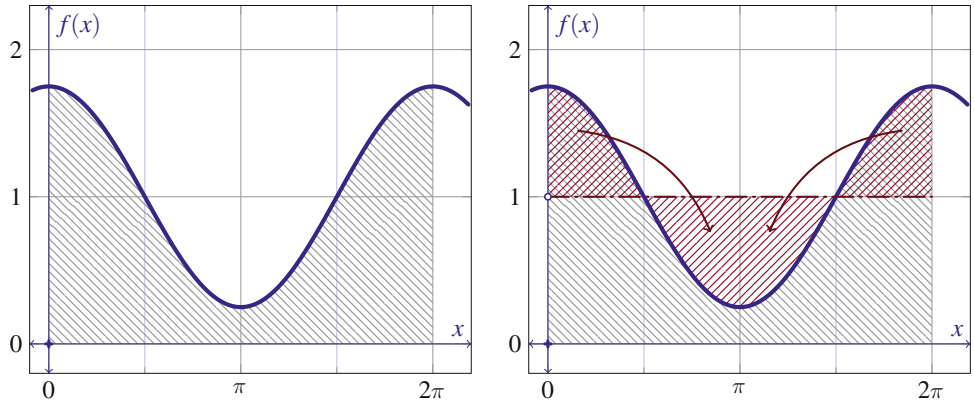


Fig. 5.19 Example P.5.108

$$
\begin{aligned}
&= \left\{ \begin{array}{l} t = 2x \quad \therefore \quad dt = 2\,dx \quad \therefore \quad dx = \dfrac{dt}{2} \\[2ex] x = 0 \quad \therefore \quad t = 0 \quad \text{and} \quad x = 2\pi \quad \therefore \quad t = 4\pi \end{array} \right\} \\
&= \frac{1}{4\pi}(2\pi - 0) + \frac{1}{8\pi}\int_0^{4\pi} \cos t \; dt = \frac{1}{2} + \frac{1}{8\pi}\big(\cancelto{0}{\sin(4\pi)} - \cancelto{0}{\sin(0)} \big) \\
&= \frac{1}{2}
\end{aligned}
$$

5.11 Improper Integrals

5.110. Given open-ended (improper) integral

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

it is not always obvious if this area is going to converge or diverge; see Fig. 5.20. One possible method to resolve improper integrals is to first calculate the area bound by finite parameter a and then to calculate the limiting value when the parameter tends to infinity, as

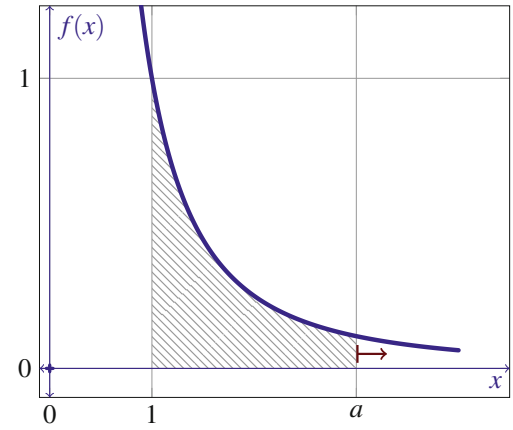
$$\lim_{a\to\infty} \int_1^a \frac{1}{x^2}\, dx = \lim_{a\to\infty} \left(-\frac{1}{x} \Big|_1^a \right) = \lim_{a\to\infty} \Big(\cancelto{0}{-\frac{1}{a}} - \Big(-\frac{1}{1}\Big) \Big) = \big(0 - (-1)\big) = 1$$

5.111. Given improper integral

$$\int_1^{\infty} \frac{1}{x}\, dx$$

after declaration of non-infinite parameter a, it follows that

Fig. 5.20 Example P.5.110



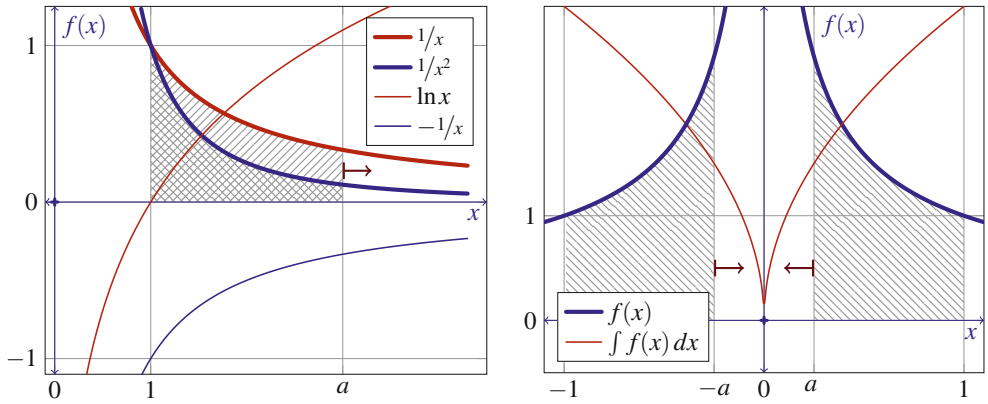


Fig. 5.21 Examples P.5.111 and P.5.112

$$\lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty}\Big(\ln|x|\Big|_1^a\Big) = \lim_{a\to\infty}(\ln a - \ln 1) = \infty - 0 = \infty$$

which is to say that this integral is divergent. Note the difference between functions $1/x$ and $1/x^2$ as illustrated in Fig. 5.21 (left). Although, by inspection, both areas seem to converge, it is clearly visible that their respective integrals behave very differently: $\ln x$ is divergent (i.e., tends to infinity) and $(-1/x)$ is convergent (i.e., tends to zero) for $x \to \infty$.

5.112. Given improper integral of a non-continuous function at $x = 0$ as

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx$$

it is resolved by two-sided limits tending to zero, as

$$\lim_{a\to 0\atop\to}\int_{-1}^{a}\frac{1}{\sqrt[3]{x^2}}\,dx+\lim_{a\to 0\atop\leftarrow}\int_{a}^{1}\frac{1}{\sqrt[3]{x^2}}\,dx=\lim_{a\to 0\atop\to}\int_{-1}^{a}x^{-2/3}\,dx+\lim_{a\to 0\atop\leftarrow}\int_{a}^{1}x^{-2/3}\,dx$$

$$=3\lim_{a\to 0\atop\to}\sqrt[3]{x}\,\Big|_{-1}^{a}+3\lim_{a\to 0\atop\leftarrow}\sqrt[3]{x}\,\Big|_{a}^{1}=3\lim_{a\to 0\atop\to}\left(\sqrt[3]{a}-\sqrt[3]{-1}\right)+3\lim_{a\to 0\atop\leftarrow}\left(\sqrt[3]{1}-\sqrt[3]{a}\right)$$

$$=3\left(\sqrt[3]{0}+1\right)+3\left(1-\sqrt[3]{0}\right)=6$$

In this case, even though there is vertical asymptote within the given interval $x\in[-1,1]$, the area converges to 3 on each side, Fig. 5.21 (right).

5.113. Given two-sided improper integral (see A.5.22),

$$\int_{-\infty}^{\infty}xe^{-x^2}\,dx$$

then the two-sided limit is

$$\lim_{a\to\infty}\int_{-a}^{a}xe^{-x^2}\,dx=\lim_{a\to\infty}\;-\frac{1}{2}e^{-x^2}\,\Big|_{-a}^{a}=-\frac{1}{2}\lim_{a\to\infty}\left(e^{-a^2}-e^{-(-a)^2}\right)\quad=-\frac{1}{2}(0-0)=0$$

which is to say that signed left-side and right-side areas are equal but opposite sign, ($\pm 1/2$).

5.114. Given one-sided improper integral

$$\int_{-\infty}^{0}\frac{1}{\sqrt{1-x}}\,dx$$

it is resolved as follows:

$$\lim_{a\to-\infty}\int_{a}^{0}\frac{1}{\sqrt{1-x}}\,dx=-2\lim_{a\to-\infty}\sqrt{1-x}\,\Big|_{a}^{0}=-2\lim_{a\to-\infty}\left(\sqrt{1-0}-\sqrt{1-a}\right)=-2(1-\infty)=\infty$$

which is to say that even though, by a quick look inspection, it may appear that the area under $f(x)$ converges, its integral diverges.

5.115. Given one-sided improper integral

$$\int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

it is resolved as

$$\lim_{a\to\infty}\int_{0}^{a}\frac{1}{1+x^2},\,dx=\lim_{a\to\infty}\arctan x\,\Big|_{0}^{a}=\lim_{a\to\infty}(\arctan a-\arctan 0)=\frac{\pi}{2}-0=\frac{\pi}{2}$$

5.116. Given one-sided improper integral, it is resolved as

$$\int_e^{\infty} \frac{1}{x \ln^3 x}\,dx = \lim_{a\to\infty}\int_e^a \frac{1}{x\ln^3 x}\,dx = \left\{ t = \ln x \quad \therefore \quad dt = \frac{dx}{x} \right\} = \lim_{a\to\infty}\int \frac{1}{t^3}\,dt$$

$$= -\frac{1}{2}\lim_{a\to\infty}\frac{1}{t^2}\,dt$$

$$= -\frac{1}{2}\lim_{a\to\infty}\frac{1}{\ln^2 x}\Bigg|_e^a$$

$$= -\frac{1}{2}\lim_{a\to\infty}\left(\frac{1}{\ln^2 a}^{\,0} - \frac{1}{\ln^2 e}\right)$$

$$= \frac{1}{2}$$

5.117. Knowing that definite integrals give the total *signed* area within the given interval, sometimes, the correct result can be intuitively deduced. Given improper integral

$$\int_{-1}^{1} \frac{1}{x}\,dx$$

note that the inverse function has vertical asymptote at $x = 0$; thus, it is logical to calculate limit $a \to 0$ after applying the Newton-Leibniz formula. That is to say,

$$\int_1^{-1} \frac{1}{x}\,dx = \lim_{a\to 0\atop\to}\int_{-1}^a \frac{1}{x}\,dx + \lim_{a\to 0\atop\leftarrow}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0\atop\to} \ln|a|\Big|_{-1}^a + \lim_{a\to 0\atop\leftarrow} \ln|a|\Big|_a^1$$

$$= \lim_{a\to 0}\left(\ln|a| - \ln|-1|^{\,0} + \ln|1|^{\,0} - \ln|a|\right) = -\infty - \infty \quad \therefore$$

(divergent, incorrect conclusion)

Mathematical software that blindly implements the Newton-Leibniz formula reports this result. However, the inverse function is odd, that is to say, symmetric; see Fig. 5.22 (left). The left- and right-side areas are equal; thus, one should conclude that the total sum must be zero. Indeed, the evaluation of undermined result should be as

$$\int_1^{-1} \frac{1}{x}\,dx = \lim_{a\to 0\atop\to}\int_{-1}^a \frac{1}{x}\,dx = \lim_{a\to 0}(\ln|a| - \ln|-1|^{\,0} + \ln|1|^{\,0} - \ln|a|)$$

$$= \lim_{a\to 0} \ln\frac{|a|}{|a|} = \lim_{a\to 0} \ln 1 = 0 \quad \therefore \quad \text{(correct)}$$

5.118. Given improper integral of a periodic function

$$\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx$$

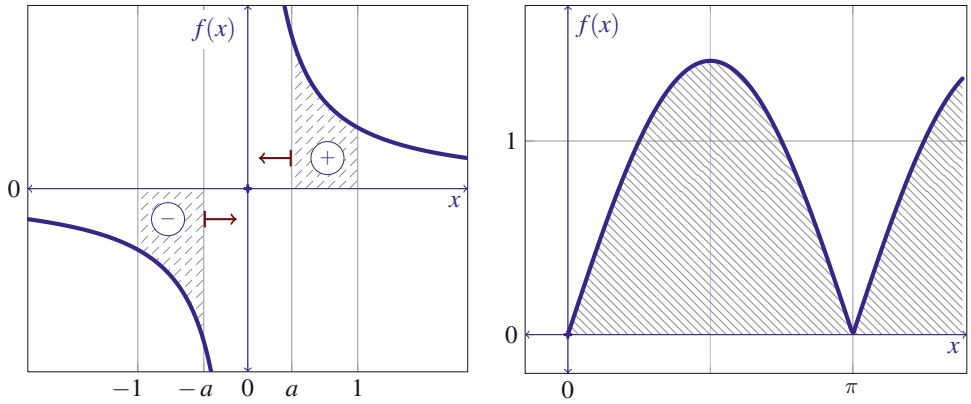


Fig. 5.22 Examples P.5.117 and P.5.118

it should be evident this integral must equal a non-zero positive value; see Fig. 5.22 (right). However, direct and blind implementation of the Newton-Leibniz formula may lead to

$$\begin{aligned}\int_0^{100\pi} \sqrt{1-\cos 2x}\, dx &= \left\{1-\cos x = 2\sin^2\frac{x}{2} \;\;\therefore\;\; 1-\cos 2x = 2\sin^2 x\right\}\\ &= \sqrt{2}\int_0^{100\pi} |\sin x|\, dx = \sqrt{2}\left|\int_0^{100\pi} \sin x\, dx\right|\\ &= \sqrt{2}\left|\cos x\Big|_0^{100\pi}\right| = \sqrt{2}\,\big|\cos(50\times 2\pi)-\cos 0\big|\\ &= \sqrt{2}\,|1-1| = 0\end{aligned}$$

However, by inspection of Fig. 5.22 (right), the area under $f(x)$ is obviously non-zero and positive (i.e., $|\sin x| \geq 0$), and $f(x)$ is periodic with the period $T = \pi$. Instead, calculation over one period gives

$$\begin{aligned}\int_0^{\pi} \sqrt{1-\cos 2x}\, dx &= \sqrt{2}\left|\int_0^{\pi} \sin x\, dx\right| = \sqrt{2}\left|\cos x\Big|_0^{\pi}\right| = \sqrt{2}\,\big|\cos(\pi)-\cos 0\,\big|\\ &= \sqrt{2}\,|-1-1| = 2\sqrt{2}\end{aligned}$$

Therefore, given hundred periods, the total area within interval $x \in [0, 100\pi]$ equals 200π.

Differential Equations

6

Systematical study of differential equations and properties of their solutions is done in accordance to their forms and techniques for solving. An introductory list of ordinary differential equation forms and terms may be as follows:

First-order differential equations take the form of

$$f(x, y, y') = 0 \quad \text{where} \quad y' \equiv \frac{dy}{dx}$$

where only the unknown function $y(x)$ and its first derivative $y'(x)$ are found in the equation. Furthermore, note two most common notations for differentiation. Leibniz's notation is explicit, that is to say, it shows explicitly the relation between the unknown function, its corresponding variable, and the order of differentiation as

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^{(iv)}y}{dx^{(iv)}}, \ldots \text{etc.}$$

while Newton's notation is implicit, i.e., it does not explicitly state the corresponding variable. Instead, the derivatives are written as y, y', y'', y''', $y^{(iv)}$, etc.

Common types of first-order equations are:

1. "Separable variables" equations take the form of

$$y' = \frac{f(x)}{g(x)}$$

2. "Homogeneous" equations take the form of

$$y' = f\left(\frac{y}{x}\right)$$

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R. Sobot, *Engineering Mathematics by Example*,
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3. "With function coefficients" equations take the form of

$$y' + f(x)\, y = g(x)$$

Homogeneous linear differential equations with the constant coefficients take the form of

$$\begin{aligned} a_1 y' + a_0 y &= 0 \quad (1^{st} \text{ order}) \\ a_2 y'' + a_1 y' + a_0 y &= 0 \quad (2^{nd} \text{ order}) \\ a_3 y''' + a_2 y'' + a_1 y' + a_0 y &= 0 \quad (3^{rd} \text{ order}) \\ &\dots \end{aligned}$$

Problems

6.1 First Order, Separable Variables

Reminder: The separation of variables technique is used to solve differential equations whose form can be written as

$$y' \equiv \frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \therefore \quad g(y)\, dy = f(x)\, dx \quad \Rightarrow \quad \int g(y)\, dy = \int f(x)\, dx$$

Therefore, the solution $y(x)$ is found by direct integration.

Derive general solutions of equations in P.6.1 to P.6.12. In the cases where the initial conditions $y(x_0) = y_0$ are given, calculate the value of the associated integration constant c.

6.1. $y' = a$

6.2. $y'' = a$

6.3. $y''' = a$

6.4. $y' = 2x,\ y(0) = 0$

6.5. $y' = y,\ y(1) = e$

6.6. $yy' + x = 0,\ y(2) = 0$

6.7. $y' + y = a$

6.8. $y - 2xy' = 1$

6.9. $y' + 2xy = x$

6.10. $\sqrt{x}\, dy = \sqrt{y}\, dx$

6.11. $x(1 + y^2) = yy'$

6.12. $y' \tan x - y = a$

6.2 First Order, Homogeneous

Reminder: The general form of "homogeneous" differential equations is

$$y' = f\left(\frac{y}{x}\right)$$

and is solved by the following change of variable technique:

$$\frac{y}{t} = t \;\; \therefore \;\; \underline{y = tx} \;\; = \left\{ \big(f(x)g(x)\big)' = f'g + fg' \right\}$$

$$\therefore$$

$$dy = t\,dx + x\,dt \;\; \Rightarrow \;\; \underline{y' = \frac{dy}{dx} = t + x\,\frac{dt}{dx}}$$

Derive general solutions of equations in P.6.13 to P.6.24. In the cases where the initial condition $y(x_0) = y_0$ is given, calculate the value of the associated integration constant c.

6.13. $y' = 1 + \dfrac{y}{x}$

6.14. $y' = \dfrac{y^2 + x^2}{xy}$

6.15. $y' = \dfrac{y^2 - x^2}{2xy}, \;\; y(1) = 1$

6.16. $(x + y)\,dx - x\,dy = 0$

6.17. $xy' - 2y = -x^2$

6.18. $y' = \dfrac{y}{x} \ln \dfrac{y}{x}$

6.19. $xy^2\,dy = (x^3 + y^3)\,dx$

6.20. $(x - 2y)\,dy = (x - y)\,dx, \;\; y(1) = 1$

6.21. $(x^2 - 2y^2)\,dx + 2xy\,dy = 0, \;\; y(3/2) = 0$

6.22. $y' - \dfrac{y}{x} = \tan \dfrac{y}{x}$

6.23. $\dfrac{y^2}{x^2} - \dfrac{y}{x} - y' + 1 = 0, \;\; y(e) = 0$

6.24. $y' = \dfrac{y + xe^{-y/x}}{x}$

6.3 First Order, Function Coefficients

Reminder: Equations in the form of

$$y' + f(x)\, y = g(x)$$

may be solved with the following steps:

1. first, the "integration factor" μ is calculated as

$$\mu = e^{\int f(x)\, dx}$$

2. then, it is assumed that

$$\frac{d}{dx}(y\,\mu) = \mu g(x)$$

3. so that the unknown function $y(x)$ is derived by integration.

$$\int d(y\,\mu) = \int \mu g(x)\, dx \quad \therefore \quad y\,\mu = \int \mu g(x)\, dx \quad \therefore \quad y = \frac{1}{\mu} \int \mu g(x)\, dx$$

Derive general solutions of equations in P.6.25 to P.6.34.

6.25. $y' - y = 1$

6.26. $y' + xy = x$

6.27. $y' - 4xy = x$

6.28. $y' - 2y - 3 = 0$

6.29. $y' + 2xy = 2xe^{-x^2}$

6.30. $y' + y\cos x = \sin(2x)$

6.31. $y' - \dfrac{2}{x+1}\, y - \dfrac{4x}{1-x^2} = 0,\ y(0) = 2$

6.32. $y' - \dfrac{2y}{x+1} - (x+1)^3 = 0$

6.33. $y'(x^2 + x) - (y+1)(2x+1) = 0$

6.34. $y' + y\tan x = \dfrac{1}{\cos x},\ y(\pi) = 1$

6.4 Linear Equations, Constant Coefficients

Reminder: "Linear differential equations with constant coefficients" are in the form of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

where a_i are constants. In general, linear differential equations with constant coefficients are solved with the following steps:

1. Its "characteristic polynomial" is

$$P_n(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r^1 + a_0 r^0 = 0$$

   where the derivative orders in differential equation are replaced with the same-order monomial powers. Therefore, the problem of solving differential equation is replaced by the problem of calculating polynomial roots $P_n(r) = 0$, i.e., $(r_1, r_2, \ldots, r_n)$.
2. Each polynomial root r_i, real or complex, is converted into one exponential term as

$$c_i\, x^n\, e^{r_i x} \quad (n = 0, 1, 2 \ldots)$$

   where c_i is the integration constant and the power of x^n term depends on the multiplicity of the particular $P(r)$ root. For example, if the root is unique, then $n = 0$. If there are two identical roots (i.e., "double root"), then the first root's exponential term is multiplied by x^0, and the second root's exponential term by x^1. Similarly, if there are three identical roots, then the first root's exponential term is multiplied by x^0, the second root's exponential by x^1, and the third root's exponential by x^2. The higher-order multiplicities follow the same pattern. By doing so, the general solution to differential equation may be written in the form of sum of the exponential terms as

$$y(x) = \sum_i c_i\, x^n\, e^{r_i x} \quad (n = 0, 1, 2 \ldots)$$

Derive general solutions of equations in P.6.35 to P.6.46.

6.35. $y'' - 9y = 0$

6.36. $y'' + y' - 2y = 0$

6.37. $-8y'' + 2y' + y = 0$

6.38. $y'' - 2y' + y = 0$

6.39. $y'' + 4y' + 4y = 0$

6.40. $4y'' - 4y' + y = 0$

6.41. $y'' + 6y' + 25y = 0$

6.42. $y'' - 2y' + 2y = 0$

6.43. $y'' - 4y' + 5y = 0$

6.44. $y''' - 3y'' + 3y' - y = 0$

6.45. $y''' - 2y'' - y + 2 = 0$

6.46. $y^{(iv)} - 2y'' = 0$

Answers

6.1 First Order, Separable Variables

6.1. $y' = a \;\;\therefore\;\; \dfrac{dy}{dx} = a \;\;\therefore\;\; dy = a\,dx \;\;\therefore\;\; \displaystyle\int dy = a \int dx$

$$\therefore\;\; \underline{y(x) = ax + c}$$

Verification: $y = ax + c \;\;\therefore\;\; y' = \dfrac{dy}{dx} = \dfrac{d}{dx}(ax + c) = a$ ✓

6.2. $y'' = a \;\;\therefore\;\; \dfrac{d^2y}{dx^2} = a \;\;\therefore\;\; \dfrac{dy'}{dx} = a \;\;\therefore\;\; \displaystyle\int dy' = a \int dx \;\;\therefore\;\; y' = ax + c_1$

$$\therefore\;\; \frac{dy}{dx} = ax + c_1 \;\;\therefore\;\; \int dy = \int (ax + c_1)\,dx \;\;\therefore\;\; \underline{y(x) = \frac{1}{2}ax^2 + c_1x + c_2}$$

Verification: $y = \dfrac{1}{2}ax^2 + c_1x + c_2 \;\;\therefore\;\; y' = \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}ax^2 + c_1x + c_2\right) = \dfrac{2}{2}ax + c_1$

$$\therefore\;\; y'' = \frac{dy'}{dx} = \frac{d}{dx}(ax + c_1) = a \;\checkmark$$

6.3. $y''' = a \;\;\therefore\;\; \dfrac{d^3y}{dx^3} = a \;\;\therefore\;\; \dfrac{dy''}{dx} = a \;\;\therefore\;\; \displaystyle\int dy'' = a \int dx \;\;\therefore\;\; y'' = ax + c_1$

$$\therefore\;\; \frac{dy'}{dx} = ax + c_1 \;\;\therefore\;\; \int dy' = \int (ax + c_1)\,dx \;\;\therefore\;\; y' = \frac{1}{2}ax^2 + c_1x + c_2$$

$$\therefore\;\; \frac{dy}{dx} = \frac{1}{2}ax^2 + c_1x + c_2 \;\;\therefore\;\; \int dy = \left(\frac{1}{2}ax^2 + c_1x + c_2\right) dx$$

$$\therefore\;\; \underline{y(x) = \frac{1}{6}ax^3 + \frac{1}{2}c_1x^2 + c_2x + c_3}$$

Verification: $y = \dfrac{1}{6}ax^3 + \dfrac{1}{2}c_1x^2 + c_2x + c_3 \;\;\therefore$

$$y' = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{6}ax^3 + \frac{1}{2}c_1x^2 + c_2x + c_3\right) = \frac{3}{6}\frac{1}{2}ax^2 + \frac{2}{2}c_1x + c_2 \;\;\therefore$$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{1}{2}ax^2 + c_1x + c_2\right) = \frac{2}{2}ax + c_1 \;\;\therefore$$

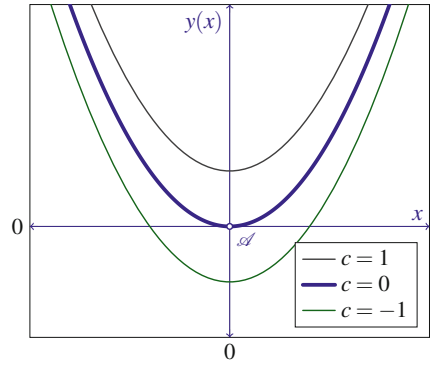
$$y''' = \frac{dy''}{dx} = \frac{d}{dx}(ax + c_1) = a \;\checkmark$$

6.4. $y' = \dfrac{dy}{dx} = 2x \;\;\therefore\;\; dy = 2x\,dx \;\;\therefore\;\; \displaystyle\int dy = 2\int x\,dx \;\;\therefore\;\; \underline{y(x) = x^2 + c}$

Verification:

$$y = x^2 + c \;\;\therefore\;\; y' = 2x \;\;\Rightarrow\;\; y' = 2x \;\;\therefore\;\; \cancel{2x = 2x} \;\;\therefore\;\; 0 = 0 \;\checkmark$$

Fig. 6.1 Example P.6.4



General solution is a family of curves, Fig. 6.1. Given the initial condition $\mathscr{A} : y(0) = 0$, its corresponding specific function is calculated as

$$\begin{aligned} y(x) &= x^2 + c \\ 0 &= (0)^2 + c \\ &\therefore \\ c = 0 \quad &\Rightarrow \quad \underline{y(x) = x^2} \end{aligned}$$

6.5. $y' = \dfrac{dy}{dx} = y \;\; \therefore \;\; \dfrac{dy}{y} = dx \;\; \therefore \;\; \displaystyle\int \frac{dy}{y} = \int dx \;\; \therefore \;\; \ln|y| = x + c_1$

$\therefore \;\; y = e^{x+c_1} \quad \{c_2 = e^{c_1}\}$

$\therefore \;\; \underline{y(x) = c_2\, e^x}$

Verification:

$$y = c_2\, e^x \;\; \therefore \;\; y' = c_2\, e^x \;\; \Rightarrow \;\; y' = y \;\; \therefore \;\; \cancel{c_2\, e^x} = \cancel{c_2\, e^x} \;\; \therefore \;\; 0 = 0 \;\; \checkmark$$

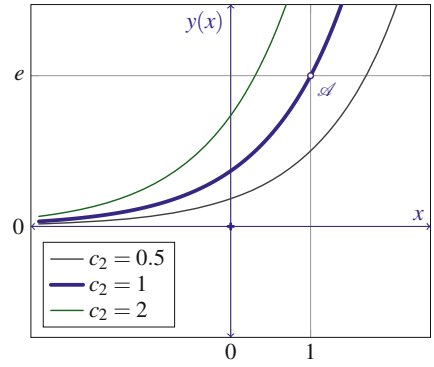
General solution is a family of curves, Fig. 6.2. Given the initial condition $\mathscr{A} : y(1) = e$, its corresponding specific function is calculated as

$$\begin{aligned} y(x) &= c_2\, e^x \\ e &= c_2\, e^1 \\ &\therefore \\ c_2 = 1 \quad &\Rightarrow \quad \underline{y(x) = e^x} \end{aligned}$$

6.6. $yy' + x = 0 \;\; \therefore \;\; y\dfrac{dy}{dx} = -x \;\; \therefore \;\; y\,dy = -x\,dx \;\; \therefore \;\; \displaystyle\int y\,dy = -\int x\,dx$

$$\therefore \;\; \frac{y^2}{\cancel{2}} = \frac{c^2}{\cancel{2}} - \frac{x^2}{\cancel{2}} \;\; \therefore \;\; \underline{y(x) = \pm\sqrt{c^2 - x^2}}$$

Fig. 6.2 Example P.6.5



which is an equation of a circle whose radius equals c; see Fig. 6.3. It is converted from Cartesian to polar coordinates as

$$r^2 \cos^2\theta + r^2 \sin^2\theta = c^2$$

$$r = \sqrt{\frac{c^2}{\cos^2\theta + \sin^2\theta}} \quad \{\cos^2\theta + \sin^2\theta = 1\}$$

$$r = c \quad \text{(i.e., radius } r \text{ is constant)}$$

Verification:

$$y = \sqrt{c^2 - x^2} \quad \therefore \quad y' = \frac{-2x}{2\sqrt{c^2 - x^2}}$$

$$\therefore$$

$$yy' + x = 0$$

$$\frac{\sqrt{c^2 - x^2}}{} \frac{-2x}{2\sqrt{c^2 - x^2}} + x = 0$$

$$0 = 0 \quad \checkmark$$

General solution is a family of curves, Fig. 6.3. Given the initial condition $\mathscr{A} : y(2) = 0$, its corresponding specific function is calculated as

$$y(x) = \pm\sqrt{c^2 - x^2}$$

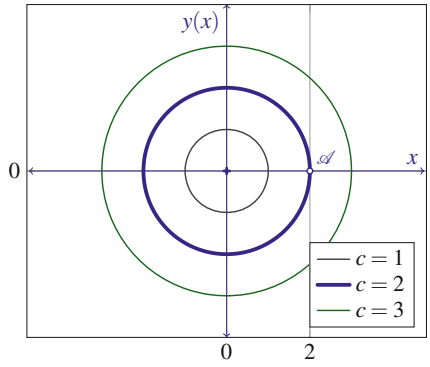
$$0 = \pm\sqrt{c^2 - 2^2}$$

$$0 = c^2 - 4 \quad \therefore \quad \sqrt{c^2} = \sqrt{4}$$

$$\therefore$$

$$|c| = 2 \quad \Rightarrow \quad \underline{y(x) = \pm\sqrt{4 - x^2}}$$

Fig. 6.3 Example P.6.6



6.7. $y' + y = a \;\;\therefore\;\; y' = \dfrac{dy}{dx} = a - y \;\;\therefore\;\; \dfrac{dy}{a-y} = dx \;\;\therefore\;\; \dfrac{dy}{y-a} = -\,dx$

$$\therefore\;\; \int \frac{dy}{y-a} = -\int dx$$

$$= \{y - a = t \;\;\therefore\;\; dy = dt\}$$

$$\therefore\;\; \int \frac{dt}{t} = -x + c_1 \;\;\therefore\;\; \ln|t| = -x + c_1 \;\;\therefore\;\; \ln|y-a| = -x + c_1$$

$$\therefore\;\; |y-a| = e^{-x+c_1} = \{e^{\pm x} > 0 \;\;\therefore\;\; |y-a| > 0 \;\;\therefore\;\; |y-a| = y - a\}$$

$$y - a = e^{-x}\, e^{c_1} = e^{-x}\, e^{c_1} = \{c = e^{c_1}\} \;\;\therefore\;\; \underline{y(x) = a + ce^{-x}}$$

Verification: $y = a + ce^{-x} \;\;\therefore\;\; y' = \dfrac{dy}{dx} = \dfrac{d}{dx}\left(a + ce^{-x}\right) = -ce^{-x}$

Therefore,

$$y' + y = a \;\;\therefore\;\; \cancel{-ce^{-x}} + a + \cancel{ce^{-x}} = a \;\;\therefore\;\; a = a \;\;\checkmark$$

6.8. $y - 2xy' = 1 \;\;\therefore\;\; y = 2xy' + 1 \;\;\therefore\;\; y - 1 = 2xy' = 2x\dfrac{dy}{dx} \;\;\therefore\;\; \dfrac{dy}{y-1} = \dfrac{1}{2}\dfrac{dx}{x}$

$$\therefore$$

$$\int \frac{dy}{y-1} = \frac{1}{2}\int \frac{dx}{x} = \{y - 1 = t \;\;\therefore\;\; dy = dt\}$$

$$\therefore\;\; \ln|y-1| = \frac{1}{2}\ln|x| + C \;\;\{C = \ln c_1\}$$

$$\therefore\;\; \ln|y-1| = \frac{1}{2}\ln|x| + \ln c_1 = \frac{1}{2}\ln(c_1|x|) = \ln(c_1|x|)^{1/2}$$

$$\therefore\;\; \{e^{\ln x} = x \;\;\text{and}\;\; e^{\pm x} > 0 \;\;\therefore\;\; x > 0\}$$

$$y - 1 = \sqrt{c_1|x|} = \sqrt{c_1}\sqrt{|x|} = c_2\sqrt{x} \;\;\therefore\;\; \underline{y(x) = 1 + c_2\sqrt{x}}$$

Verification: $y = 1 + c_2\sqrt{x} \;\;\therefore\;\; y' = \dfrac{dy}{dx} = \dfrac{d}{dx}\left(1 + c_2\sqrt{x}\right) = \dfrac{c_2}{2}\sqrt{x}$

Therefore,

$$y - 2xy' = 1 \quad \therefore \quad 1 + c_2\sqrt{x} - 2\,\frac{c_2}{2}\sqrt{x} = 1 \quad \therefore \quad 1 + c_2\sqrt{x} - c_2\sqrt{x} = 1 \quad \therefore \quad 0 = 0 \ \checkmark$$

6.9. $y' + 2xy = x \quad \therefore \quad \dfrac{dy}{dx} = x - 2xy \quad \therefore \quad \dfrac{dy}{dx} = x(1-2y) \quad \therefore \quad \dfrac{dy}{1-2y} = x\,dx$

$$\therefore \quad \int \frac{dy}{1-2y} = \int x\,dx = \left\{1 - 2y = t \quad \therefore \quad -2\,dy = dt \quad \therefore \quad dy = -\frac{1}{2}\,dt\right\}$$

$$-\frac{1}{2}\int \frac{dt}{t} = \frac{x^2}{2} - \frac{c_1}{2} \quad \therefore \quad \ln|1-2y| = c_1 - x^2 \quad \therefore \quad 1 - 2y = e^{c_1 - x^2}$$

$$\therefore \quad \underline{y(x) = \frac{1}{2}\left(c_2\,e^{-x^2} + 1\right)}$$

Verification: $y = \dfrac{1}{2}\left(c_2\,e^{-x^2} + 1\right) \quad \therefore \quad y' = -c_2\,x\,e^{-x^2}$

Therefore,

$$y' + 2xy = x \quad \therefore \quad -c_2\,x\,e^{-x^2} + 2x\,\frac{1}{2}\,c_2\,e^{-x^2} + 2x\,\frac{1}{2} = x \quad \therefore \quad 0 = 0 \ \checkmark$$

6.10. $\sqrt{x}\,dy = \sqrt{y}\,dx \quad \therefore \quad \displaystyle\int \frac{dy}{\sqrt{y}} = \int \frac{dx}{\sqrt{x}} \quad \therefore \quad 2\sqrt{y} = 2\sqrt{x} + C \quad \therefore \quad \sqrt{y} = \sqrt{x} + c_1$

$$\underline{y(x) = (\sqrt{x} + c_1)^2 = x + 2c_1\sqrt{x} + c_2}$$

Verification: $y = x + 2c_1\sqrt{x} + c_2 \quad \therefore \quad y' = \dfrac{dy}{dx} = \dfrac{d}{dx}\left(x + 2c_1\sqrt{x} + c_2\right) = 1 + \dfrac{2c_1}{2\sqrt{x}}$

Therefore,

$$\sqrt{x}\,dy = \sqrt{y}\,dx \quad \therefore \quad y' = \frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x}} \quad \therefore \quad 1 + \frac{2c_1}{2\sqrt{x}} = \frac{\sqrt{(\sqrt{x} + c_1)^2}}{\sqrt{x}}$$

$$1 + \frac{c_1}{\sqrt{x}} = \frac{\sqrt{x} + c_1}{\sqrt{x}} \quad \therefore \quad 1 + \frac{c_1}{\sqrt{x}} = 1 + \frac{c_1}{\sqrt{x}} \quad \therefore \quad 0 = 0 \ \checkmark$$

6.11. $x(1+y^2) = yy' \quad \therefore \quad x = \dfrac{y}{1+y^2}\,\dfrac{dy}{dx}$

$$\therefore \quad \int x\,dx = \int \frac{y}{1+y^2}\,dy \quad \left\{1 + y^2 = t \quad \therefore \quad 2y\,dy = dt\right\}$$

$$\frac{1}{2}\int \frac{dt}{t} = \int x\,dx \quad \therefore \quad \frac{1}{2}\ln(1+y^2) = \frac{x^2}{2} + \frac{c_1}{2} \quad \therefore$$

$$1 + y^2 = e^{x^2 + c_1} = c_2\,e^{x^2}$$

$$\underline{y(x) = \sqrt{c_2\,e^{x^2} - 1}}$$

Verification: $y = \sqrt{c_2\, e^{x^2} - 1} \;\;\therefore\;\; y' = \dfrac{2 c_2\, x\, e^{x^2}}{2\sqrt{c_2\, e^{x^2} - 1}}$

Therefore,

$$x(1+y^2) = yy' \;\;\therefore\;\; x\left(1 + \left(\sqrt{c_2\, e^{x^2} - 1}\right)^2\right) = \left(\sqrt{c_2\, e^{x^2} - 1}\right) \frac{c_2\, x\, e^{x^2}}{\sqrt{c_2\, e^{x^2} - 1}}$$

$$\therefore\;\; x\left(1 + c_2\, e^{x^2} - 1\right) = c_2\, x\, e^{x^2} \;\;\therefore\;\; c_2\, x\, e^{x^2} = c_2\, x\, e^{x^2} \;\;\therefore\;\; 0 = 0 \;\checkmark$$

6.12. $y' \tan x - y = a \;\;\therefore\;\; \dfrac{dy}{dx}\tan x = a + y \;\;\therefore\;\; \dfrac{dy}{a+y} = \dfrac{1}{\tan x}\, dx = \dfrac{\cos x}{\sin x}\, dx$

$$\therefore\;\; \int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\, dx$$

$$= \Big\{ a + y = t \;\;\therefore\;\; dy = dt, \quad \text{and} \quad \sin x = r \;\;\therefore\;\; \cos x\, dx = dr \Big\}$$

$$\ln|a+y| = \ln|\sin x| + \ln c_1 = \ln|c_1 \sin x| \;\;\therefore\;\; \underline{y(x) = c_1 \sin x - a}$$

Verification: $y = c_1 \sin x - a \;\;\therefore\;\; y' = c_1 \cos x$

Therefore,

$$y' \tan x - y = a \;\;\therefore\;\; c_1 \cos x \frac{\sin x}{\cos x} - c_1 \sin x + a = a \;\;\therefore\;\; 0 = 0 \;\checkmark$$

6.2 First Order, Homogeneous

6.13. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$\left.\begin{aligned} y' = 1 + \frac{y}{x} \quad &\therefore\;\; y' = \frac{dy}{dx} = 1 + \frac{y}{x} = \Big\{ t = \frac{y}{x} \Big\} = 1 + t \\ y = x\, t \quad &\therefore\;\; y' = \frac{dy}{dx} = \big\{ (fg)' = f'g + fg' \big\} = x't + xt' = t + x\,\frac{dt}{dx} \end{aligned}\right\} \Rightarrow$$

$$1 + x\,\frac{dt}{dx} = 1 + 1 \;\;\therefore\;\; \int dt = \int \frac{dx}{x} \;\;\therefore\;\; t = \ln x + \ln c \;\;\therefore\;\; \frac{y}{x} = \ln cx \;\;\therefore\;\; \underline{y(x) = x\, \ln cx}$$

Verification: $y = x\, \ln cx \;\;\therefore\;\; y' = \ln cx + x\, 1\, \dfrac{c}{cx} = \ln cx + 1$

Therefore,

$$y' = 1 + \frac{y}{x} \;\;\therefore\;\; \ln cx + 1 = 1 + \frac{x \ln cx}{x} \;\;\therefore\;\; 0 = 0 \;\checkmark$$

6.14. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$\left.\begin{aligned} y' = \frac{x^2+y^2}{xy} \quad &\therefore\ y' = \frac{dy}{dx} = \frac{y}{x} + \frac{x}{y} = \left\{t = \frac{y}{x}\right\} = t + \frac{1}{t} \\ y = x\,t \quad &\therefore\ y' = \frac{dy}{dx} = \left\{(fg)' = f'g + fg'\right\} = x't + xt' = t + x\,\frac{dt}{dx} \end{aligned}\right\} \Rightarrow$$

$$\cancel{t} + \frac{1}{t} = \cancel{t} + x\,\frac{dt}{dx}$$

$$\frac{1}{t} = x\,\frac{dt}{dx} \quad \therefore \quad \int \frac{dx}{x} = \int t\,dt \quad \therefore \quad \ln|x| + \ln c = \frac{t^2}{2}$$

$$t = \sqrt{2\,\ln(c\,|x|)} \quad \therefore \quad \frac{y}{x} = \sqrt{2\,\ln(c\,|x|)} \quad \{x > 0\} \quad \therefore \quad \underline{y = x\sqrt{2\,\ln(c\,x)}}$$

Note that the integration constant c may take any convenient form, including the logarithm of a constant.

Verification:

$$y = x\sqrt{2\,\ln(c\,x)} \quad \therefore \quad y' = \sqrt{2\,\ln(c\,x)} + \frac{\cancel{x}\,1}{\cancel{2}\sqrt{2\,\ln(c\,x)}}\,\frac{\cancel{2}c}{\cancel{c}\,\cancel{x}} = \sqrt{2\,\ln(c\,x)} + \frac{1}{\sqrt{2\,\ln(c\,x)}}$$

Therefore,

$$y' = \frac{x^2+y^2}{xy}$$

$$\sqrt{2\,\ln(c\,x)} + \frac{1}{\sqrt{2\,\ln(c\,x)}} = \frac{x^2 + \left(x\sqrt{2\,\ln(c\,x)}\right)^2}{x^2\sqrt{2\,\ln(c\,x)}}$$

$$\sqrt{2\,\ln(c\,x)} + \frac{1}{\sqrt{2\,\ln(c\,x)}} = \frac{x^2 + x^2\,2\,\ln(c\,x)}{x^2\sqrt{2\,\ln(c\,x)}}$$

$$\sqrt{2\,\ln(c\,x)} + \frac{1}{\sqrt{2\,\ln(c\,x)}} = \frac{\cancel{x^2}\,1}{\cancel{x^2}\sqrt{2\,\ln(c\,x)}} + \frac{\cancel{x^2}\,2\,\ln(c\,x)}{\cancel{x^2}\sqrt{2\,\ln(c\,x)}}$$

$$\sqrt{2\,\ln(c\,x)} + \frac{1}{\sqrt{2\,\ln(c\,x)}} = \frac{1}{\sqrt{2\,\ln(c\,x)}} + \sqrt{\frac{\left(2\,\ln(c\,x)\right)^{\cancel{2}}}{\cancel{2\,\ln(c\,x)}}}$$

$$\cancel{\sqrt{2\,\ln(c\,x)}} + \frac{1}{\cancel{\sqrt{2\,\ln(c\,x)}}} = \frac{1}{\cancel{\sqrt{2\,\ln(c\,x)}}} + \cancel{\sqrt{2\,\ln(c\,x)}}$$

$$0 = 0 \quad \checkmark$$

6.15. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$\left.\begin{aligned} y' = \frac{y^2 - x^2}{2xy} \quad &\therefore \quad \frac{dy}{dx} = \frac{y}{2x} - \frac{x}{2y} = \left\{t = \frac{y}{x}\right\} = \frac{1}{2}\left(t - \frac{1}{t}\right) \\ y = x\,t \quad &\therefore \quad y' = \frac{dy}{dx} = \left\{(fg)' = f'g + fg'\right\} = x't + xt' = t + x\,\frac{dt}{dx} \end{aligned}\right\} \Rightarrow$$

$$t + x\,\frac{dt}{dx} = \frac{1}{2}\left(t - \frac{1}{t}\right) = \frac{t^2 - 1}{2t}$$

$$x\,\frac{dt}{dx} = \frac{t^2 - 1}{2t} - t = \frac{t^2 - 1 - 2t^2}{2t} = -\frac{t^2 + 1}{2t}$$

$$\therefore \quad \int \frac{dx}{x} = -\int \frac{2t}{t^2 + 1}\,dt = \left\{t^2 + 1 = r \quad \therefore \quad 2t\,dt = dr, \quad (t^2 + 1 \neq 0)\right\}$$

$$\therefore \quad \ln|x| = -\int \frac{dr}{r} = -\ln|r| = -\ln[t^2 + 1] + \ln c \quad \therefore \quad \ln|x| = \ln\frac{c}{t^2 + 1}$$

$$\therefore \quad \underline{x = \frac{c}{t^2 + 1}} \quad \text{and} \quad y = t\,x \quad \therefore \quad \underline{y = \frac{c\,t}{t^2 + 1}}$$

This is a parametric solution, i.e., $y = y(t)$ and $x = x(t)$. The parametric variable t may be eliminated and the explicit $y(x)$ solution derived, for example, as follows.

$$\left.\begin{aligned} x^2 &= \frac{c^2}{(t^2 + 1)^2} \quad \therefore \quad (t^2 + 1)^2 = \frac{c^2}{x^2} \quad \therefore \quad t^2 = \frac{c}{x} - 1 \\ y^2 &= \frac{c^2 t^2}{(t^2 + 1)^2} \end{aligned}\right\} \quad \therefore$$

$$y^2 = \frac{\cancel{c^2}\,(c/x - 1)}{\cancel{c^2}/x^2} = \frac{(c - x)/\cancel{x}}{1/\cancel{x^2}} = x(c - x) = c\,x - x^2 \qquad \therefore$$

$$\underline{y^2 + x^2 - c\,x = 0} \quad \text{or,} \quad \underline{y = \sqrt{c\,x - x^2}}$$

which is an equation of a shifted circle whose diameter equals c; see Fig. 6.4. It is converted from Cartesian to polar coordinates as

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

$$\therefore$$

$$y^2 + x^2 = c\,x$$

$$\cancel{r^2}(\cos^2\theta + \sin^2\theta) = c\,\cancel{r}\cos\theta$$

$$r = \frac{c\,\cos\theta}{\cos^2\theta + \sin^2\theta}$$

$$= c\,\cos\theta$$

Verification: $y = \sqrt{c\,x - x^2} \;\;\therefore\;\; y' = \dfrac{c - 2x}{2\sqrt{c\,x - x^2}}$ Therefore,

$$y' = \frac{y^2 - x^2}{2xy}$$

$$\frac{c - 2x}{2\sqrt{c\,x - x^2}} = \frac{\left(\sqrt{c\,x - x^2}\right)^2 - x^2}{x\,2\sqrt{c\,x - x^2}}$$

$$c - 2x = \frac{\left(c\,x - x^2\right) - x^2}{x}$$

$$c - 2x = \frac{c\,x - 2x^2}{x}$$

$$c - 2x = c - 2x$$

$$0 = 0 \;\checkmark$$

General solution is a family of curves, Fig. 6.4. Given the initial condition $\mathscr{A} : y(1) = 1$, its corresponding specific function is calculated as

$$y = \sqrt{c\,x - x^2}$$

$$1 = \sqrt{c\,(1) - 1^2}$$

$$1 = c - 1$$

$$\therefore\;\; c = 2 \;\;\Rightarrow\;\; \underline{y(x) = \sqrt{2x - x^2}}$$

6.16. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$(x + y)\,dx - x\,dy = 0$$

$$x\,dx + y\,dx = x\,dy \;\;\therefore\;\; (x + y)\,dx = x\,dy \;\;\therefore\;\; \frac{x + y}{x} = \frac{dy}{dx}$$

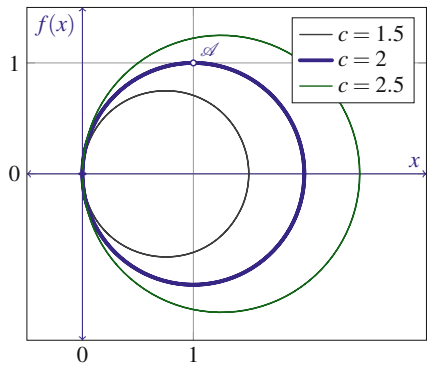


Fig. 6.4 Example P.6.15

$$1+\frac{y}{x}=\frac{dy}{dx}\;\;\therefore\;\;\left\{\frac{y}{x}=t\;\;\therefore\;\;y=x\,t\;\;\therefore\;\;y'=\frac{dy}{dx}=t+x\,t'\right\}$$

$$1+\cancel{t}=\cancel{t}+x\,t'\;\;\therefore\;\;1=x\,\frac{dt}{dx}\;\;\therefore\;\;\int dt=\int\frac{dx}{x}$$

$$\therefore\;\;t=\ln(c_1\,x)\quad\text{thus,}\quad \underline{y=x\ln(c_1\,x)}$$

Verification:

$$y=x\ln(c_1\,x)\;\;\therefore\;\;y'=\ln(c_1\,x)+\cancel{x}\,\frac{\cancel{c_1}}{\cancel{c_1}\,\cancel{x}}=\ln(c_1\,x)+1$$

Therefore,

$$(x+y)\,dx-x\,dy=0$$

$$1+\frac{y}{x}=\frac{dy}{dx}=y'$$

$$\cancel{1}+\frac{\cancel{x}\ln(c_1\,x)}{\cancel{x}}=\cancel{\ln(c_1\,x)}+\cancel{1}$$

$$0=0\;\checkmark$$

6.17. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$xy'-2y=-x^2\;\;\therefore\;\;y'-2\frac{y}{x}=-x\;\;\therefore\;\;y'=2\frac{y}{x}-x$$

so that,

$$\left.\begin{aligned}y'=\frac{dy}{dx}=2\frac{y}{x}-x\;\;&=\left\{t=\frac{y}{x}\right\}=2t-x\\ y=x\,t\;\;\therefore\;\;y'\;\;&=\frac{dy}{dx}=\left\{(fg)'=f'g+fg'\right\}=x't+xt'=t+x\,\frac{dt}{dx}\end{aligned}\right\}\Rightarrow$$

$$t+x\,\frac{dt}{dx}=2t-x\;\;\therefore\;\;x+x\,\frac{dt}{dx}=2t-t\;\;\therefore\;\;x\,dx+x\,dt=t\,dx\;\;\therefore\;\;x\,dx+x\,dt-t\,dx=0$$

$$\therefore\;\;(x-t)\,dx+x\,dt=0\;\;\therefore\;\;\frac{x-t}{x}\,dx=-dt\;\;\therefore\;\;1-\frac{t}{x}=-\frac{dt}{dx}$$

And, the second time

$$t'=\frac{dt}{dx}=\frac{t}{x}-1=\left\{\frac{t}{x}=p\right\}=p-1\;\;\therefore\;\;\cancel{p}+x\frac{dp}{dx}=\cancel{p}-1\;\;\therefore\;\;\int dp=-\int\frac{dx}{x}$$

$$\therefore\;\;p=-\ln|c_1x|\;\;\therefore\;\;\frac{t}{x}=-\ln|c_1x|\;\;\therefore\;\;\frac{y}{x^2}=-\ln|c_1x|\;\;\therefore\;\;\underline{y=-x^2\ln|c_1x|}$$

Verification:

$$y = -x^2 \ln|c_1 x| \quad \therefore \quad y' = -2x \ln|c_1 x| - x$$

$$y' = 2\frac{y}{x} - x - 2x \ln|c_1 x| - x = 2\frac{-x^2 \ln|c_1 x|}{x} - x$$

$$0 = 0 \ \checkmark$$

6.18. $y' = \dfrac{y}{x} \ln \dfrac{y}{x} = \left\{ t = \dfrac{y}{x} \quad \therefore \quad y' = t + x\dfrac{dt}{dx} \right\} = t \ln t$

$$\therefore$$

$$t + x\frac{dt}{dx} = t \ln t \quad \therefore \quad x\frac{dt}{dx} = t(\ln t - 1) \quad \therefore \quad \int \frac{dt}{t(\ln t - 1)} = \int \frac{dx}{x}$$

$$= \left\{ r = \ln t \quad \therefore \quad dr = \frac{dt}{t} \right\} \quad \therefore \quad \ln|c_1 x| = \int \frac{dr}{r-1} = \ln|r-1| = \ln|\ln t - 1|$$

$$\therefore$$

$$\ln|c_1 x| = \ln|\ln \frac{y}{x} - 1| \quad \therefore \quad c_1 x = \ln \frac{y}{x} - 1 \quad \therefore \quad 1 + c_1 x = \ln \frac{y}{x} \quad \therefore \quad \frac{y}{x} = e^{1+c_1 x}$$

$$\therefore \quad \underline{y = x\, e^{1+c_1 x}}$$

6.19. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$xy^2\, dy = (x^3 + y^3)\, dx \quad \therefore \quad y' = \frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} = \left(\frac{x}{y}\right)^2 + \frac{y}{x} \quad \therefore \quad \{ y = t\,x \quad \therefore \quad y' = t + xt' \}$$

$$\therefore \quad t + x\,t' = \frac{1}{t^2} + t \quad \therefore \quad x\,\frac{dt}{dx} = \frac{1}{t^2}$$

$$\therefore \quad \int t^2\, dt = \int \frac{dx}{x} \quad \therefore \quad \frac{t^3}{3} = \ln(c_1\,|x|) \quad \therefore \quad \frac{y^3}{3x^3} = \ln(c_1\,|x|)$$

$$\therefore \quad \underline{y = x\sqrt[3]{3\ln(c_1\,|x|)}}$$

Verification:

$$y = x\sqrt[3]{3\ln(c_1\,|x|)} \quad \therefore \quad y' = \sqrt[3]{3\ln(c_1\,|x|)} + \frac{1}{\left(3x\ln(c_1|x|)\right)^{2/3}}$$

Therefore,

$$y' = \left(\frac{x}{y}\right)^2 + \frac{y}{x}$$

$$\sqrt[3]{3\ln(c_1|x|)} + \frac{1}{\left(3x\ln(c_1|x|)\right)^{2/3}} = \frac{x^2}{x^2\ 1\ \left(3x\ln(c_1|x|)\right)^{2/3}} + \frac{x\sqrt[3]{3\ln(c_1|x|)}}{x}$$

$$0 = 0 \ \checkmark$$

6.20. $(x-2y)\,dy=(x-y)\,dx \;\;\therefore\;\; y'=\dfrac{dy}{dx}=\dfrac{x-y}{x-2y} \;\;\therefore\;\; \dfrac{dy}{dx}=\dfrac{\cancel{x}(1-y/x)}{\cancel{x}(1-2\,y/x)}$

$$\therefore\;\; \{y=t\,x \;\;\therefore\;\; y'=t+xt'\} \;\;\therefore\;\; t+x\frac{dt}{dx}=\frac{1-t}{1-2t}$$

$$\therefore\;\; x\frac{dt}{dx}=\frac{1-t}{1-2t}-t=\frac{2t^2-2t+1}{1-2t} \;\;\therefore\;\; \int\frac{1-2t}{2t^2-2t+1}\,dt=\int\frac{dx}{x}$$

where

$$\int\frac{1-2t}{2t^2-2t+1}\,dt=\{u=2t^2-2t+1 \;\;\therefore\;\; du=(4t-2)\,dt=-2(1-2t)\,dt\}$$

$$-\frac{1}{2}\int\frac{1}{u}\,du=-\frac{1}{2}\ln|2t^2-2t+1|=-\frac{1}{2}\ln\left|2\left(\frac{x}{y}\right)^2-2\left(\frac{x}{y}\right)+1\right|$$

so that

$$-\frac{1}{2}\ln\left|2\left(\frac{y}{x}\right)^2-2\left(\frac{y}{x}\right)+1\right|=\ln c_1x$$

$$\ln\left|2\left(\frac{y}{x}\right)^2-2\left(\frac{y}{x}\right)+1\right|=\ln(c_1x)^{-2}$$

$$2\left(\frac{y}{x}\right)^2-2\left(\frac{y}{x}\right)+1=\frac{1}{c_1^2x^2} \;\;\left\{c_2^2=\frac{1}{c_1^2}\right\}$$

$$\underline{2y^2-2yx+x^2=c_2^2}$$

which is an equation of an ellipse; see Fig. 6.5. It is converted from Cartesian to polar coordinates as

$$x=r\cos(\theta),\quad y=r\sin(\theta)$$

$$\therefore$$

$$2y^2-2yx+x^2=c_2^2$$

$$2(r\cos(\theta))^2-2r\cos(\theta)\,r\sin(\theta)+(r\sin(\theta))^2=c_2^2$$

$$r^2\big(2\cos^2(\theta)-2\cos(\theta)\,\sin(\theta)+\sin^2(\theta)\big)=c_2^2$$

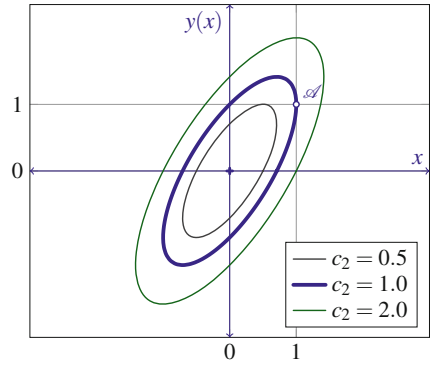
$$\therefore$$

$$r=\sqrt{\frac{c_2^2}{2\cos^2(\theta)-2\cos(\theta)\,\sin(\theta)+\sin^2(\theta)}}=\sqrt{\frac{c_2^2}{\cos^2(\theta)-2\cos(\theta)\,\sin(\theta)+\underbrace{\cos^2(\theta)+\sin^2(\theta)}_{1}}}$$

$$=\sqrt{\frac{c_2^2}{\cos^2(\theta)-2\cos(\theta)\,\sin(\theta)+1}}$$

General solution is a family of curves, Fig. 6.5. Given the initial condition $\mathscr{A}$: $y(1)=1$, its corresponding specific function is calculated as

Fig. 6.5 Example P.6.20



$$2y^2 - 2yx + x^2 = c_2^2$$
$$2(1)^2 - 2(1)(1) + 1^2 = c_2^2$$
$$\therefore$$
$$c_2 = 1 \;\Rightarrow\; \underline{y(x) = 2y^2 - 2yx + x^2 = 1}$$

6.21. Given equation may be converted into explicit $f(x/y)$ form, then the change of variable method is applied as

$$(x^2 - 2y^2)\,dx + 2xy\,dy = 0 \;\;\therefore\;\; \frac{2xy}{x^2 - 2y^2}\,dy = -\,dx \;\;\therefore\;\; \frac{2}{\dfrac{x^2}{xy} - \dfrac{2y^2}{xy}}\,dy = -\,dx$$

$$\therefore\;\; \{y = t\,x \;\;\therefore\;\; y' = t + x\,t'\} \;\;\therefore\;\; \frac{2}{1/t - 2t}\,(t + x\,t') = -1$$

$$\therefore\;\; \frac{t}{2t^2 - 1}\,(t + x\,t') = \frac{1}{2} \;\;\therefore\;\; t + x\,\frac{dt}{dx} = \frac{1}{2}\,\frac{2t^2 - 1}{t}$$

$$\therefore\;\; x\,\frac{dt}{dx} = \frac{1}{2}\,\frac{2t^2 - 1}{t} - t = -\frac{1}{2t} \;\;\therefore\;\; 2\int t\,dt = -\int \frac{dx}{x}$$

$$\therefore\;\; t^2 = -\ln|cx| \;\;\therefore\;\; \left(\frac{y}{x}\right)^2 = -\ln|cx| \;\;\therefore\;\; \underline{y^2 + x^2 \ln|cx| = 0}$$

Verification:

$$y^2 + x^2 \ln|cx| = 0 \;\;\therefore\;\; y^2 = -x^2 \ln|cx| \;\;\therefore\;\; y = |x|\sqrt{-\ln|cx|}$$

$$\therefore\;\; \frac{dy}{dx} = -\frac{2x \ln|cx| + x^2(1/x)}{2x\sqrt{-\ln|cx|}} = -\frac{2\ln|cx| + 1}{2\sqrt{-\ln|cx|}}$$

At the same time,

$$\frac{dy}{dx} = -\frac{x^2 - 2y^2}{2xy} = -\frac{x^2 - 2(-x^2 \ln|cx|)}{2x\,|x|\sqrt{-\ln|cx|}} = -\frac{x^2(2\ln|cx| + 1)}{2x^2\sqrt{-\ln|cx|}} \quad \checkmark$$

It is converted from Cartesian to polar coordinates as

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

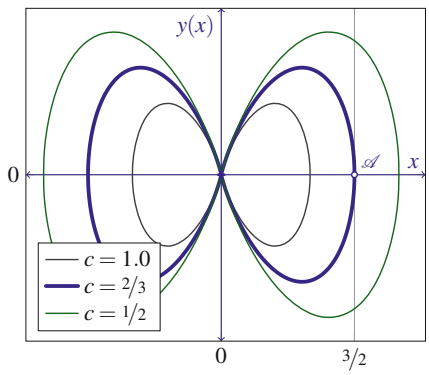
$$\therefore$$

$$\begin{aligned} y^2 + x^2 \ln|cx| &= 0 \\ r^2\sin^2 x + r^2\cos^2 x \ln|cr\cos x| &= 0 \\ r^2\sin^2 x &= -r^2\cos^2 x \ln|cr\cos x| \\ \tan^2 x &= -\ln|cr\cos x| \quad \{c_1 = 1/|c|\} \\ |r| &= c_1 \frac{e^{-\tan^2 x}}{|\cos x|} \quad (x \in (0, 2\pi)) \end{aligned}$$

General solution is a family of curves, Fig. 6.6. Given the initial condition $\mathscr{A} : y(3/2) = 0$, its corresponding specific function is calculated as

$$\begin{aligned} y^2 + x^2 \ln|cx| &= 0 \\ 0^2 + (3/2)^2 \ln|c(3/2)| &= 0 \\ \ln|c(3/2)| &= 0 \\ c\,(3/2) &= 1 \\ &\therefore \\ c = 2/3 \quad \Rightarrow \quad y(x) &= |x|\sqrt{-\ln\left|\frac{2}{3}x\right|} \end{aligned}$$

Fig. 6.6 Example P.6.21



6.22. $y' - \frac{y}{x} = \tan\frac{y}{x} \;\;\therefore\;\; \{y = t\,x \;\;\therefore\;\; y' = t + x\,t'\}$

$$t + x\frac{dt}{dx} = t + \tan t \;\;\therefore\;\; \int \frac{dt}{\tan t} = \int \frac{dx}{x}$$

$$\therefore\;\; \ln|cx| = \int \frac{\cos t}{\sin t}\,dt \;\;\{u = \sin t \Rightarrow du = \cos t\,dt\}$$

$$\ln|cx| = \ln|\sin t| \;\;\therefore\;\; |cx| = \left|\sin\frac{y}{x}\right| \;\;\therefore\;\; \underline{y(x) = x\,\arcsin(cx)}$$

Verification:

$$y = x\,\arcsin(cx) \;\;\therefore\;\; y' = \arcsin(cx) + \frac{cx}{\sqrt{1-(cx)^2}}$$

$$y' - \frac{y}{x} = \tan\frac{y}{x}$$

$$\cancel{\arcsin(cx)} + \frac{cx}{\sqrt{1-(cx)^2}} - \frac{\cancel{x}\,\cancel{\arcsin(cx)}}{\cancel{x}} = \tan\frac{\cancel{x}\,\arcsin(cx)}{\cancel{x}}$$

$$\frac{cx}{\sqrt{1-(cx)^2}} = \tan(\arcsin(cx)) \;\;\left\{\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}\right\}$$

$$\frac{cx}{\cancel{\sqrt{1-(cx)^2}}} = \frac{cx}{\cancel{\sqrt{1-(cx)^2}}}$$

$$0 = 0 \;\checkmark$$

6.23. $\frac{y^2}{x^2} - \frac{y}{x} - y' + 1 = 0 \;\;\therefore\;\; \{y = t\,x \;\;\therefore\;\; y' = t + x\,t'\}$

$$t + x\,t' = t^2 - t + 1 \;\;\therefore\;\; x\frac{dt}{dx} = t^2 - 2t + 1 \;\;\therefore\;\; \int \frac{dt}{(t-1)^2} = \int \frac{dx}{x}$$

$$\{u = t - 1 \;\;\therefore\;\; du = dt\}$$

$$\ln(cx) = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{t-1} = -\frac{1}{y/x - 1} = -\frac{x}{y - x}$$

$$\therefore\;\; y\ln(cx) - x\ln(cx) = -x$$

$$\underline{y(x) = x - \frac{x}{\ln(cx)}}$$

Verification:

$$y = x - \frac{x}{\ln(cx)} \;\;\therefore\;\; y' = 1 - \frac{\ln(cx) - 1}{\ln^2(cx)}$$

Therefore,

$$\frac{y^2}{x^2} - \frac{y}{x} - y' + 1 = 0$$

$$\frac{1}{x^2}\left(x-\frac{x}{\ln(cx)}\right)^2-\frac{1}{x}\left(x\,1-\frac{x\,1}{\ln(cx)}\right)-1+\frac{\ln(cx)-1}{\ln^2(cx)}+1=0$$

$$\frac{1}{x^2}\left(x^2\,1-\frac{2x^2}{\ln(cx)}+\frac{x^2\,1}{\ln^2(cx)}\right)-1+\frac{1}{\ln(cx)}+\frac{1}{\ln(cx)}-\frac{1}{\ln^2(cx)}=0$$

$$1-\frac{2}{\ln(cx)}+\frac{1}{\ln^2(cx)}-1+\frac{2}{\ln(cx)}-\frac{1}{\ln^2(cx)}=0$$

$$0=0\ \checkmark$$

General solution is a family of curves, Fig. 6.7. Given the initial condition $\mathscr{A}: y(e)=0$, its corresponding specific function is calculated as

$$y=x-\frac{x}{\ln(cx)}$$

$$0=e-\frac{e}{\ln(c)+\ln(e)}$$

$$e\,1=\frac{e\,1}{\ln(c)+1}\quad\therefore\quad \ln(c)+1=1\,0\quad\therefore\quad \ln(c)=0$$

$$\therefore$$

$$c=1\quad\Rightarrow\quad \underline{y(x)=x-\frac{x}{\ln x}}$$

6.24. $y'=\dfrac{y+xe^{-y/x}}{x}=\dfrac{y}{x}+e^{-y/x}\quad\therefore\quad \{y=t\,x\quad\therefore\quad y'=t+x\,t'\}$

$$t+x\,t'=t+e^{-t}\quad\therefore\quad \int e^t\,dt=\int\frac{dx}{x}\quad\therefore\quad \ln|cx|=e^t=e^{y/x}$$

$$\ln(\ln(cx))=\frac{y}{x}\quad\therefore\quad \underline{y(x)=x\ln(\ln(cx))}$$

Verification:

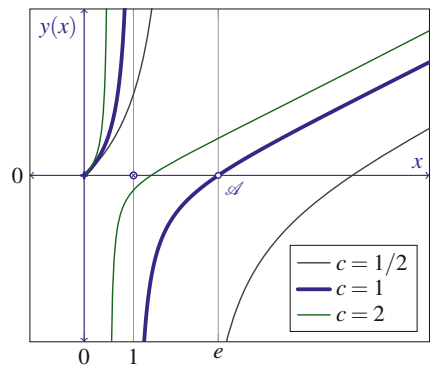


Fig. 6.7 Example P.6.23

$$y = x\ln(\ln(cx)) \quad \therefore \quad y' = \ln(\ln(cx)) + \cancel{x}\frac{1}{\cancel{x}\ln(cx)} = \ln(\ln(cx)) + \frac{1}{\ln(cx)}$$

Therefore,

$$y' = \frac{y}{x} + e^{-y/x}$$

$$\ln(\ln(cx)) + \frac{1}{\ln(cx)} = \frac{1}{\cancel{x}}\,\cancel{x}\ln(\ln(cx)) + e^{-\cancel{x}\ln(\ln(cx))/\cancel{x}}$$

$$\ln(\ln(cx)) + \frac{1}{\ln(cx)} = \ln(\ln(cx)) + \frac{1}{e^{\ln(\ln(cx))}}$$

$$\cancel{\ln(\ln(cx))} + \cancel{\frac{1}{\ln(cx)}} = \cancel{\ln(\ln(cx))} + \cancel{\frac{1}{(\ln(cx))}}$$

$$0 = 0 \quad \checkmark$$

6.3 First Order, Function Coefficients

6.25. Given $y' - y = 1 = 1$, it can be declared that

$$f(x) = -1 \quad \text{and} \quad g(x) = 1 \quad \therefore \quad \int f(x)\,dx = -\int dx = -x \quad \therefore \quad \underline{\mu = e^{-x}}$$

Therefore,

$$\frac{d}{dx}\left(e^{-x}\,y\right) = e^{-x} \quad \therefore \quad \int d\left(e^{-x}\,y\right) = \int e^{-x}\,dx$$

$$e^{-x}\,y = \int e^{-x}\,dx \quad \left\{-x = t \quad \therefore \quad -dx = dt \quad \therefore \quad dx = -dt\right\}$$

$$= -\int e^{t}\,dt = -e^{t} = -e^{-x} + c_1$$

$$\therefore$$

$$e^{-x}\,y = -e^{-x} + c_1 \quad \therefore \quad \underline{y = -1 + c_1 e^{x}}$$

Verification:

$$y = -1 + c_1 e^{x} \quad \therefore \quad y' = c_1 e^{x}$$

Therefore,

$$c_1 e^{x} - \left(-1 + c_1 e^{x}\right) = 1$$

$$\cancel{c_1 e^{x}} + 1 - \cancel{c_1 e^{x}} = 1$$

$$0 = 0 \quad \checkmark$$

6.26. Given $y' + xy = x$, it can be declared that

$$f(x) = x \quad \text{and} \quad g(x) = x \quad \therefore \quad \int f(x)\,dx = \int x\,dx = \frac{x^2}{2} \quad \therefore \quad \underline{\mu = e^{x^2/2}}$$

$$\frac{d}{dx}\left(e^{x^2/2}\,y\right) = x\,e^{x^2/2} \quad \therefore \quad \int d\left(e^{x^2/2}\,y\right) = \int x\,e^{x^2/2}\,dx$$

$$e^{x^2/2}\,y = \int x\,e^{x^2/2}\,dx \quad \left\{\frac{x^2}{2} = t \quad \therefore \quad x\,dx = dt\right\}$$

$$= \int e^t\,dt = e^t = e^{x^2/2} + c_1$$

$$\therefore$$

$$e^{x^2/2}\,y = e^{x^2/2} + c_1 \quad \therefore \quad \underline{y = 1 + c_1 e^{-x^2/2}}$$

Verification:

$$y = 1 + c_1 e^{-x^2/2} \quad \therefore \quad y' = -c_1 x\,e^{-x^2/2}$$

$$y' + xy = x$$

$$\cancel{-c_1 x\,e^{-x^2/2}} + \cancel{x} + \cancel{c_1 x e^{-x^2/2}} = \cancel{x}$$

$$0 = 0 \quad \checkmark$$

6.27. Given $y' - 4xy = x$, it can be declared that

$$f(x) = -4x \quad \text{and} \quad g(x) = x \quad \therefore \quad \int f(x)\,dx = -4\int x\,dx = -4\frac{x^2}{2} = -2x^2 \quad \therefore \quad \underline{\mu = e^{-2x^2}}$$

Therefore,

$$\frac{d}{dx}\left(e^{-2x^2}\,y\right) = x\,e^{-2x^2} \quad \therefore \quad \int d\left(e^{-2x^2}\,y\right) = \int x\,e^{-2x^2}\,dx$$

$$e^{-2x^2}\,y = \int x\,e^{-2x^2}\,dx \quad \left\{-2x^2 = t \quad \therefore \quad -4x\,dx = dt \quad \therefore \quad x\,dx = -\frac{1}{4}\,dt\right\}$$

$$= -\frac{1}{4}\int e^t\,dt = -\frac{1}{4}e^t = -\frac{1}{4}e^{-2x^2} + c_1 \quad \therefore$$

$$e^{-2x^2}\,y = -\frac{1}{4}e^{-2x^2} + c_1 \quad \therefore \quad \underline{y = -\frac{1}{4} + c_1 e^{2x^2}}$$

Verification:

$$y = -\frac{1}{4} + c_1 e^{2x^2} \quad \therefore \quad y' = 4c_1 x\, e^{2x^2}$$

Therefore,

$$y' - 4xy = x$$
$$4c_1 x\, e^{2x^2} - 4x\left(-\frac{1}{4} + c_1 e^{2x^2}\right) = x$$
$$\cancel{4c_1 x\, e^{2x^2}} + \cancel{4}\,\cancel{x}\,\frac{1}{\cancel{4}} - \cancel{4x\, c_1 e^{2x^2}} = \cancel{x}$$
$$0 = 0 \quad \checkmark$$

6.28. Given $y' - 2y - 3 = 0 \quad \therefore \quad y' - 2y = 3$, it can be declared that

$$f(x) = -2 \quad \text{and} \quad g(x) = 3 \quad \therefore \quad \int f(x) = -2\int dx = -2x \quad \therefore \quad \underline{\mu = e^{-2x}}$$

Therefore,

$$\frac{d}{dx}\left(y\, e^{-2x}\right) = 3\, e^{-2x} \quad \therefore \quad \int d\left(y\, e^{-2x}\right) \quad = 3\int e^{-2x}\, dx \quad \therefore \quad y\, e^{-2x} = -\frac{3}{2}e^{-2x} + c_1$$
$$\underline{y = -\frac{3}{2} + c_1\, e^{2x}}$$

Verification:

$$y = -\frac{3}{2} + c_1\, e^{2x} \quad \therefore \quad y' = 2c_1\, e^{2x}$$

Therefore,

$$y' - 2y = 3$$
$$2c_1\, e^{2x} - 2\left(-\frac{3}{2} + c_1\, e^{2x}\right) = 3$$
$$\cancel{2c_1\, e^{2x}} + \cancel{2}\,\frac{3}{\cancel{2}} - \cancel{2c_1\, e^{2x}} = 3$$
$$0 = 0 \quad \checkmark$$

6.29. Given $y' + 2xy = 2xe^{-x^2}$, it can be declared that

$$f(x) = 2x \quad \text{and} \quad g(x) = 2x\, e^{-x^2} \quad \therefore \quad \int f(x)\, dx = 2\int x\, dx = \cancel{2}\,\frac{x^2}{\cancel{2}} \quad \therefore \quad \underline{\mu = e^{x^2}}$$

Therefore,

$$y\,e^{x^2} = 2\int \cancel{e^{x^2}}\,x\,\cancel{e^{-x^2}}\,dx = x^2 + c_1 \quad \therefore \quad \underline{y = e^{-x^2}\left(x^2 + c_1\right)}$$

Verification:

$$y = e^{-x^2}\left(x^2 + c_1\right) \quad \therefore \quad y' = -2x\,e^{-x^2}\left(x^2 + c_1\right) + 2x\,e^{-x^2}$$

Therefore,

$$\begin{aligned} y' + 2xy &= 2xe^{-x^2} \\ \cancel{-2x\,e^{-x^2}}\left(x^2 + c_1\right) + \cancel{2x\,e^{-x^2}} + \cancel{2x}\left(\cancel{e^{-x^2}}\left(x^2 + c_1\right)\right) &= \cancel{2x\,e^{-x^2}} \\ \cancel{-x^2} - \cancel{c_1} + \cancel{1} + \cancel{x^2} + \cancel{c_1} &= \cancel{1} \\ 0 &= 0 \quad \checkmark \end{aligned}$$

6.30. Given $y' + y\cos x = \sin(2x)$, it can be declared that

$$f(x) = \cos x \quad \text{and} \quad g(x) = \sin(2x) \quad \therefore \quad \int f(x)\,dx = \int \cos x\,dx = \sin x \quad \therefore \quad \underline{\mu = e^{\sin x}}$$

Therefore,

$$y\,e^{\sin x} = \int \sin(2x)\,e^{\sin x}\,dx = \{\text{see A.5.41}\} = 2e^{\sin x}\left(\sin x - 1\right) + c$$

$$\therefore$$

$$\underline{y(x) = 2\left(\sin x - 1\right) + ce^{-\sin x}}$$

Verification:

$$y = 2\left(\sin x - 1\right) + ce^{-\sin x} \quad \therefore \quad y' = 2\cos x - c\cos x\,e^{-\sin x}$$

Therefore,

$$\begin{aligned} y' + y\cos x &= \sin(2x) \\ 2\cos x - \cancel{c\cos x\,e^{-\sin x}} + 2\cos x\left(\sin x - 1\right) + \cancel{c\cos x\,e^{-\sin x}} &= 2\cos x\sin(x) \\ \cancel{2\cos x} + \cancel{2\cos x\sin x} - \cancel{2\cos x} &= \underline{2\cos x\sin(x)} \\ 0 &= 0 \quad \checkmark \end{aligned}$$

6.31. Given

$$y' - \frac{2}{x+1}\,y - \frac{4x}{1-x^2} = 0 \quad \therefore \quad y' + \frac{-2}{x+1}\,y = \frac{4x}{1-x^2}$$

it can be declared that

$$f(x) = -\frac{2}{x+1} \quad \text{and} \quad g(x) = \frac{4x}{1-x^2} \quad \therefore \quad \int f(x)\,dx = -2\int \frac{dx}{x+1} = -2\ln|x+1|$$

$$\therefore$$

$$\underline{\mu = e^{-2\ln|x+1|} = \frac{1}{(x+1)^2}}$$

Therefore,

$$\int d\left(y\,\frac{1}{(x+1)^2}\right) = 4\int \frac{x}{(1-x)(1+x)^3}\,dx = \{\text{see Ch. 5.67}\}$$

$$= \frac{1}{4}\ln\frac{(x+1)^2}{(x-1)^2} - \frac{x}{(x+1)^2} + c$$

$$\frac{y}{(x+1)^2} = \frac{1}{4}\ln\frac{(x+1)^2}{(x-1)^2} - \frac{x}{(x+1)^2} + c$$

$$y = \frac{(x+1)^2}{4}\ln\frac{(x+1)^2}{(x-1)^2} - \cancel{(x+1)^2}\,\frac{x}{\cancel{(x+1)^2}} + c(x+1)^2$$

$$\therefore$$

$$\underline{y(x) = \frac{(x+1)^2}{4}\ln\frac{(x+1)^2}{(x-1)^2} - x + c(x+1)^2}$$

and its derivative is

$$y'(x) = \frac{x+1}{2}\ln\frac{(x+1)^2}{(x-1)^2} + \frac{(x+1)^2}{4}\,\frac{\frac{\cancel{-4(x+1)}}{\cancel{(x-1)^3}}}{\frac{\cancel{(x+1)^2}}{\cancel{(x-1)^2}}} - 1 + 2c(x+1)$$

$$y'(x) = \frac{x+1}{2}\ln\frac{(x+1)^2}{(x-1)^2} + \frac{(x+1)^2}{4}\,\frac{-4}{(x+1)(x-1)} - 1 + 2c(x+1)$$

Verification:

$$y' - \frac{2}{x+1}\,y = \frac{4x}{1-x^2}$$

$$\frac{\cancel{x+1}}{2}\ln\frac{(x+1)^2}{\cancel{(x-1)^2}} + \frac{(x+1)^2}{\cancel{4}\,1}\,\frac{\cancel{-4}\,1}{(x+1)(x-1)} - 1 + \cancel{2c(x+1)}$$

$$-\frac{\cancel{2}}{\cancel{x+1}}\,\frac{\cancel{(x+1)^2}}{\cancel{4}\,2}\ln\frac{(x+1)^2}{\cancel{(x-1)^2}} + \frac{2}{x+1}x - \frac{1}{\cancel{x+1}}\cancel{2c(x+1)^2} = \frac{4x}{1-x^2}$$

$$-\frac{(x+1)^2}{(x+1)(x-1)} - 1 + \frac{2x}{x+1} = \frac{4x}{1-x^2}$$

$$\frac{\cancel{4x}}{\cancel{1-x^2}} = \frac{\cancel{4x}}{\cancel{1-x^2}}$$

$$0 = 0 \quad \checkmark$$

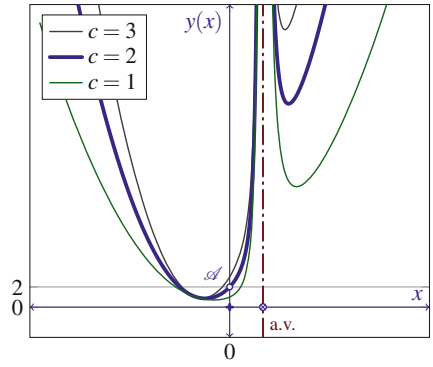


Fig. 6.8 Example P.6.31

General solution is a family of curves, Fig. 6.8. Given the initial condition $\mathscr{A} : y(0) = 2$, its corresponding specific function is calculated as

$$y = \frac{(x+1)^2}{4} \ln \frac{(x+1)^2}{(x-1)^2} - x + c(x+1)^2$$

$$2 = \frac{(0+1)^2}{4} \ln \frac{(0+1)^2}{(0-1)^2} - 0 + c(0+1)^2 \qquad (0)$$

$$2 = c(0+1)^2 \quad \therefore \quad c = 2$$

$$\Rightarrow \quad \underline{y(x) = \frac{(x+1)^2}{4} \ln \frac{(x+1)^2}{(x-1)^2} - x + 2(x+1)^2}$$

6.32. Given $y' - (2y)/(x+1) - (x+1)^3 = 0$, it can be declared that

$$y' - \frac{2}{x+1}\, y - (x+1)^3 = 0 \quad \therefore \quad y' - \frac{2}{x+1}\, y = (x+1)^3$$

$$f(x) = -\frac{2x}{(x+1)} \quad \text{and} \quad g(x) = (x+1)^3$$

Therefore (see P.5.9),

$$\int f(x)\, dx = -\int \frac{2}{x+1}\, dx = -2 \ln |x+1| = \ln \frac{1}{(x+1)^2} \quad \therefore$$

$$\mu = \exp\left(\ln \frac{1}{(x+1)^2} \right) = \frac{1}{(x+1)^2}$$

That is to say,

$$\int d\left(y\, \frac{1}{(x+1)^2} \right) = \int \frac{1}{(x+1)^2} (x+1)^3\, dx = \{x+1 = t,\, dx = dt\}$$

$$y\, \frac{1}{(x+1)^2} = \frac{(x+1)^2}{2} + c_1 \quad \therefore \quad \underline{y = \frac{(x+1)^4}{2} + c_1 (x+1)^2}$$

Or, alternatively, calculating integral without the change of variables technique,

$$\int (x+1)\,dx = \int x\,dx + \int dx = \frac{x^2}{2} + x + c \quad \therefore \quad \underline{y = (x+1)^2\left(\frac{x^2}{2} + x + c\right)}$$

note the relation $c = c_1 + 1/2$.

Verification: $y = \dfrac{(x+1)^4}{2} + c_1(x+1)^2 \quad \therefore \quad y' = 2(x+1)^3 + 2c_1(x+1)$

Therefore,

$$y' - \frac{2y}{x+1} = (x+1)^3$$

$$2(x+1)^3 + 2c_1(x+1) - \frac{2}{x+1}\left(\frac{(x+1)^4}{2} + c_1(x+1)^2\right) = (x+1)^3$$

$$2(x+1)^3 + 2c_1(x+1) - (x+1)^3 - 2c_1(x+1) = (x+1)^3$$

$$0 = 0 \quad \checkmark$$

6.33. Given equation may be transformed as

$$y'(x^2+x) - (y+1)(2x+1) = 0$$

$$y'(x^2+x) = y(2x+1) + (2x+1)$$

$$y' - \frac{2x+1}{x^2+x}\,y = \frac{2x+1}{x^2+x}$$

$$\therefore$$

$$f(x) = -\frac{2x+1}{x^2+x} \quad \text{and} \quad g(x) = \frac{2x+1}{x^2+x}$$

$$\int f(x)\,dx = -\int \frac{2x+1}{x^2+x}\,dx \quad \{t = x^2+x \quad \therefore \quad dt = (2x+1)\,dx\} = -\int \frac{dt}{t} = -\ln|t|$$

$$= \ln\left|\frac{1}{x^2+x}\right| \quad \therefore$$

$$\mu = \exp\left(\ln\left|\frac{1}{x^2+x}\right|\right) = \frac{1}{|x^2+x|}$$

That is to say,

$$d\left(y\,\frac{1}{|x^2+x|}\right) = \frac{2x+1}{x^2+x}\,\frac{1}{|x^2+x|}\,dx$$

$$\{t = x^2+x \quad \therefore \quad dt = (2x+1)\,dx\}$$

$$= \int \frac{dt}{t\,|t|} = -\frac{1}{|t|} \quad \{\text{see derivative and integral chapters}\} = -\frac{1}{|x^2+x|} + c$$

$$\therefore$$

$$y = -\cancel{|x^2+x|}\frac{1}{\cancel{|x^2+x|}} + c\,|x^2+x|$$

$$y(x) = -1 + c\,|x^2+x|$$

Verification:

$$y = -1 + c\,|x^2+x|$$

$$y' = \frac{c\,|x^2+x| + 2xc\,|x^2+x|}{x^2+x} = c\,|x^2+x|\,\frac{2x+1}{x^2+x}$$

Therefore,

$$y' - \frac{2x+1}{x^2+x}\,y = \frac{2x+1}{x^2+x}$$

$$\cancel{c\,|x^2+x|\frac{2x+1}{x^2+x}} + \cancel{\frac{2x+1}{x^2+x}} - \cancel{\frac{2x+1}{x^2+x}c\,|x^2+x|} = \cancel{\frac{2x+1}{x^2+x}}$$

$$0 = 0 \quad \checkmark$$

6.34. Given $y' + y\tan x = 1/\cos x$, it can be declared that

$$y' + y\tan x = \frac{1}{\cos x} \quad \therefore \quad f(x) = \tan x \quad \text{and} \quad g(x) = \frac{1}{\cos x}$$

so that

$$\int f(x)\,dx = \int \tan x\,dx = \{\text{see Ch. 5.15}\} = -\ln|\cos x| = \ln\frac{1}{|\cos x|}$$

$$\therefore$$

$$\mu = e^{\ln\frac{1}{|\cos x|}} = \frac{1}{|\cos x|}$$

That is to say,

$$\cancel{\int} d\left(y\,\frac{1}{|\cos x|}\right) = \int \frac{1}{\cos x}\,\frac{1}{|\cos x|}\,dx \quad \{\text{see Ch. 5.46}\} = \frac{\sin x}{|\cos x|} + c$$

$$\therefore$$

$$y(x) = \cancel{|\cos x|}\,\frac{\sin x}{\cancel{|\cos x|}} + c|\cos x| = \sin x + c|\cos x|$$

$$\therefore$$

$$y' = \frac{-c\ \sin x|\cos x| + \cos^2 x}{\cos x} = \cos x - \frac{c\ \sin x|\cos x|}{\cos x}$$

Verification:

$$\begin{aligned}
y' + y\tan x &= \frac{1}{\cos x} \\
\frac{-c\ \sin x|\cos x| + \cos^2 x}{\cos x} + (\sin x + c|\cos x|)\frac{\sin x}{\cos x} &= \frac{1}{\cos x} \\
\frac{-c\ \sin x|\cos x| + \cos^2 x + \sin x \sin x + c\sin x|\cos x|}{\cos x} &= \frac{1}{\cos x} \\
\frac{\cos^2 x + \sin^2 x}{\cos x} &= \frac{1}{\cos x} \\
0 &= 0 \quad \checkmark
\end{aligned}$$

General solution is a family of curves, Fig. 6.9. Given the initial condition $\mathscr{A} : y(\pi) = 1$, its corresponding specific function is calculated as

$$\begin{aligned}
y &= \sin x + c|\cos x| \\
1 &= \sin\pi + c|\cos\pi| \\
\therefore\ \ c &= 1 \ \Rightarrow\ \underline{y(x) = \sin x + |\cos x|}
\end{aligned}$$

6.4 Linear Equations, Constant Coefficients

6.35. Given $y'' - 9y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$\begin{aligned}
y'' - 9y = 0 \ \ \therefore\ \ P_2(r) &= r^2 - 9 = (r-3)(r+3) = 0 \\
\therefore\ \ r_1 = 3,\ \ r_2 &= -3\ \ (r_1 \neq r_2)
\end{aligned}$$

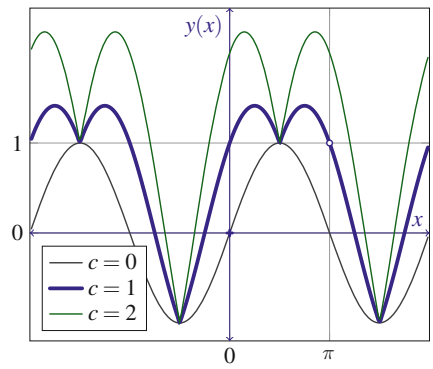


Fig. 6.9 Example P.6.34

Therefore, as the two roots are unique, the solution is written as

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{3x} + c_2 e^{-3x}$$

Verification:

$$\begin{aligned} y(x) &= c_1 e^{3x} + c_2 e^{-3x} \\ y'(x) &= 3c_1 e^{3x} - 3c_2 e^{-3x} \\ y''(x) &= 9c_1 e^{3x} + 9c_2 e^{-3x} \end{aligned}$$

Therefore,

$$\begin{aligned} y'' - 9y &= 0 \\ \left(9c_1 e^{3x} + 9c_2 e^{-3x}\right) - 9\left(c_1 e^{3x} + c_2 e^{-3x}\right) &= 0 \\ \cancel{9c_1 e^{3x}} + \cancel{9c_2 e^{-3x}} - \cancel{9c_1 e^{3x}} - \cancel{9c_2 e^{-3x}} &= 0 \\ 0 &= 0 \quad \checkmark \end{aligned}$$

6.36. Given $y'' + y' - 2y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$\begin{aligned} y'' + y' - 2 = 0 \quad \therefore \quad P_2(r) = r^2 + r - 2 = (r+2)(r-1) = 0 \\ \therefore \quad r_1 = -2, \quad r_2 = 1 \quad (r_1 \neq r_2) \end{aligned}$$

Therefore, as the two roots are unique, the solution is written as

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{-2x} + c_2 e^{x}$$

Verification:

$$\begin{aligned} y(x) &= c_1 e^{-2x} + c_2 e^{x} \\ y'(x) &= -2c_1 e^{-2x} + c_2 e^{x} \\ y''(x) &= 4c_1 e^{-2x} + c_2 e^{x} \end{aligned}$$

Therefore,

$$\begin{aligned} y'' + y' - 2 &= 0 \\ 4c_1 e^{-2x} + c_2 e^{x} - 2c_1 e^{-2x} + c_2 e^{x} - 2c_1 e^{-2x} - 2c_2 e^{x} &= 0 \\ 0 &= 0 \quad \checkmark \end{aligned}$$

6.37. Given $-8y'' + 2y' + y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$-8y'' + 2y' + y = 0 \quad \therefore \quad P_2(r) = -8r^2 + 2r + 1 = -8r^2 + 4r - 2r + 1 = -4r(2r-1) - (2r-1)$$
$$= -(4r+1)(2r-1) = 0$$
$$\therefore \quad r_1 = -\frac{1}{4}, \quad r_2 = \frac{1}{2} \quad (r_1 \neq r_2)$$

Therefore, as the two roots are unique, the solution is written as

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{-x/4} + c_2 e^{x/2}$$

Verification:

$$y(x) = c_1 e^{-x/4} + c_2 e^{x/2}$$
$$y'(x) = -\frac{1}{4} c_1 e^{-x/4} + \frac{1}{2} c_2 e^{x/2}$$
$$y''(x) = \frac{1}{16} c_1 e^{-x/4} + \frac{1}{4} c_2 e^{x/2}$$

Therefore,

$$-8y'' + 2y' + y = 0$$
$$-8\left(\frac{1}{16} c_1 e^{-x/4} + \frac{1}{4} c_2 e^{x/2}\right) + 2\left(-\frac{1}{4} c_1 e^{-x/4} + \frac{1}{2} c_2 e^{x/2}\right) + \left(c_1 e^{-x/4} + c_2 e^{x/2}\right) = 0$$
$$-\frac{1}{2} c_1 e^{-x/4} - \frac{1}{2} c_1 e^{-x/4} + c_1 e^{-x/4} - 2c_2 e^{x/2} + c_2 e^{x/2} + c_2 e^{x/2} = 0$$
$$0 = 0 \quad \checkmark$$

6.38. Given $y'' - 2y' + y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$y'' - 2y' + y = 0 \quad \therefore \quad \therefore \quad P_2(r) = r^2 - 2r + 1 = (r-1)(r-1) = 0$$
$$\therefore \quad r_1 = 1, \quad r_2 = 1 \quad (r_1 = r_2 = r)$$

Therefore, as there is duplicity of the roots, the solution is written as a linear combination

$$y(x) = c_1 e^{rx} + x\, c_2 e^{rx} = c_1 e^x + x\, c_2 e^x = e^x (c_1 + x\, c_2)$$

Otherwise, the two solution terms would not be independent.

Verification:

$$y(x) = c_1 e^x + x c_2 e^x = e^x (c_1 + x\, c_2)$$
$$\{[f\, g]' = f' g + f g'\}$$
$$y'(x) = c_1 e^x + c_2 e^x + x c_2 e^x = e^x (c_1 + c_2 (x+1))$$

$$y''(x) = c_1e^x + c_2e^x + c_2e^x + xc_2e^x = e^x(c_1 + c_2(x+2))$$

Therefore,

$$\begin{aligned} y'' - 2y' + y = 0 &= 0 \\ \left[e^x(c_1 + c_2(x+2))\right] - 2\left[e^x(c_1 + c_2(x+1))\right] + \left[e^x(c_1 + x\,c_2)\right] &= 0 \quad \div e^x \\ c_1 + xc_2 + 2c_2 - 2c_1 - 2xc_2 - 2c_2 + c_1 + x\,c_2 &= 0 \\ 0 &= 0 \ \checkmark \end{aligned}$$

6.39. Given second-order homogenous differential equation with constant coefficients $y'' + 4y' + 4y = 0$, roots of its characteristic polynomial $P(r)$ are

$$\begin{aligned} y'' + 4y' + 4y = 0 \ \ \therefore \ \ P_2(r) &= r^2 + 4r + 4 = (r+2)(r+2) = 0 \\ \therefore \ \ r_1 = -2, \ \ r_2 &= -2 \ \ (r_1 = r_2 = r) \end{aligned}$$

Therefore, as there is duplicity of the roots, the solution is written as a linear combination

$$y(x) = c_1e^{rx} + x\,c_2e^{rx} = c_1e^{-2x} + x\,c_2e^{-2x} = e^{-2x}(c_1 + x\,c_2)$$

which can be confirmed by the verification.

6.40. Given second-order homogenous differential equation with constant coefficients $y'' + 4y' + 4y = 0$, roots of its characteristic polynomial $P(r)$ are

$$\begin{aligned} 4y'' - 4y' + y = 0 \ \ \therefore \ \ P_2(r) &= 4r^2 - 4r + 1 = 4r^2 - 2r - 2r + 1 \\ &= 2r(2r-1) - (2r-1) = (2r-1)^2 = 0 \\ \therefore \ \ r_1 = \frac{1}{2}, \ \ r_2 &= \frac{1}{2} \ \ (r_1 = r_2 = r) \end{aligned}$$

Therefore, as there is duplicity of the roots, the solution is written as a linear combination

$$y(x) = c_1e^{rx} + x\,c_2e^{rx} = c_1e^{x/2} + x\,c_2e^{x/2} = e^{x/2}(c_1 + x\,c_2)$$

which can be confirmed by the verification.

6.41. Given $y'' + 6y' + 25y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$\begin{aligned} y'' + 6y' + 25y = 0 \ \ \therefore \ \ P_2(r) &= r^2 + 6r + 25 \\ r_{1,2} = \frac{-6 \pm \sqrt{36 - 4 \times 25}}{2} &= -3 \pm 4i \qquad \therefore \\ r_1 &= -3 + 4i \\ r_2 &= -3 - 4i \ \ (r_1 \neq r_2^*) \end{aligned}$$

In the case of unique roots, even if complex conjugate, the solution is written as

$$\begin{aligned}
y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(-3+4i)x} + c_2 e^{(-3-4i)x} \\
&= e^{-3x}\left[c_1 e^{4xi} + c_2 e^{-4xi}\right] \\
&\left\{e^{ix} = \cos x + i \sin x\right\} \\
&= e^{-3x}\left[c_1(\cos 4x + i \sin 4x) + c_2(\cos(-4x) + i \sin(-4x))\right] \\
&\left\{\cos(-x) = \cos x, \quad \sin(-x) = -\sin x\right\} \\
&= e^{-3x}\left[c_1(\cos 4x + i \sin 4x) + c_2(\cos 4x - i \sin 4x)\right] \\
&= e^{-3x}\left[(c_1 + c_2)\cos 4x + i(c_1 - c_2)\sin 4x\right] \\
&\left\{c_1 + c_2 = d_1, \quad i(c_1 - c_2) = d_2\right\} \\
&= e^{-3x}\left(d_1 \cos 4x + d_2 \sin 4x\right)
\end{aligned}$$

As an illustration, assuming a simple case of $d_1 = d_2 = 1$, the plot of $y(x)$ module is shown in Fig. 6.10. As it is bound by e^{-3x}, for $x < 0$, the amplitude of $y(x)$ tends to infinity, while for $x \geq 0$, it is damped by the e^{-3x} term and therefore tends to zero.

6.42. Given $y'' - 2y' + 2y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$\begin{aligned}
y'' - 2y' + 2y = 0 \quad &\therefore \quad P_2(r) = r^2 - 2r + 2 \\
r_{1,2} &= \frac{2 \pm \sqrt{4 - 4 \times 8}}{2} = 1 \pm i \qquad \therefore \\
r_1 = 1 + i \quad &\text{and} \quad r_2 = 1 - i \quad (r_1 \neq r_2^*)
\end{aligned}$$

In the case of unique roots, even if complex conjugate, the solution is written as

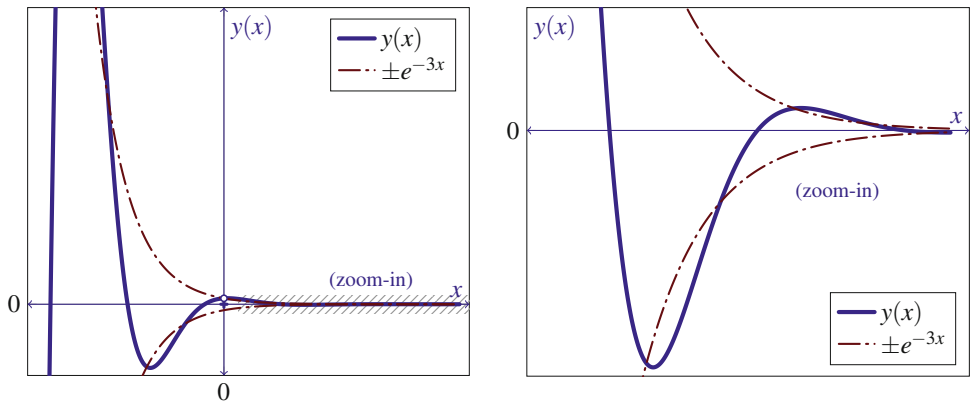


Fig. 6.10 Example P.6.41

$$
\begin{aligned}
y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(-3+4i)x} + c_2 e^{(-3-4i)x} \\
&= e^{-3x}\left[c_1 e^{4xi} + c_2 e^{-4xi}\right] \\
&\left\{e^{ix} = \cos x + i \sin x\right\} \\
&= e^{-3x}\left[c_1(\cos 4x + i \sin 4x) + c_2(\cos(-4x) + i \sin(-4x))\right] \\
&\left\{\cos(-x) = \cos x, \quad \sin(-x) = -\sin x\right\} \\
&= e^{-3x}\left[c_1(\cos 4x + i \sin 4x) + c_2(\cos 4x - i \sin 4x)\right] \\
&= e^{-3x}\left[(c_1 + c_2)\cos 4x + i(c_1 - c_2)\sin 4x\right] \\
&\left\{c_1 + c_2 = D_1, \quad i(c_1 - c_2) = D_2\right\} \\
&= e^{-3x}\left(D_1 \cos 4x + D_2 \sin 4x\right) \quad [D_{1,2} \in \mathbb{C}]
\end{aligned}
$$

6.43. Given $y'' - 2y' + 2y = 0$ second-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$
\begin{aligned}
y'' - 4y' + 5y = 0 = 0 \quad \therefore \quad P_2(r) &= r^2 - 4r + 5 \\
r_{1,2} = \frac{4 \pm \sqrt{16 - 4 \times 5}}{2} &= 2 \pm i \qquad \therefore \\
r_1 = 2 + i \quad \text{and} \quad r_2 &= 2 - i \quad (r_1 \neq r_2^*)
\end{aligned}
$$

In the case of unique roots, even if complex conjugate, the solution is written as

$$
\begin{aligned}
y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} x = c_1 e^{(2+i)x} + c_2 e^{(2-i)x} \\
&= e^{2x}\left[c_1 e^{ix} + c_2 e^{-ix}\right] \\
&\left\{e^{ix} = \cos x + i \sin x\right\} \\
&= e^{2x}\left[c_1(\cos x + i \sin x) + c_2(\cos(-x) + i \sin(-x))\right] \\
&\left\{\cos(-x) = \cos x, \quad \sin(-x) = -\sin x\right\} \\
&= e^{2x}\left[c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)\right] \\
&= e^{2x}\left[(c_1 + c_2)\cos x + i(c_1 - c_2)\sin x\right] \\
&\left\{c_1 + c_2 = d_1, \quad i(c_1 - c_2) = d_2\right\} \\
&= e^{2x}\left(d_1 \cos x + d_2 \sin x\right)
\end{aligned}
$$

6.44. Given $y''' - 3y'' + 3y' - y = 0$ third-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$
\begin{aligned}
y''' - 3y'' + 3y' - y = 0 \quad \therefore \quad P_3(r) &= r^3 - 3r^2 + 3r - 1 = \left\{\text{see Vol. I}\right\} = (r-1)^3 \\
\therefore \quad r_1 = r_2 = r_3 = r &= 1
\end{aligned}
$$

Therefore, as there is triplicity of the roots, the solution is written as a linear combination

$$y(x) = c_1 e^{rx} + x\, c_2 e^{rx} + x^2\, c_2 e^{rx} = c_1 e^x + x\, c_2 e^x + x^2\, c_3 e^x = e^x (c_1 + c_2\, x + c_3\, x^2)$$

which can be confirmed by the verification.

6.45. Given $y''' - 2y'' - y + 2 = 0$ third-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$y''' - 2y'' - y + 2 = 0 \quad \therefore \quad P_3(r) = r^3 - 2r^2 - r + 2 = \{\text{see Vol. I}\} = (r+1)(r-1)(r-2)$$
$$\therefore \quad r_1 = -1,\ r_2 = 1,\ r_3 = 2$$

Therefore, all roots are unique; the solution is written as a linear combination

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x} = c_1 e^{-x} + c_2 e^x + c_3 e^{2x}$$

which can be confirmed by the verification.

6.46. Given $y^{(iv)} - 2y'' = 0$ fourth-order homogenous differential equation with constant coefficients, roots of its characteristic polynomial $P(r)$ are

$$y^{(iv)} - 2y'' = 0 \quad \therefore \quad P_3(r) = r^4 - 2r^2 = r^2(r^2 - 2) = (r + \sqrt{2})(r - \sqrt{2})(r - 0)(r - 0)$$
$$\therefore \quad r_1 = 0,\ r_2 = 0,\ r_3 = \sqrt{2},\ r_4 = -\sqrt{2}$$

Therefore, as two roots are unique and two are equal, the solution is written as a linear combination

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x} + c_4 e^{r_4 x} = c_1 e^0 + c_2\, x e^0 + c_3 e^{\sqrt{2}} + c_4 e^{-\sqrt{2}}$$
$$= c_1 + c_2\, x + c_3 e^{\sqrt{2}} + c_4 e^{-\sqrt{2}}$$

which can be confirmed by the verification.

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